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An explicit formula for Dirichlet's L-Function

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Abstract

An Explicit Formula for Dirichlet's L-Function

by Shannon M. Hyder

The Riemann zeta function has a deep connection to the distribution of primes. In 1911 Landau proved that, for every fixed $x > 1$,

$$\sum_{0<\gamma\leq T} x^\rho = -\frac{T}{2\pi}\Lambda(x) + O(\log T)$$

as $T \to \infty$. Here $\rho = \beta + i\gamma$ denotes a complex zero of the zeta function and $\Lambda(x)$ is an extension of the usual von Mangoldt function, so that $\Lambda(x) = \log p$ if x is a positive integral power of a prime p and $\Lambda(x) = 0$ for all other real values of x. Landau's remarkable explicit formula lacks uniformity in x and therefore has limited applications to the theory of the zeta function. In 1993 Gonek proved a version of Landau's explicit formula that is uniform in both variables x and T. This explicit formula was used to estimate various sums involving the zeros of the zeta function, such as the discrete mean value formula for the zeta function. The purpose of this thesis is to obtain a generalization of Landau's and Gonek's explicit formulas in terms of the zeros of the Dirichlet L-function. To accomplish this, we employ the argument principal, Cauchy's residue theorem, and an inequality of Selberg.

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Contents

Chapter 1

The Functions $\zeta(s)$ and $L(s,\chi)$

1.1 Definition of $\zeta(s)$ and Generalization

In this chapter we define the Riemann zeta function as a function of a complex variable and identify its basic properties. In Riemann's memoir the zeta function is defined as a Dirichlet series and as an Euler product. If s is the complex variable $\sigma + it$ with $\mathrm{Re}(s) = \sigma > 1$, then the zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.1}$$

If $\sigma \geq \sigma_0 > 1$, then the Dirichlet series (1.1) converges uniformly and absolutely, since

$$\begin{aligned}\left|\sum_{n=1}^{\infty} \frac{1}{n^{\sigma+it}}\right| &\leq \sum_{n=1}^{\infty} \frac{1}{|n^{\sigma+it}|} = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n^{\sigma_0}} < 1 + \int_1^{\infty} \frac{du}{u^{\sigma_0}} \\ &= 1 + \frac{1}{\sigma_0 - 1}.\end{aligned}$$

By Weierstrass' theorem, $\zeta(s)$ is holomorphic for $\sigma > 1$. Euler had given many results in number theory using product formulas. One of these product formulas was the formula for $\zeta(s)$. Namely, Euler used

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{1.2}$$

to show that the series $\sum_p 1/p$ diverges. Here, both the product and the sum are taken over all primes p. The Euler identity (1.2), also valid in the region $\sigma > 1$, shows the remarkable connection that exists between $\zeta(s)$ and the primes.

There are many different ways in which $\zeta(s)$ can be generalized. We shall describe only one of these generalizations, which we will use to answer the question considered in this thesis. We commence by letting m be a positive integer and using $\chi(n)$ to denote a Dirichlet character modulo m. If we set $s = \sigma + it$ with $\sigma > 1$, then we may define the Dirichlet L-function by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

The function $L(s, \chi)$ is a holomorphic function in the region $\sigma > 1$. In addition, analogous to the Euler product for $\zeta(s)$, since $\chi(n)$ is multiplicative, $L(s, \chi)$ satisfies the identity

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

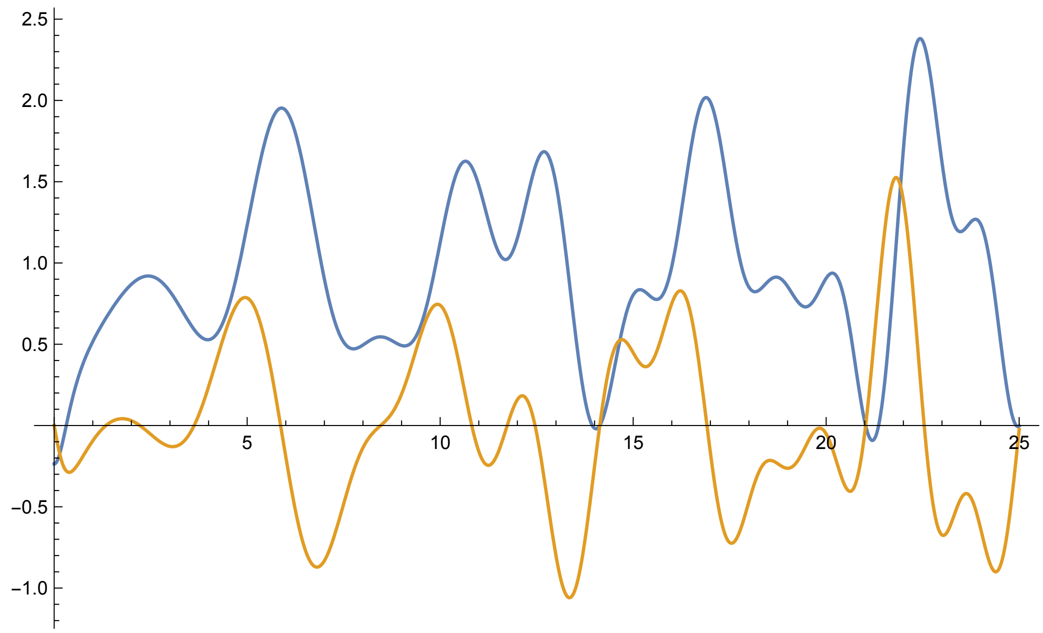
where p again runs over all the primes, in the region $\sigma > 1$. One can see this by supposing that $\chi(n)$ is the principle character modulo m; i.e., $\chi_0(n) = 1$ for $(n, m) = 1$ and $\chi_0(n) = 0$ for $(n, m) > 1$. Then

