5-1985

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Departmental Honors Thesis

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Spring, 1985

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CHAPTER 1 Introduction To General Relativity

Section 1.1 Introduction

The General Theory of Relativity (GRT) gives rise to many interesting questions, one of which is a question about how test particles orbit various relativistic mass configurations. A test particle is defined as an electrically neutral particle small enough that its self gravitating energy, as calculated using standard Newtonian theory, can be neglected when compared with the particle's rest mass, i.e., $M/R \ll 1$. (Here M is the mass of the particle in meters, and R is the radius of the particle again in meters. The mass of the sun in these units is 1.47 km.) For example, the earth, which can be considered a test particle for the sun, travels about the sun in an elliptic orbit. What would happen if the sun suddenly became a black-hole? How would the earth's orbit be affected? How would the path of any test particle orbiting this new black-hole be affected? It is the purpose of this paper to answer these questions not only for the case of the spherical black-hole (which has been extensively studied), but also for the case of an axisymmetric mass distribution idealized by a long thin rod-like mass (which will henceforth be referred to as a line mass). The classical case and relativistic case concerning orbits about the line mass will be discussed, along with some interesting features of the relativistic line mass orbits. Some background information concerning GRT will be needed before the problem of the line mass is approached.

Newton believed in an absolute space, and he considered this to be the absolute frame of reference for all motion. Thus, a space traveler could tell his speed exactly relative to the absolute space. For Newton space was flat, or Euclidean in nature. In Principia Newton wrote, "Absolute space, in its own nature, without relation to anything external, remains always similiar and immovable."[1] And later, "Absolute, true, and mathematical time, of itself and from its own nature, flows equably without relation to any thing external."[2] GRT postulates just the opposite: that space and time are irrevocably linked (into spacetime), and that gravitating masses affect spacetime. Thus, GRT begins by denying that there is an absolute, flat, space-time continuum. Yet, as will be shown later, the equations of motion in GRT must reduce to the appropiate equations of motion in the Newtonian limit of the physical system. Hence, there remains a necessary link between the classical Newtonian physics and the physics of general relativity.

1.2 Basic Principles of General Relativity

The term "general relativity" is actually a misnomer, for the theory is primarily concerned with the invariance of fundamental physical principles in any reference frame-i.e., system of reference. These reference systems are simply coordinate systems which may or may not be inertial. The General Principle (GP) of relativity states that all systems of reference are equivalent with respect to the formulation of the fundamental laws of physics.[3] Thus, whether one is travelling near the speed of light relative to a given inertial frame or accelerating in a non-inertial frame, or in a frame near a gravitating mass, the fundamental laws of physics will not be different for any system. Laws of conservation of energy and momentum, quantum electrodynamics, etc., are all valid in GRT. The mathematical formulation of the basic laws may be slightly (or even greatly) altered, but the laws themselves are still operational.

A second principle on which GRT rests is the Equivalence Principle

(EP). The EP states that at every point in spacetime in an arbitrary gravitational field it is possible to choose a locally inertial coordinate system, such that within a sufficiently small region of the point in question, the laws of nature take the same form as in an unaccelerated Cartesian coordinate system.[4] Thus, within a sufficiently small region of a particular point in spacetime the fundamental laws of physics do not change form, and gravitation has no effect on the motion or other physical processes of the particles. When the GP and EP are taken together, two consequences naturally follow. First, suppose that the reference frame is very far away from any large gravitating mass (the weak field assumption), travelling at a velocity much slower than the speed of light, that the frame is inertial, and that the gravitational field is static.[5] Then the relativistic effects must reduce to classically predicted phenomena. This is the Newtonian limit which was introduced earlier. An example of the Newtonian limit will be given in a following chapter. Second, as a consequence of the EP the chosen coordinate system about a gravitating mass must be locally Euclidean. A distinguishing characteristic of Riemannian geometry is that it is locally Euclidean. Thus, Riemannian geometry is the proper geometry for GRT. Historically, Riemannian geometry was developed before GRT; Einstein used the Riemannian mathematics of curved space to describe the physics of GRT.[6]

