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A fractional boundary value problem

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We consider the boundary value problem consisting of the fractional differential equation

$$-D_{0+}^{\alpha}z + aD_{0+}^{\beta}z = w(t)f(t,z), \quad 0 < t < 1, \tag{1}$$

and the boundary conditions

$$z(0) = z'(0) = ... = z^{(n-2)}(0) = 0, \quad z(1) = k, \tag{2}$$

where $k \in \mathbb{R}$, $a \in \mathbb{R}$, $n \in \mathbb{N}$, $0 \leq \beta \leq n-2$, $n-1 < \alpha < n$, $w \in L[0,1]$ with $w(t) \geq 0$, and $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$. Here, $D_{0+}^{\gamma}h$ is the γ-th left Riemann-Liouville fractional derivative of $h : [0,1] \to \mathbb{R}$ defined by

$$(D_{0+}^{\gamma}h)(t) = \frac{1}{\Gamma(l-\gamma)} \frac{d^l}{dt^l} \int_0^t (t-s)^{l-\gamma-1} h(s)ds, \ l = \lfloor \gamma \rfloor + 1,$$

whenever the right-hand side exists with $\Gamma(\cdot)$ being the Gamma function.

Fractional differential equation models such as this are relevant in various science and engineering fields. Problems like this have been solved by first finding the Green's function for z in similar equations such as

$$-D_{0+}^{\alpha}z = f(t,z).$$

The Green's function (denoted by G) was developed as a method to solve boundary value problems with linear homogeneous equations (i.e. when $f(t,z) = 0$). However, when the right side of equation is a function of t, say $w(t)f(t,z(t))$, the unique solution to the problem is given by

$$u(t) = \int_0^1 G(t,s)w(s)f(s,z(s))ds,$$

where G is the solution to the similar problem with the right hand side equal to zero.

We try to solve problem (1)-(2) by splitting it into the two following problems:

$$\begin{cases} -D^{\alpha}_{0+}u + aD^{\beta}_{0+}u = w(t)f(t,u),\ 0 < t < 1, \\ u(0) = u'(0) = ... = u^{(n-2)}(0) = 0, \quad u(1) = 0, \end{cases} \tag{3}$$

$$\begin{cases} -D^{\alpha}_{0+}v + aD^{\beta}_{0+}v = 0,\ 0 < t < 1, \\ v(0) = v'(0) = ... = v^{(n-2)}(0) = 0, \quad v(1) = k. \end{cases} \tag{4}$$

Note that if $u(t)$ is a solution of (3) and $v(t)$ is a solution of (4), then $z(t) = u(t)+v(t)$ is a solution of (1)-(2).

The solution of (3) is found by using the Green's Function: $G : [0,1] \times [0,1] \to \mathcal{R}$, which was obtained in [3]:

$$G(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1}E_{\alpha-\beta,\alpha}[at^{\alpha-\beta}]E_{\alpha-\beta,\alpha}[a(1-s)^{\alpha-\beta}]}{E_{\alpha-\beta,\alpha}[a]}, & 0 \le t \le s \le 1, \\ \dfrac{[t(1-s)]^{\alpha-1}E_{\alpha-\beta,\alpha}[at^{\alpha-\beta}]E_{\alpha-\beta,\alpha}[a(1-s)^{\alpha-\beta}]}{E_{\alpha-\beta,\alpha}[a]} & \\ \qquad -(t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}[a(t-s)^{\alpha-\beta}], & 0 \le s \le t \le 1, \end{cases} \tag{5}$$

with $|a| < \Gamma(\alpha - \beta + 1)$. Here, the Green's Function is given in terms of the known Mittag-Leffler function $E_{\gamma,\delta} : \mathcal{C} \to \mathcal{C}$:

$$E_{\gamma,\delta}[z] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\gamma n + \delta)}.$$

Using this function, we find the solution to (3) to have the form:

$$u(t) = \int_0^1 G(t,s)w(s)f(s,u(s))ds$$

The solution to (4) is easily seen to be:

$$v(t) = \frac{kt^{\alpha-1}E_{\alpha-\beta,\alpha}(at^{\alpha-\beta})}{E_{\alpha-\beta,\alpha}(a)}.$$

Note that by Lemma 4.3 of [3], the Mittag-Leffler function, E, is increasing, so $|v(t)| \leq k$ for $t \in [0,1]$.

The solution to (1)-(2) is therefore $z(t) = u(t) + v(t)$.