$$\begin{aligned} L(s, \chi_0) = \prod_p \left(1 - \frac{\chi_0(p)}{p^s}\right)^{-1} &= \prod_{(p,m)=1} \left(1 - \frac{1}{p^s}\right)^{-1} \\ &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{p|m} \left(1 - \frac{1}{p^s}\right) \\ &= \zeta(s) \prod_{p|m} \left(1 - \frac{1}{p^s}\right). \end{aligned}$$

This simple calculation shows that $L(s, \chi)$ differs from $\zeta(s)$ only by the factor

$$\prod_{p|m} \left(1 - \frac{1}{p^s}\right).$$

The function $L(s,\chi)$ plays an important role in problems that concern the existence of primes in a given arithmetic progression. The picture below shows a parametric plot of $L(s,\chi)$ with modulus 10 along the critical line $\sigma = 1/2$. This parametric plot is obtained by superimposing the plot of $\mathrm{Re}(L(s,\chi))$ (blue curve) on top of $\mathrm{Im}(L(s,\chi))$ (orange curve).



The functions $\zeta(s)$ and $L(s,\chi)$ have many similar basic properties. One such property is that $\zeta(s)$ and $L(s,\chi)$ can be extended to the entire complex plane. This means that there exists analytic functions $F(s)$ and $G(s,\chi)$ which are defined for the entire complex plane that have the property that $\zeta(s) = F(s)$ and $L(s,\chi) = G(s,\chi)$ in the region $\sigma > 1$. For $\zeta(s)$ Riemann used contour integration to establish the identity

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty (x^{s/2-1} + x^{-s/2-1/2})\omega(x)\,dx, \tag{1.3}$$

where

$$\omega(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x},$$

which holds in the region $\sigma > 1$. The Gamma function is defined through Euler's integral

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1}\, dt,$$

which only applies for $\sigma > 0$. Since $\omega(x) = O(e^{-\pi x})$ as x approaches infinity, the improper integral on the right-hand side of (1.3) converges absolutely and uniformly in the region $\sigma > K$ for some absolute constant K. By Weierstrass' theorem, this integral is holomorphic in the entire complex plane.

1.2 Functional Equations of $\zeta(s)$ and $L(s,\chi)$

In (1.3) the Gamma function has a first-order pole at the point $s = 0$ and $\Gamma(1/2) = \sqrt{\pi}$. Because the right-hand side of (1.3) is unchanged if s is replaced by $1 - s$, (1.3) gives both the analytic property for $\zeta(s)$ onto the entire complex plane and the functional equation for $\zeta(s)$. Thus, by (1.3),

$$F(s) = \pi^{s/2}\Gamma^{-1}\left(\frac{s}{2}\right)\left(\frac{1}{s(s-1)} + \int_1^{\infty} (x^{s/2-1} + x^{-s/2-1/2})\omega(x)\, dx\right),$$

which shows that $F(s)$ is holomorphic on the entire complex plane except for the point $s = 1$, where it has a simple pole with residue 1. If, now, $\xi(s)$ is defined to be the entire function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

then it is easy to see that

$$\xi(s) = \xi(1-s)$$

or, equivalently,

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Next, an analytic continuation of $L(s,\chi)$ is achieved by the following result: If we let $\chi(n)$ be a non-principal character modulo m and set

$$S(x) = \sum_{n \le x} \chi(n),$$

then we would have for $\sigma > 1$

$$L(s,\chi) = s\int_1^\infty S(x)x^{-s-1}\,dx.$$

Clearly, this result gives an analytic continuation of $L(s,\chi)$ to the half-plane $\sigma > 0$. To extend $L(s,\chi)$ to the entire complex plane we will need another result: If $\chi(n)$ is a primitive character modulo k and

$$\delta = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1, \end{cases}$$

and

$$\psi(s,\chi) = (\pi k^{-1})^{-(s+\delta)/2}\Gamma\left(\frac{s+\delta}{2}\right)L(s,\chi),$$

then

$$\psi(1-s,\overline{\chi}) = \frac{i^\delta\sqrt{k}}{g(\chi)}\psi(i,\chi),$$

where $g(\chi)$ is the Gauss sum

$$g(\chi) = \sum_{n=1}^{k}\chi(n)e^{2\pi in/k}.$$

Here, we note that $L(s,\chi)$ is an entire function.