1.3 Mathematical Background

In this section the mathematical tools will be developed which will be used to describe the physics of GRT. Tensors are necessary in the development of the theory, and will be briefly introduced here. For a more extensive treatment of tensor analysis one may consult the literature.[7] Tensors are simply generalizations of scalars and vectors. A scalar is a "rank" zero tensor. A vector is a rank one tensor. A three dimensional vector is a rank 1 tensor, and has 3^1 or 3 components. A second rank three-dimensional tensor has 3^2 or nine components, and a general tensor of order n has M^n components (where M is the dimensionality of the tensor). In GRT tensors are almost always four-dimensional (3 spatial coordinates, one time coordinate).

An example of a tensor from classical mechanics is the stress tensor. If a beam is carrying a load, the stresses and strains in the beam at any point can be expressed in tensor form. The stress tensor is described by

$$Pab=\begin{pmatrix} Pxx & Pxy & Pxz \\ Pyx & Pyy & Pyz \\ Pzx & Pzy & Pzz \end{pmatrix} \qquad (1.1)$$

This is a rank 3 tensor with 3^2 (or nine) components. The forces (per unit area) Pxx, Pyy, Pzz are tensions, while the others are shear forces (per unit area).

A background in Euclidean geometry is the necessary starting point for understanding non-Euclidean geometry. In rectangular Cartesian coordinates $x^i(i=1,2,3)$ the distance dl between two points $P(x^i)$ and $Q(x^i+dx^i)$ is given by the Pythagorean theorem

$$dl^2=(dx^1)^2+(dx^2)^2+(dx^3)^2 \qquad (1.2)$$

Of course, in Cartesian coordinates $x^1=x$, $x^2=y$, $x^3=z$. The Einstein summation convention, which states that repeated indices are to be summed, will be used throughout this paper. Thus, (1.2) becomes

$$dl^2=dx^idx^i \qquad (1.3)$$

A natural extension of Euclidean geometry is psudo-Euclidean geometry which is used in special relativity. The name for psudo-Euclidean geometry is Minkowski space, which incorporates space and time into spacetime. Separation between events, rather than points, is considered. Thus (1.2) must be modified to account for time differences in the measurament of dl^2. The interval between two events, A(t,x,y,z) and B(t+dt,x+dx,y+dy,z+dz) is given by the interval between two events (the squared length of separation between two events),

$$ds^2=(dx^1)^2+(dx^2)^2+(dx^3)^2-c^2(dx^4)^2 \tag{1.4}$$

where $x^1=x$, $x^2=y$, $x^3=z$, $x^4=t$. Equation (1.4) or an equation of the form of (1.4) is usually referred to as the convention element, or metric for the geometry which is being studied. In dimensional units where $c=1$ (i.e., the speed of light is taken to be 1) equation (1.4) becomes

$$ds^2=(dx^1)^2+(dx^2)^2+(dx^3)^2-(dx^4)^2 \tag{1.5}$$

Using the notion of a tensor and the Einstein summation convention equation (1.5) can be expressed as

$$ds^2=N_{ij}dx^idx^j \tag{1.6}$$

where

$$N_{ij}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \tag{1.7}$$

N is called the Minkowski metric tensor.

From equation (1.5) one can classify directions and vectors in spacetime. When ds^2 is negative the vectors are said to be time like; when ds^2 is positive the vectors are said to be space like; and when ds^2 equals zero the vectors are said to be null or light like.

In 1854 Riemann proposed a study of generalized spaces which did not necessarily have to be Euclidean or psudo-Euclidean in nature.