Remark: As found in [3], for G defined in (5), when $|a| < \Gamma(\alpha - \beta + 1)$, $G(0,\cdot) = G(1,\cdot) = G(\cdot,0) = G(\cdot,1) = 0$, and $|G(t,s)| \leq \overline{G}(s)$ on $[0,1] \times [0,1]$ where

$$\overline{G}(s) = \begin{cases} (1-s)^{\alpha-1}E_{\alpha-\beta,\alpha}[a(1-s)^{\alpha-\beta}], & a \in [0, \Gamma(\alpha-\beta+1)), \\ \dfrac{(1-s)^{\alpha-1}E_{\alpha-\beta,\alpha}[a(1-s)^{\alpha-\beta}]}{\Gamma(\alpha)E_{\alpha-\beta,\alpha}[a]}, & a \in (-\Gamma(\alpha-\beta+1), 0). \end{cases} \tag{6}$$

We need to define the constant

$$U = \int_0^1 \overline{G}(s)w(s)ds \tag{7}$$

with $\overline{G}$ defined in (6). We take $X = C[0,1]$ with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$, $u \in X$, to be our Banach space.

We have developed two main theorems on the existance and uniqueness of solutions for our problem.

Theorem 1: Let $|a| < \Gamma(\alpha - \beta + 1)$. Assume there exists $r > 0$ such that

$$|f(t,x)| \leq \frac{r-k}{U} \quad \text{for} \quad (t,x) \in [0,1] \times [-r,r], \tag{8}$$

where k is the constant given in the right hand boundary condition given in (2). Then, BVP (1)-(2) has at least one solution z with $\|z\| \leq r$.

Theorem 2: Let $|a| < \Gamma(\alpha - \beta + 1)$. Assume that f satisfies the Lipschitz condition in x

$$|f(t,x_1) - f(t,x_2)| \le B|x_1 - x_2| \quad \text{for } (t,x_1),\ (t,x_2) \in [0,1] \times \mathbb{R},$$

with $B \in (0, 1/U)$. Then, BVP (1)-(2) has a unique solution.

Proofs:

For the purpose of these proofs, we need to define an operator $T: X \to X$ by

$$(Tz)(t) = \int_0^1 G(t,s)w(s)f(s,u(s))ds + \frac{kt^{\alpha-1}E_{\alpha-\beta,\alpha}(at^{\alpha-\beta})}{E_{\alpha-\beta,\alpha}(a)},$$

where G is the Green's function used to solve for $u(t)$. It is clear that z is a solution of BVP (1)-(2) if and only if $z \in X$ is a fixed point of T. There is a standard argument to show that T is completely continuous.

Proof of Theorem 1: Let $\Omega \subset X$ be the set defined by

$$\Omega = \{z \in X \mid \|z\| \le r\},$$

where r is given in Theorem 1.

For any $z \in \Omega$, we have $\|z(t)\| \le r$ on $[0,1]$, and so

$$\begin{aligned}
|(Tz)(t)| =& \left| \int_0^1 G(t,s)w(s)f(s,u(s))ds + \frac{kt^{\alpha-1}E_{\alpha-\beta,\alpha}(at^{\alpha-\beta})}{E_{\alpha-\beta,\alpha}(a)} \right| \\
\le& \left| \int_0^1 G(t,s)w(s)f(s,u(s))ds \right| + \left| \frac{kt^{\alpha-1}E_{\alpha-\beta,\alpha}(at^{\alpha-\beta})}{E_{\alpha-\beta,\alpha}(a)} \right| \\
\le& \int_0^1 |G(t,s)|w(s)|f(s,u(s))|ds + k \\
\le& \int_0^1 \frac{\overline{G}(s)w(s)(r-k)}{U} ds + k = \frac{r-k}{U}U + k \\
=& r - k + k = r, \quad t \in [0,1].
\end{aligned}$$

Hence, $\|Tu\| \le r$. Therefore, $T\Omega \subset \Omega$.