1.3 Connection Between $\zeta(s)$ and the Distribution of Primes

Let $\pi(x)$ be the number of primes less than or equal to x. Taking logarithms on both sides of (1.2) and using the summation by parts formula, we obtain

$$\begin{aligned}\log\zeta(s) &= -\sum_p \log\left(1-\frac{1}{p^s}\right)\\ &= -\int_1^\infty \log\left(1-\frac{1}{x^s}\right)\,d\pi(x) \qquad (1.4)\\ &= s\int_1^\infty \frac{\pi(x)}{x(x^{s-1})}\,dx.\end{aligned}$$

This identity leads to an important approach in the study of $\pi(x)$ through $\log\zeta(s)$. As it happens, a more convenient function to study the distribution of primes is the Tchebyshev function $\psi(x)$. To introduce this function, we put

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, \\ 0 & \text{otherwise,} \end{cases}$$

and use it to show that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \tag{1.5}$$

where $\Lambda(n)$ is the von Mangoldt function. The sum of its values over all $n \leq x$ is $\psi(x)$. To derive (1.5), we compute

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= -\frac{d}{ds}\left(\log\zeta(s)\right) \\ &= \frac{d}{ds}\left(-\log\prod_p\left(1-\frac{1}{p}\right)^{-1}\right) \\ &= \frac{d}{ds}\left(\sum_p \log\left(1-\frac{1}{1-p^s}\right)\right) \\ &= \sum_p \frac{d}{ds}\left(\log\left(1-\frac{1}{1-p^s}\right)\right) \\ &= \sum_p \log p \left(p^{-s}+p^{-2s}+\cdots\right) \\ &= \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}. \end{aligned}$$

This is because the series over p are uniformly convergent for $\sigma > 1+\delta$ with $\delta > 0$.

We now obtain an identity analogous to (1.4) which contains $\psi(x)$. Using integration by parts, we obtain

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} = \int_1^{\infty} x^{-s}\,d\psi(x) = s\int_1^{\infty} x^{-s-1}\psi(x)\,dx. \tag{1.6}$$

Identities (1.4) and (1.6) suggest a profound connection between the distribution of primes and the analytic properties of $\zeta(s)$.

Next, to investigate the distribution of primes in a fixed arithmetic progression, we repeat the above calculation and replace $\zeta(s)$ by $L(s,\chi)$. Let q be a positive integer, let $(a,q)=1$, and let χ be a Dirichlet character. The generalized Tchebyshev functions are

$$\psi(x;q,a) = \sum_{\substack{n\le x \\ n\equiv a \pmod q}} \Lambda(n)$$

and

$$\psi(x,\chi) = \sum_{n\le x} \Lambda(n)\chi(n).$$

Next we have

$$\begin{aligned}
-\frac{L'(s,\chi)}{L(s,\chi)} &= \frac{d}{ds}(\log L(s,\chi)) \\
&= \sum_p \frac{d}{ds}\left(\log\left(1-\frac{\chi(p)}{p^s}\right)\right) \\
&= \sum_p \frac{\chi(p)\log p}{p^s}\left(1-\frac{\chi(p)}{p^s}\right)^{-1} \\
&= \sum_p \log p\left(\frac{\chi(p)}{p^s}+\frac{\chi(p)}{p^{2s}}+\cdots\right) \\
&= \sum_n^\infty \frac{\Lambda(n)\chi(n)}{n^s}.
\end{aligned}$$

Thus

$$-\frac{L'(s,\chi)}{L(s,\chi)} = \int_1^\infty x^{-s}\, d\psi(s,\chi) = s\int_1^\infty x^{-s-1}\psi(s,x)\, dx.$$

1.4 Expression for $\psi(x)$ in Terms of $\zeta(s)$

Equation (1.6) expresses the logarithmic derivative of $\zeta(s)$ in terms of the ψ-function. We now take up the inverse problem of finding an expression for ψ-function in terms of the zeros of $\zeta(s)$.

Theorem 1 *Let $2\le T\le x$. Then we have*

$$\psi(x) = \sum_{n\le x}\Lambda(n) = x - \sum_{|\mathrm{Im}(\rho)|\le T}\frac{x^\rho}{\rho} + O\left(\frac{\log^2 x}{T}\right),$$

where $\rho = \beta + i\gamma$ runs through the nontrivial zeros of $\zeta(s)$.