Riemann was interested in characterizing curved spaces by modifying (1.6) He proposed that the components of the metric tensor could be functions of the coordinates of the space which they were to characterize. In mathematical terms Riemann proposed to study the geometry of n-dimensional space characterized by n coordinates $u_i(i=1,2,...,n)$. The infinitesimal distance between two points in this n-dimensional space with coordinate differences du^i is given by

$$ds^2=g_{ik}du^i du^k \qquad (1.8)$$

where $g_{ik}(u_i...u_n)$ are arbitrarily prescribed functions of the coordinates. The g_{ik} is a tensor of rank 2 with n^2 components. The g_{ik} tensor (the metric tensor) is connected with general relativity through the EP.[8] Mathematically, the EP states that all effects of a gravitational field can be described in terms of derivatives $\partial E^a/\partial X^u$ of the functions $E^a(x)$ that define the transformation from the laboratory coordinates x^u to the locally inertial coordinate E^a.[9] These partial derivatives are the components of the metric tensor. As a classical example, consider the line element of the surface of a sphere of radius a, given by

$$ds^2=a^2 d\beta^2+a^2 (\sin^2\beta)d\phi^2 \qquad (1.9)$$

where $g_{\beta\beta}=a^2$, $g_{\phi\phi}=a^2(\sin^2\beta)$, all other $g_{uv}=0$.

The g_{uv} stand for the components of the covariant metric tensor. The contravariant form is given by g^{uv}, with the condition that $g_{uv}g^{uv}=1$ for all components of the metric. As an extension of the Einstein summation convention, contravariant and covariant symbols of the same variable are summed. For example

$$g_{ai}g^{bi}=\Sigma^n_{i=1}\, g_{a1}g^{b1}+g_{a2}g^{b2}+...+g_{an}g^{bn} \qquad (1.10)$$

From the metric tensor Christoffel symbols of the first and second kinds can be found. These are given by

$$[ij,k]=(g_{ik,j}+g_{jk,i}-g_{ij,k})/2 \qquad (1.11)$$

$$\{ij,k\}=g^{ia}[jk,a] \qquad (1.12)$$

Formula (1.11) gives the Christoffel symbols of the first kind, while formula (1.12) gives the Christoffel symbols of the second kind. The commas following the metric components ($g_{ij,k}$) are equivalent to derivatives; thus, $,=\partial/\partial x^u$. The Christoffel symbols of the second kind are extremely important for the purpose of finding the equations of orbital paths. It is assumed that test masses orbiting a large gravitating mass will follow geodesic orbits. A geodesic is simply the shortest distance from one point to another in spacetime. Thus in Euclidean space a geodesic is always a straight line, but in Riemannian geometry the geodesic may be a curved line. Physically, this is due to the deformation of spacetime by a gravitating mass. The equivalence principle implies that test particles will move in orbital paths (either opened or closed) about a gravitating mass and that the geodesic equations are the equations of orbital path. Thus, the geodesic equations are the equations of motion for a test particle. After the Christoffel symbols of the second kind have been found, construction of the geodesic equations can be easily carried out by using the formula

$$d^2x^u/dt^2+\{ab,u\}(dx^a/dt)(dx^b/dt)=0 \qquad (1.13)$$

Generally, the line element in general relativity has the form

$$ds^2=g_{uv}dx^udx^v. \qquad (1.14)$$

This can be integrated

$$I=(1/2)\int g_{\mu\nu}(dx^{\mu}/d\beta)(dx^{\nu}/d\beta)d\beta \qquad (1.15)$$

where β is an arbitrary parameter. Taking the variation of I, the geodesic equations can again be found. This method of finding the geodesic equations is called the geodesic Lagrangian method.[10] Using this method yields the same results as in (1.12) with the advantage that it resembles the classical Lagrangian method of variations, and that it is less time consuming. Either method yields the geodesic equations, which can be used to describe orbits about a gravitating body.

Other tensors can be found using the Christoffel symbols (which arose from the metric tensor). The first is the Riemann tensor defined by

$$R_{ijkm}=[jm,i]_{,k}-[jk,i]_{,m}+\{jm,a\}[im,a]-\{jm,a\}[ik,a] \qquad (1.16)$$

Also,

$$R^{a}{}_{jkm}=g^{ai}R_{ijkm} \qquad (1.17)$$

The Riemann tensor is a 4^4 rank tensor and thus has 256 components, twenty of which are independent. This is due to the symmetry relations given by

$$R_{ijkm}=-R_{jikm}=-R_{ijmk}=R_{kmij}$$

$$R_{iabc}+R_{ibca}+R_{icab}=0 \qquad (1.18)$$

From the Riemann tensor the Ricci tensor can be obtained by contraction of the first and third indices of (1.17). Thus

$$R_{ij}=R^{a}{}_{iaj} \qquad (1.19)$$

$$R^{i}{}_{j}=g^{ia}R_{aj} \qquad (1.20)$$

In expanded form

$$R_{ij}=\{bj,b\}_{,i}-\{ij,b\}_{,b}+\{ti,b\}\{bj,t\}-\{tb,b\}\{ij,t\} \qquad (1.21)$$