By Schauder fixed point theorem, T has a fixed point z in Ω. Hence, our BVP has at least one solution $z(t)$ with $\|z\| \leq r$. □

Proof of Theorem 2: Let $z_1,\, z_2 \in X$ be solutions of (0.1). Then $z_1 = u_1 + v_1$ and $z_2 = u_2 + v_2$, where u_1, u_2 are solutions of (0.3) and v_1, v_2 are solutions of (0.4). For $t \in [0,1]$,

$$\begin{aligned}
|(Tz_1 - Tz_2)(t)| &= \Bigg| \int_0^1 G(t,s)w(s)f(s,u_1(s)))ds + \frac{kt^{\alpha-1}E_{\alpha-\beta,\alpha}(at^{\alpha-\beta})}{E_{\alpha-\beta,\alpha}(a)} \\
&\quad - \int_0^1 G(t,s)w(s)f(s,u_2(s)))ds - \frac{kt^{\alpha-1}E_{\alpha-\beta,\alpha}(at^{\alpha-\beta})}{E_{\alpha-\beta,\alpha}(a)} \Bigg| \\
&= \left| \int_0^1 G(t,s)w(s)f(s,u_1(s)))ds - \int_0^1 G(t,s)w(s)f(s,u_2(s)))ds \right| \\
&= \left| \int_0^1 G(t,s)w(s)\left(f(s,u_1(s)) - f(s,u_2(s))\right)ds \right| \\
&\leq \int_0^1 \overline{G}(s)w(s)\left|f(s,u_1(s)) - f(s,u_2(s))\right| ds \\
&\leq \int_0^1 \overline{G}(s)w(s)B\left|u_1(s) - u_2(s)\right| ds \leq BU\|u_1 - u_2\|.
\end{aligned}$$

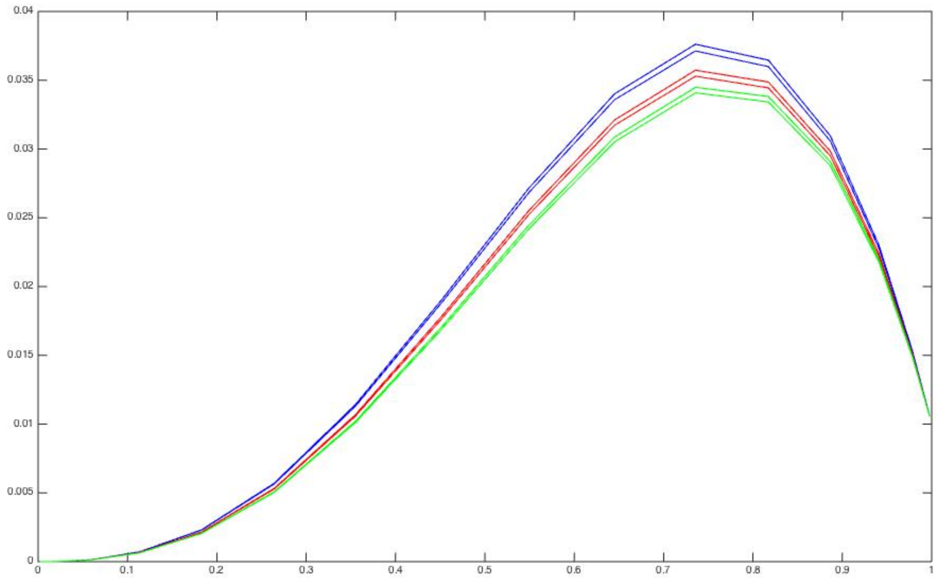
Since $BU < 1$, T is a contraction mapping. By the contraction mapping principle, T has a unique fixed point. Therefore, our BVP has a unique solution. □

Examples:

Example 1: Consider the BVP

$$\begin{cases} -D_{0+}^{3.6}z + aD_{0+}^{1.4}z = \left[\cos(20t) + 1 + e^{\sin(100t)}\right]\left[\frac{1}{2}\sin(z) + \frac{z}{z+1} + e^t\right] \\ z(0) = z'(0) = \cdots = z^{(n-2)} = 0, \quad z(1) = 0.01 \end{cases} \tag{9}$$

Here, $f(t,z) = \frac{1}{2}\sin(z) + \frac{z}{z+1} + e^t$ and $w(t) = \cos(20t) + 1 + e^{\sin(100t)}$. Furthermore, $\Gamma(3.2) = 2.424$. The following figure represents the solution to this problem using an iterative process with $a = 2.4$ shown in green, $a = 0.4$ shown in red, and $a = -2.4$ shown in blue.