Proof. For $\mathrm{Re}(s) > 1$, we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \frac{1}{s-1} \sum_{n=1}^{\infty} \left(\frac{1}{s-\rho_n} + \frac{1}{\rho_n} \right) - \sum_{n=1}^{\infty} \left(\frac{1}{s-2n} - \frac{1}{2n} \right) - B_0,$$

where $\rho_n = \beta_n + i\gamma_n$ runs through all of the nontrivial zeros of $\zeta(s)$ and B_0 is some constant. On taking $b = 1/\log x$, we obtain

$$\psi(x) = \frac{1}{2\pi i} \int_{b-iT_1}^{b+iT_1} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^s}{s} \, ds + O\left(\frac{\log^2 x}{T} \right),$$

where $T \le T_1 \le T+1$ and T_1 is choosen in such a way that the distance from the line $\mathrm{Im}(s) = T_1$ to the nearest zero of $\zeta(s) \gg 1/\log T$. We consider the integral

$$I = \frac{1}{2\pi i} \int_{\Gamma} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^s}{s} \, ds,$$

where Γ is a rectangle with vertices at $b \pm iT_1$ and $-5 \pm iT_1$.

By Cauchy's theorem, we have

$$I = x - \sum_{|\mathrm{Im}(\rho)| \le T_1} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)}$$

We now estimate the integrals over the line segments $[b+iT_1, -1/2+iT_1]$, $[-1/2+iT_1, -1/2-iT_1]$, and $[-1/2-iT_1, b-iT_1]$. The integrals over the line segments $[b+iT_1, -1/2+iT_1]$ and $[-1/2-iT_1, b-iT_1]$ are equal in absolute value. They are bounded by

$$\frac{1}{2\pi} \left| \int_{-1/2+iT_1}^{b+iT_1} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^s}{s} \, ds \right| < \frac{x}{T_1} \int_{-1/2}^{b} \left| -\frac{\zeta'(\sigma+iT_1)}{\zeta(\sigma+iT_1)} \right| d\sigma$$

The integral over the line segment $[-1/2-iT_1, -1/2+iT_1]$ is bounded by

$$\frac{1}{2\pi} \left| \int_{-1/2-iT_1}^{-1/2+iT_1} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^s}{s} \, ds \right| \le \frac{1}{\sqrt{x}} \int_{-T_1}^{T_1} \left| -\frac{\zeta'(-1/2+it)}{\zeta(-1/2+it)} \right| \frac{dt}{1/\log x + |t|}.$$

We now estimate $|-\zeta'(\sigma+it)/\zeta(\sigma+it)|$, where either $-1/2 \le \sigma \le b$ and $t = T_1$, or otherwise $\sigma = -1/2$ and $2 \le |t| \le T_1$. We have

$$\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} = \sum_{|t-\gamma| \le 1} \frac{1}{\sigma - \sigma_n + i(t-\gamma_n)} + O(\log|t| + 2).$$

The sum above is at most $O(\log^2 x)$, and the total number of zeros of $\zeta(s)$ for which $|t-\gamma_n| \leq 1$ is at most $O(\log|t|+2)$. On the other hand, if $t = T_1$ and $-1/2 \leq \sigma \leq b$, then $|T_1 - \gamma_n| \gg 1/\log t$. Altogether, the four estimates give the theorem.

1.5 Expression for $\psi(x,\chi)$ in Terms of $L(s,\chi)$

We obtain an expression for the generalized ψ-function in terms of the zeros of $L(s,\chi)$.

Theorem 2 *Let $2 \leq k \leq T \leq x$. Let, further, χ be a primitive character modulo k. Then we have*

$$\psi(x,\chi) - \psi(k,\chi) = - \sum_{|\mathrm{Im}(\rho_\chi)|\leq T} \frac{x^{\rho_\chi} - k^{\rho_\chi}}{\rho_\chi} + O\left(\frac{x\log^2 x}{T}\right),$$

where $\rho_x = \beta_\chi + i\gamma_\chi$ runs over the nontrivial zeros of $L(s,\chi)$.

Proof. Taking a number T_1 with $T \leq T_1 \leq T+1$ such that the distance between T_1 and $\mathrm{Im}(\rho_\chi)$ is greater than $c\log kT$, where ρ_χ runs through the zeros of $L(s,\chi)$, and considering the rectangle Γ with vertices at $b \pm iT_1$ and $-1/2 \pm iT1$, where $b = 1 + 1/\log x$, we have

$$\frac{1}{2\pi i}\int_\Gamma \left(-\frac{L'(s,\chi)}{L(s,\chi)}\right)\frac{x^s - k^s}{s}\,ds = - \sum_{|\mathrm{Im}(\rho_\chi)|\leq T_1} \frac{x^{\rho_\chi} - k^{\rho_\chi}}{\rho_\chi} + \theta\log x,$$

where $|\theta| \leq 1$. Then

$$\psi(x,\chi) = \frac{1}{2\pi i}\int_{b-iT_1}^{b+iT_1}\left(-\frac{L'(s,\chi)}{L(s,\chi)}\right)\frac{x^s}{s} + O\left(\frac{x\log^2 x}{T_1}\right).$$