The Ricci tensor can also be obtained by another method which proves to be more convenient to implement. It should be clear that g_{uv} is a 4x4 matrix. Thus, the determinant of g_{uv} can be found. Let $g=\det((g_{uv}))$ be the determinant of g_{uv}. It should also be evident that

$$\{ih,i\}=(1/2)g^{ij}g_{ij,h} \qquad (1.22)$$

which can easily be obtained from (1.11) and (1.12). Using the theory of determinants, $g^{ij}=\Delta^{ij}/g$, where Δ^{ij} is the cofactor of g_{ij}. Since Δ^{ij} is a subdeterminant of g and does not contain the variable g_{ij} , then $\partial g/\partial g_{ij}=\Delta^{ij}$. Replacing Δ^{ij} by this expression, an alternate expression for g^{ij} is obtained:

$$g^{ij}=(1/g)\partial g/\partial g_{ij}. \qquad (1.23)$$

Now an alternate expression for $\{ih,i\}$ becomes evident, namely

$$\{ih,i\}=(1/2g)(\partial g/\partial g_{ik})(\partial g_{ik}/x^h)=(1/2g)\partial g/\partial x^h=(1/2)\partial/\partial x^h(\ln|g|)$$
$$=\partial/\partial x^h(\ln\sqrt{-g}) \qquad (1.24)$$

where $g<0$. Equation (1.15) thus becomes

$$R_{ij}=\{ij,a\}_{,a}-(\ln\sqrt{-g})_{,i,j}+\{ti,b\}\{bj,t\}-\{ij,t\}(\ln\sqrt{-g})_{,t} \qquad (1.25)$$

Equation (1.25) gives the same answers as (1.21), without the necessity of calculating the Riemann tensor.[11] The power of (1.25) will be seen in chapter 2.

Thus far the mathematics has been stated; now the physical motivation for the mathematics must be understood. The theory of gravitation is a field theory, much like that of electromagnetism. In classical eletromagnetism the field equations are given by Maxwell's equations. Similarly in relativity the field equations are given by

$$R_{ij}=\varkappa g_{ij} \qquad (1.26)$$

where $\varkappa$ is a constant. Riemannian manifolds of any dimension n and satisfying (1.26) are called Einstein spaces.[12] A condition used in this paper is that the gravitational masses to be studied are in free, or "empty" space (i.e., no other large gravitating masses are nearby). This condition is satisfied when $\varkappa=0$.[13] Thus the proper form for the field equations is

$$R_{ij}=0 \tag{1.27}$$

The method employed in finding a specific line element ds^2 in GRT is to begin with a general form of the line element. One then proceeds to find the Riemannian and Ricci tensors. Application of (1.27) gives rise to the necessary conditions for the specific line element. Then one can find ds^2 specifically for a given physical situation. One can then find specific geodesic equations, and thus orbits of test particles can be found.

CHAPTER 2 The Spherical Solution

2.1 The Metric of Spherical Space

Perhaps the best way to illustrate the mathematics of general relativity is through the example of the Schwarzschild solution to the Einstein field equations for spherically symmetric Einstein space. The spherical solutions to the field equations (1.21) were found by Schwarzschild in 1916.[1] The spherical solution is perhaps the most important solution to the field equations. In spherical coordinates r measures radial distance from the origin; ø measures the angle from the x-axis; and ß measures the angle from the z-axis (see figure 1). The metric of the Euclidean spherical coordinate system is given by