Example 2: Consider the BVP

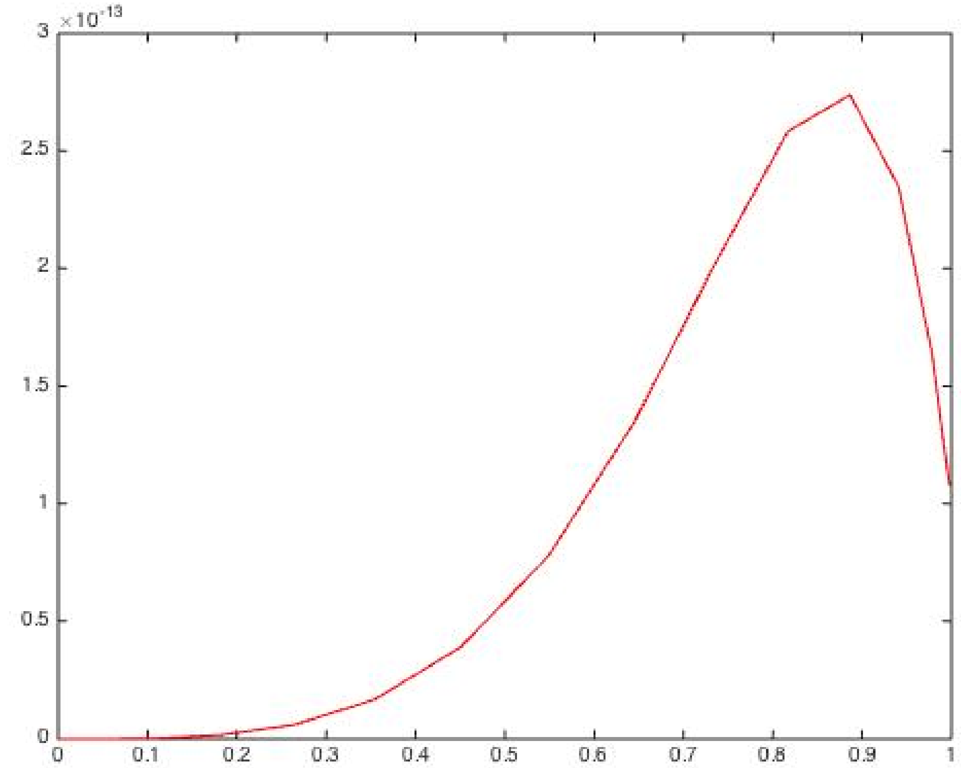
$$\begin{cases} -D_{0+}^{4.5}z + 1.32D_{0+}^{2.7}z = \left[\frac{z^2}{z^2+1}\right][\cos(t)+1] \\ z(0) = z'(0) = \cdots = z^{(n-2)} = 0, \quad z(1) = 10^{-13} \end{cases} \tag{10}$$

Here, $f(t,z) = \frac{z^2}{z^2+1}$ and $w(t) = \cos(t)+1$. Note that $|a| = 1.32 < 1.6765 \approx \Gamma(2.8)$. We claim that this problem has a unique solution. By (0.6) and (0.7), $1/U \approx 24.5834$. Then, for $(t, z_1),\ (t, z_2) \in [0,1] \times \mathbb{R}$:

$$\begin{aligned} |f(t,z_1) - f(t,z_2)| &= \left| \frac{z_1^2}{z_1^2+1} - \frac{z_2^2}{z_2^2+1} \right| \\ &= \left| \frac{z_1^2(z_2^2+1) - z_2^2(z_1^2+1)}{(z_1^2+1)(z_2^2+1)} \right| \\ &= \left| \frac{z_1^2 - z_2^2}{(z_1^2+1)(z_2^2+1)} \right| \\ &= \left| \frac{(z_1-z_2)(z_1+z_2)}{(z_1^2+1)(z_2^2+1)} \right| \\ &= |z_1 - z_2| \left| \frac{z_1+z_2}{(z_1^2+1)(z_2^2+1)} \right| \end{aligned}$$

$$
\begin{aligned}
=&|z_1 - z_2| \left| \frac{z_1}{(z_1^2+1)(z_2^2+1)} + \frac{z_2}{(z_1^2+1)(z_2^2+1)} \right| \\
<&|z_1 - z_2| \left| \frac{z_1}{z_1^2+1} + \frac{z_2}{z_2^2+1} \right| \\
<&|z_1 - z_2||1+1| \\
=&2|z_1 - z_2|.
\end{aligned}
$$

Therefore, f satisfies the Lipschitz condition with $B = 2 \in (0, 1/U)$. Hence by Theorem 2, BVP (0.10) has a unique solution. The numerical solution is given below.



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