We first estimate the integrals over the upper, lower, and left sides of Γ. The integrals over the upper and lower sides of Γ have the same bound which is derived as follows:

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int_{-1/2+iT_1}^{b+iT_1}\left(-\frac{L'(s,\chi)}{L(s,\chi)}\right)\frac{x^s}{s}\,ds\right|
&\le \frac{e}{2\pi}\int_{-1/2}^{b}\left|-\frac{L'(\sigma+iT_1,\chi)}{L(\sigma+iT_1,\chi)}\right|\frac{x}{T_1}\,d\sigma\\
&= \frac{e}{2\pi}\int_{-1/2}^{b}\left|\sum_{|T_1-\gamma_\chi|\le 1}\frac{1}{\sigma-\sigma_n+i(T_1-\gamma_\chi)}+O(\log kT_1)\right|\frac{x}{T_1}\,d\sigma\\
&= O\left(\frac{x\log^2 kT_1}{T_1}\right)\\
&= O\left(\frac{x\log^2 x}{T_1}\right).
\end{aligned}$$

We next estimate the integral over the left side of Γ as follows:

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int_{-T_1}^{T_1}\left(-\frac{L'(-1/2+iT,\chi)}{L(-1/2+iT,\chi)}\right)\frac{x^{-1/2+iT}}{-1/2+iT}\,dt\right|
&\le \frac{2}{x}\int_{-T_1}^{T_1}\left|\frac{L'(-1/2+iT,\chi)}{L(-1/2+iT,\chi)}\right|\frac{dt}{1/2+|t|}\\
&= O\left(\frac{\log^2 kT}{\sqrt{x}}\right).
\end{aligned}$$

Combining all estimates in the equation for $\psi(x,\chi)$ yields the theorem.

Chapter 2

Generalization of the Landau–Gonek Explicit Formula

In 1993 Gonek [4] proved a uniform version of an explicit formula of Landau. This remarkable formula has applications to the distribution of primes in short intervals and to the pair correlation of zeros of $\zeta(s)$. In this chapter we prove a generalization of this formula for Dirichlet L-functions.

2.1 An Explicit Formula of Landau

A remarkable formula of Landau [5] asserts that for every fixed $x > 1$

$$\sum_{0<\gamma\le T} x^{\rho} = -\frac{T}{2\pi}\Lambda(x) + O(\log T) \tag{2.1}$$

as $T \to \infty$, where $\Lambda(x) = \log p$ if x is a positive integral power of a prime p, and $\Lambda(x) = 0$ for all other real values of x. The use of (2.1) is limited by its slack of uniformity in x. Gonek proved a version of (2.1) that is uniform in x and T. We can summarize his result as follows:

Theorem 3 (Gonek) *For $x, T > 1$*

$$\sum_{0<\gamma<T} x^\rho = -\frac{T}{2\pi}\Lambda(x) + O(x\ \log 2xT\ \log\log 3x)$$
$$+ O\left(\log x\ \min\left(T, \frac{x}{\langle x\rangle}\right)\right) + O\left(\log 2T\ \min\left(T, \frac{1}{\log x}\right)\right), \tag{2.2}$$

where $\langle x\rangle$ denotes the distance from x to the nearest prime power other than x.

It should be noted that (2.1) is obtained from (2.2) by fixing x and letting T approach infinity. In addition, if x is an integer such that $2 \le x \ll T$, then the last two O-terms in (2.2) are absorbed by the first O-term. In this case (2.2) reduces to

$$\sum_{0<\gamma\le T} x^\rho = -\frac{T}{2\pi}\Lambda(x) + O(x\ \log 2xT\ \log\log 3x). \tag{2.3}$$

Also from (2.2) and the symmetry of the zeros about the line $\mathrm{Re}\, s = 1/2$, for $0 < x < 1$

$$\sum_{0<\gamma\le T} x^\rho = \sum_{0<\gamma\le T} x^{1-\bar\rho} = x\overline{\sum_{0<\gamma\le T}\left(\frac{1}{x}\right)^\rho}. \tag{2.4}$$

A consequence of Theorem 3 is the following corollary:

Corollary 3.1 (Gonek) *Let $0 < x < 1$ and $T > 1$. Then*

$$\sum_{0<\gamma\le T} x^\rho = -\frac{Tx}{2\pi}\Lambda\left(\frac{1}{x}\right) + O\left(\log\frac{2T}{x}\log\log\frac{3}{x}\right)$$
$$+ O\left(\log\frac{1}{x}\min\left(xT, \frac{1}{\langle 1/x\rangle}\right)\right) + O\left(x\log 2T\ \min\left(T, \frac{1}{\log 1/x}\right)\right).$$

In his paper Gonek used Theorem 3 to estimate various sums involving ρ. In particular, he derived a discrete mean value formula for $\zeta(s)$ in the theorem below.