$$dq^2=dr^2+r^2(d\o^2+\sin^2\o\, d\beta^2) \qquad (2.1).$$

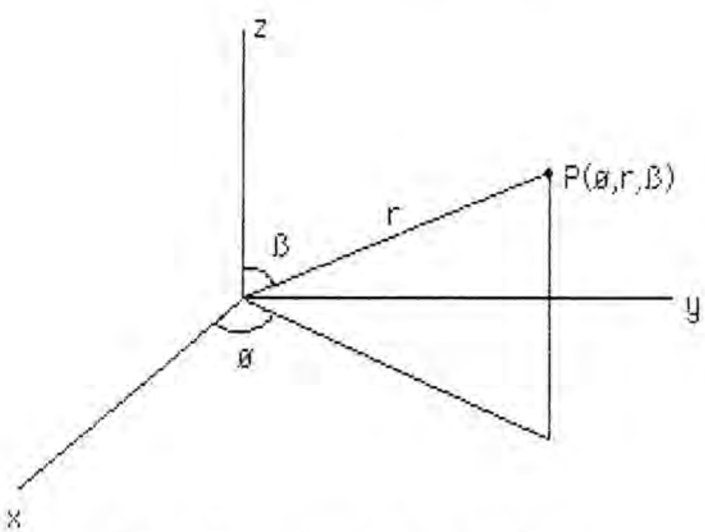

figure 2.1: spherical symmetry

Which coordinates in (2.1) will be deformed if a mass of relativistic size is placed at the origin? This question can be answered through the thoughtful construction of a coordinate system about the origin. Let weightless spheres of radius r be constructed about the origin. These spheres are constructed of a weightless lattice work of rulers so that gravity does not compress them. Spherical symmetry implies that when a massive body is placed at the origin the radial distances will be distorted, but the spheres themselves will not be distorted. This is due to their assumed weightlessness. Thus, the only thing that can now be said about r is that a sphere at $r=r_0$ will have a surface area of $4\pi r_0{}^2$.

Time will also be distorted about the massive body. The relativistic line element will thus be given by an equation which takes into account the radial and time deformations. If in Minkowski space-time the line element is given by

$$ds^2=dt^2-dq^2 \tag{2.2}$$

then in Einstein spacetime the line element must be given by

$$ds^2=e^A dt^2-e^B dr^2-r^2(d\varnothing^2+\sin^2\varnothing d\beta^2) \tag{2.3}$$

where $A=A(r)$, and $B=B(r)$ are functions which must be determined.

The derivation of A and B will be briefly outlined in the following few paragraphs. The first quantities which need to be found are the Christoffel symbols of the second kind (see equation 1.6). After calculations, the surviving Christoffel symbols are:

$$\{tt,r\}=-(1/2)e^{A-B}A'$$

$$\{rt,t\}=(1/2)A'$$

$$\{rr,r\}=(1/2)B'$$

$$\{r\varnothing,\varnothing\}=\{r\beta,\beta\}=1/r$$

$$\{\varnothing\varnothing,r\}=-e^{-B}r$$

$$\{\varnothing\beta,\beta\}=\cot\varnothing$$

$$\{\beta\beta,r\}=-e^{-B}r\sin 2\varnothing$$

$$\{\beta\beta,\varnothing\}=-\sin\varnothing\cos\varnothing.$$

With knowledge of the Christoffel symbols the Ricci tensor components can be calculated by using equation (1.25) where $g=-r^4e^{A+B}\sin^2\theta$. Thus, the components of the Ricci tensor are:

$$R_{11}=(1/2)A''-(1/4)A'B'+(1/4)A'^2-B'/r \tag{2.4}$$

$$R_{22}=e^{-B}[1+(1/2)r(A'-B')]-1 \tag{2.5}$$

$$R_{33}=R_{22}\sin^2\theta \tag{2.6}$$

$$R_{44}=-e^{A-B}[(1/2)A''-(1/4)A'B'+(1/4)A'^2+A'/r] \tag{2.7}$$

$$R_{ij}=0 \quad \text{where } i\neq j \tag{2.8}$$

Through the application of the field equations $R_{ij}=0$ for all i,j, then A and B can be found. Notice that by using (2.4) and (2.7) in conjunction

$$A'=-B' \tag{2.9}$$

and thus, upon integration

$$A=-B+k \tag{2.10}$$

where k is a constant of integration. By changing the time scale from t to $te^{-k/2}$, the constant in (2.10) is absorbed in the line element given by (2.3). Hence

$$A=-B$$

Equation (2.4) can now be used, yielding

$$e^A[1+rA']=1$$

If the substitution $e^A=\delta$ is made, then

$$\delta+r\delta'=1$$

which yields upon integration

$$\delta=1-2m/r$$

where 2m is a constant of integration at this point. Since δ is a solution of (2.4) and (2.7) as used in conjunction, it is necessary to show that δ

satisfies both (2.4) and (2.7) individually, and it turns out that σ is a valid solution to the problem. The line element is thus given by

$$ds^2=(1-2m/r)dt^2-(1-2m/r)^{-1}dr^2-r^2(d\phi^2+\sin^2\phi d\beta^2) \qquad (2.11)$$

From comments made in Rindler[2] it is obvious that m is the mass of the central body. The system of units used here and throughout this paper are those of the natural unit system $G=c=1$. (G is the gravitational constant; c is the speed of light.) Hence m is given in meters. In this system of units the mass of the sun is 1.47 km, and the mass of the earth is 0.44 cm.

It should also be noted that the spatial part of (2.11) satisfies the conditions of the Newtonian limit as described in chapter 1. As r approaches infinity ds^2 approaches the Newtonian line element as given in (2.3).

The metric given in equation (2.11) is the starting point for finding orbits of test particles in spherical Einstein space. The reader should at this point consider a brief review of the classical orbital problem of a test particle about a spherical distribution of mass. In the literature this problem is referred to as Kepler's problem.[3]

SECTION 2.2: Orbits About a Relativistic Spherical Mass

It is the purpose of this section to describe orbits about a relativistic spherical mass. Using the Lagrangian variational principle

$$\partial\int ds=0 \qquad (2.12)$$

on the line element given in (2.11), the integral becomes

$$\partial\int\{(1-2m/r)t'^2-(1-2m/r)^{-1}r'^2-r^2[\phi'^2+\sin^2\phi\beta'^2]\}ds=0 \qquad (2.13)$$

where ′ represents differentiation with respect to s. The geodesic equations for t, ϕ, and β can be extracted from (2.13) using the

Lagrange-Euler equation

$$d/ds(\partial L/\partial q')=\partial L/\partial q.$$

The geodesic equations are then given by

$$d/ds(r2ø')=r^2\sin ø \cos ø\, ß'^2 \qquad (2.14)$$

$$d/ds(r^2\sin^2ø\, ß')=0 \qquad (2.15)$$

$$d/ds[(1-2m/r)t']=0 \qquad (2.16)$$

The geodesic equation for r' can be found most easily by dividing (2.11) by ds^2, which gives rise to

$$1=(1-2m/r)t'^2-(1-2m/r)^{-1}r'^2-r^2(ø'^2+\sin^2ø ß'^2) \qquad (2.17)$$

Using the above formulae (2.14-2.17) it is possible to find orbits about the mass. First, initial conditions should be noted. In classical spherical orbits a particle that is moving in a central force field will move in a plane and will not leave that plane of motion. This is due to the symmetry of the problem. Does the same behavior exist in the relativistic case? Suppose that at some initial s, $ø=\pi/2$ and $ø'=0$. From (2.14) it is obvious that for all s

$$ø=\pi/2 \qquad (2.18)$$

since the chosen initial conditions give a unique solution for (2.14).

Second, from (2.15) and (2.18) it is clear that

$$r^2ß'=h=\text{const} \qquad (2.19)$$

Third, from (2.16)

$$(1-2m/r)t'=l=\text{const} \qquad (2.20)$$

Fourth, (2.18), (2.19), and (2.20) can be substituted into (2.17) which yields

$$1=l^2(1-2m/r)^{-1}-(1-2m/r)-1r'^2-h^2/r^2 \qquad (2.21)$$

Equation (2.21) is the equation of motion for r=r(s).