Theorem 4 (Gonek) *Assume the Riemann hypothesis. For T large and α real with $|\alpha| \le (\log T)/2\pi$, we have*

$$\sum_{0<\gamma\le T}\left|\zeta\left(\frac{1}{2} + i\left(\frac{\gamma + 2\pi\alpha}{\log T}\right)\right)\right|^2 = \left(1 - \left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2\right)\frac{T}{2\pi}\log^2 T + O(T\log^{7/4} T), \tag{2.5}$$

where the constant implied by the O-term is absolute.

Formulas (2.1) and (2.2) have been employed by a number of authors to study $\zeta(s)$. For example, in 2002 Murty and Zaharescu [6] employed a version of these formulas to derive explicit formulas for the pair correlation of zeros of functions in the Selberg class. In 2005 Ford and Zaharescu [3] used an analogue of these formulas to study the distribution of the imaginary part of zeros of $\zeta(s)$. Namely they investiaged the distribution of $\{\alpha\gamma\}$, the fractional parts of $\alpha\gamma$, when α is a fixed nonzero real number and γ runs through the imaginary parts of the nontrivial zeros of $\zeta(s)$.

In relation to the works above, it would be interesting to obtain a generalization of (2.2) for $L(s, \chi)$. We can summarize our result as follows:

Theorem 5 *Let χ be a principal character modulo k with $k \geq 2$. Let $T, x \geq 2$. Let, further, n_x be the nearest prime power to x. We have*

$$\sum_{\substack{\gamma_\chi \\ |\gamma_\chi| \leq T}} x^{\rho_\chi} = -\frac{\Lambda(n_x)\chi(n_x)}{\pi} \frac{\sin(T\log(x/n_x))}{\log(x/n_x)} + O(x \log^2 xkT) + O\left(\frac{\log kT}{\log x}\right),$$

where the sum is over the nontrivial zeros $\rho_\chi = \beta_\chi + i\gamma_\chi$ of $L(s, \chi)$.

2.2 Proof of the Generalized Explicit Formula

We start with the formula

$$\sum_{\gamma_\chi} x^{\rho_\chi} = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{L'(s,\chi)}{L(s,\chi)} x^s \, ds,$$

where the sum is over all zeros ρ_χ of $L(s, \chi)$ and $b = 1 + 1/\log x$. Let

$$N(T, \chi) = |\{\rho_\chi = \beta_\chi + i\gamma_\chi \colon L(\rho_\chi, \chi) = 0,\ 0 \leq \beta_\chi \leq 1, |\gamma_\chi| \leq T\}|.$$

By an application of the argument principal, as $T \to \infty$

$$N(T, \chi) = \frac{T}{2\pi} \log \frac{kT}{2\pi} - \frac{T}{2\pi} + O(\log kT).$$

(See Ahlfors' classical book [1], Section 5.2, pages 151–153; see also Davenport's classical [2], Chapter 16, Equation (1), page 101.)

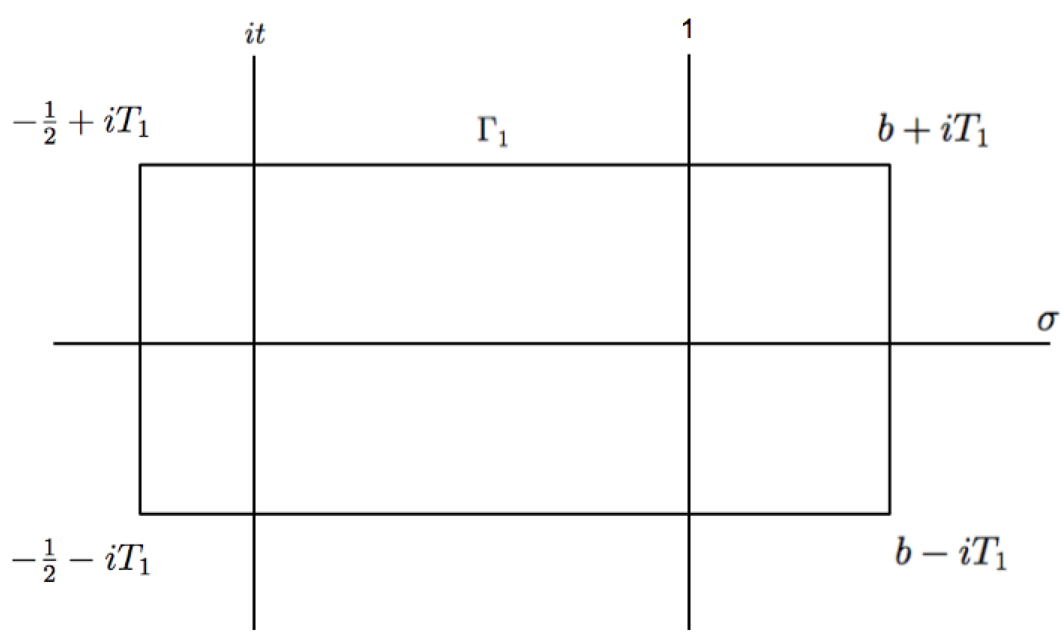
Consequently,

$$N(T+1,\chi) - N(T,\chi) = O(\log kT).$$

Hence, there exists a number T_1 satisfying $T \le T_1 \le T+1$ such that $|T_1 - \gamma_\chi| > 1/c \log kT$ for all zeros ρ_χ, with the constant c absolute. It suffices to consider the case that T_1 does not coincide with the ordinate of any zero ρ_χ. As we will see below, by changing T_1 by a bounded amount alters the size of the sum over all zeros ρ_χ by at most $O(\log kT)$, and hence does not influence the size of the final error term.

Moving the portion $|t| \le T_1$ of the line of integration to the left of the abscissa $\mathrm{Re}(s) = b$, we get the rectangle Γ_1, orientated counterclockwise, with vertices $-1/2 + b \pm iT_1$ and $b \pm iT_1$. We may assume that the left-hand side of Γ_1 does not run through a neighborhood of a trivial zero. Since the integrand is holomorphic within and on the boundary of Γ_1, hence by Cauchy's residue theorem

$$\sum_{|\gamma_\chi| \le T_1} x^{\rho_\chi} = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{L'(s,\chi)}{L(s,\chi)} x^s \, ds.$$



We now estimate the integrals over the upper, lower, and right sides of Γ_1. The first two integrals have the same bound with our choice of T_1. We have the representation

$$\frac{L'(s,\chi)}{L(s,\chi)} = \sum_{\substack{\rho_\chi \\ |T_1-\gamma_\chi|\le 1}} \frac{1}{s-\rho_\chi} + O(\log kT)$$

uniformly for $-1 \le \sigma \le 2$. (See Davenport's classical book [2] Chapter 16, Equation (4), page 102.) On noting that the number of zeros ρ_χ occurring in the formula is at most $O(\log kT)$, we obtain

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\int_{-1/2+b-iT_1}^{b+iT_1} \frac{L'(s,\chi)}{L(s,\chi)} x^s\, ds\right| \\
&\qquad \le \frac{cx}{2\pi}\int_{-1/2+b}^{b} \left| \sum_{\substack{\rho_\chi \\ |T_1-\gamma_\chi|\le 1}} \frac{1}{(\sigma-\beta_\chi)+i(T_1-\gamma_\chi)} + O(\log kT_1)\right| d\sigma \\
&\qquad = O(x\log^2 kT).
\end{aligned}$$

The integral over the right side of Γ_1 is bounded by expanding $L'(b+iT,\chi)/L(b+iT,\chi)$ into its Dirichlet series and then integrating term by term as follows:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-T_1}^{T_1} \frac{L'(b+it,\chi)}{L(b+it,\chi)} x^{b+it}\, dt &= \frac{-1}{2\pi}\int_{-T_1}^{T_1} \sum_{m=1}^{\infty} \Lambda(m)\chi(m)\left(\frac{x}{n_x}\right)^{b+it} dt \\
&= \frac{-\Lambda(n_x)}{2\pi}\left(\frac{x}{n_x}\right)^{b}\int_{-T_1}^{T_1}\left(\frac{x}{n_x}\right)^{it} dt \\
&\quad - \frac{1}{2\pi}\sum_{\substack{m=1 \\ m\ne n_x}}^{\infty} \Lambda(m)\chi(m)\left(\frac{x}{n_x}\right)^{b}\int_{-T_1}^{T_1}\left(\frac{x}{n_x}\right)^{it} dt \\
&= -\frac{\Lambda(n_x)\chi(n_x)}{\pi}\frac{\sin(T\log(x/n_x))}{\log(x/n_x)} \\
&\quad + O\left(\sum_{\substack{m=1 \\ m\ne n_x}}^{\infty}\left(\frac{x}{n_x}\right)^{b}\frac{\Lambda(m)}{\log(x/n_x)}\right).
\end{aligned}$$

This is because

$$\begin{aligned}
&\left|\frac{\Lambda(n_x)\chi(n_x)}{\pi}\frac{\sin(T\log(x/n_x))}{\log(x/n_x)}\left(\frac{x}{n_x}\right)^b - \frac{\Lambda(n_x)\chi(n_x)}{\pi}\frac{\sin(T\log(x/n_x))}{\log(x/n_x)}\right| \\
&\qquad = \left|\frac{\Lambda(n_x)\chi(n_x)}{\pi}\frac{\sin(T\log(x/n_x))}{\log(x/n_x)}\left(\left(\frac{x}{n_x}\right)^b - 1\right)\right| \\
&\qquad \le \frac{\Lambda(n_x)}{\pi}\frac{|(x/n_x)^b - 1|}{|\log(x/n_x)|} \\
&\qquad = O(\Lambda(n_x)) \\
&\qquad \le \log n_x,
\end{aligned}$$

which can be absorbed in the O-term above.

We now estimate the sum in the O-term by decomposing it into three subsums extended over the following sets of m: $m < x/2$, $x < x/2 \le m \le 2x$ but $m \ne n_x$, and $m > 2x$. For values of m which satisfy $m < x/2$ or $m > 2x$, it is clear that $|\log(x/m)| \ge \log 2$ and hence there contribution is

$$\begin{aligned}
\sum_{m<x/2}\left(\frac{x}{m}\right)^b \frac{\Lambda(m)}{|\log(x/m)|} + \sum_{m>2x}\left(\frac{x}{m}\right)^b \frac{\Lambda(m)}{|\log(x/m)|} &< -\frac{x^b}{\log 2}\frac{\zeta'(b)}{\zeta(b)} \\
&< ex\frac{\log x}{\log 2}.
\end{aligned}$$

For values of m which satisfy $x/2 \le m \le 2x$ but $m \ne n_x$, we consider those terms with $x < 2 \le m < n_x < x$. Using the inequality

$$\frac{1}{\log\lambda} < \frac{\lambda}{\lambda - 1} < 1 + \frac{\sqrt{\lambda}}{\lambda - 1}, \quad \text{for } \lambda > 1,$$

due to Selberg, we obtain

$$\begin{aligned}
\sum_{x/2\le m<n_x}\left(\frac{x}{m}\right)^b \frac{\Lambda(m)}{|\log(x/m)|} &\le 2^b \sum_{x/2\le m<n_x}\Lambda(m)\left(1 + \frac{\sqrt{mx}}{x - m}\right) \\
&\le 2^b\log x + 2^b\log^2 x \\
&= O(\log^2 x).
\end{aligned}$$

Those terms with $x < n_x < m \le 2x$ are treated similarly.

To estimate the integral over the left side of Γ_1, we use the functional equation of $L(s,\chi)$ to relate the quotients $L'(1-b+iT,\chi)/L(1-b+iT,\chi)$ and $L'(b-iT,\overline{\chi})/L(b-iT,\overline{\chi})$. Its logarithmic derivative gives

$$\frac{L'(s,\chi)}{L(s,\chi)} = -\frac{L'(1-s,\overline{\chi})}{L(1-s,\overline{\chi})} - \log\frac{k}{2\pi} - \frac{\Gamma'(1-s,\overline{\chi})}{\Gamma(1-s,\overline{\chi})} + \frac{\pi}{2}\cot\frac{\pi(s+\kappa)}{2},$$

where

$$\kappa = \frac{1-\chi(-1)}{2}.$$

By Stirling's formula, we have

$$\begin{aligned} -\frac{1}{2\pi}\int_{-T_1}^{T_1} \frac{L'(1-b+it,\chi)}{L(1-b+it,\chi)} x^{1-b+it}\,dt = {} & \frac{x^{1-b}}{2\pi}\int_{-T_1}^{T_1} \frac{L'(b-it,\overline{\chi})}{L(b-it,\overline{\chi})} x^{it}\,dt \\ & + \frac{x^{1-b}}{2\pi}\int_{-T_1}^{T_1} \log\frac{k(|t|+2)}{2\pi} x^{it}\,dt \\ & + O(x\log kT). \end{aligned}$$

This is because $\cot\pi(s+\kappa)/2$ is bounded in Γ_1. The first integral on the right side is bounded as follows:

$$\begin{aligned} \left|\frac{x^{1-b}}{2\pi}\int_{-T_1}^{T_1} \frac{L'(b-it,\overline{\chi})}{L(b-it,\overline{\chi})}\right| dt & \le \frac{x}{2\pi e}\int_{-T_1}^{T_1} \left|\frac{L'(b+it,\chi}{L(b+it,\chi)}\right| dt \\ & = O(x\log^2 kT), \end{aligned}$$

since

$$|L(b-it,\overline{\chi})| = |L(b+it,\chi)|$$

and the integrand is $O(\log(k|t|+2))$. We may next integrate by parts to find that the second integral is $O(\log kT/\log x)$.

Finally, we piece things together and note that the trivial zeros ϱ_χ at $s=\kappa-2n$, $n\in\mathbb{N}\cup\{0\}$, is $O(1)$. Theorem 5 is thus completely proved.

Remark. We point out that the integral

$$\int_{-T_1}^{T_1} \frac{L'(b-it,\overline{\chi})}{L(b-it,\overline{\chi}} x^{it}\,dt$$

can be bounded in the same way the integral over the right side of Γ_1 was estimated, to give the main term

$$-\frac{x\Lambda(n_x)\overline{\chi}(n_x)}{\pi}\frac{\sin T_1 \log n_x x}{\log n_x x}.$$

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