As in the classical Kepler problem (2.21) can be simplified and written in a form which gives more insight into the actual movements of test particles. A far more informative equation can be obtained if $r=r(\beta)$ can be found. If $\cdot$ represents differentiation with respect to β, then

$$r(dr/d\beta)=(dr/ds)(ds/d\beta)=\dot{r}/\dot{\beta} \qquad (2.22)$$

Through substitution into (2.21) the equation of motion becomes

$$(1-2m/r)=l^2-(h^2/r^4)r^2-(h^2/r^2)(1-2m/r) \qquad (2.23)$$

Again, by making the substutions

$$u=1/r \qquad (2.24)$$

$$r=-\dot{u}/u^2 \qquad (2.25)$$

equation (2.21) becomes

$$(du/d\beta)^2=(u)^2=[((l^2-1)/h^2)+(2m/h^2)u-u^2+2mu^3] \qquad (2.26)$$

Equation (2.26) can be integrated immediately, yielding

$$\beta(u)=\beta_0+\int 1/\{ [(l^2-1)/h^2]+(2m/h^2)u-u^2+2mu^3 \}^{1/2}\, du \qquad (2.27)$$

where β_0 is an initial β (usually taken to be zero). Thus the orbits of test-particles in the spherical case can be found exactly.

Unfortunately, (2.27) is not extremely enlightening, and therefore (2.26) should be reconsidered. Equation (2.26) can be rewritten in the form

$$(du/d\beta)^2=f(u) \qquad (2.28)$$

where

$$f(u)=2mu^3-u^2+(2m/h^2)u+(l^2-1)/h^2 \qquad (2.29)$$
$$=2m(u-u_1)(u-u_2)(u-u_3)$$

u_1,u_2,u_3 being zeros of $f(u)$, and $u_1<u_2<u_3$ if they are all real. The restriction is made that the test particles at the perhelia give

$$du/d\beta=0 \ , \quad f(u)=0 \qquad (2.30)$$

From (2.28) $f(u)\geq 0$ throughout the orbit; by (2.29) $f(u)>0$ for large u. This implies that all roots are real. Since f(u) can only be positive, it follows that in figure 2.2a the orbits reside between u_1 and u_2, and are elliptic orbits. If f(u) is graphed as in figure 2.2b. the orbit is hyperbolic in nature (remember that u=1/r).

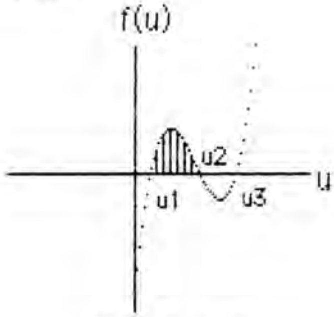

figure 2.2a

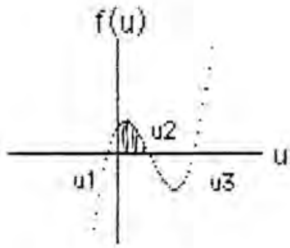

figure 2.2b

The general solution of (2.28) involves Jacobian elliptic functions.[4] If

$$x=(1/2)\beta\sqrt{\{2m(u_3-u_1)\}},\ y=\sqrt{\{(u-u_1)/(u_2-u_1)\}}$$

$$k=\sqrt{\{(u_2-u_1)/(u_3-u_1)\}} \tag{2.31}$$

Then (2.28) becomes

$$(dy/dx)^2=(1-y^2)(1-k^2y^2) \tag{2.32}$$

The general solution to (2.32) is

$$y=\mathrm{sn}(x+c) \tag{2.33}$$

where c is an arbitrary constant. Thus all geodesic orbits having perhelia satisfy

$$u-u_1=(u_2-u_1)sn^2((\emptyset/2)\surd[2m(u_3-u_1)] + c) \qquad (2.34)$$

where $u=1/r$, and the modulus of the elliptic function is k as given in (2.31).

The effective potential for the spherical mass distribution can be found using equations (2.11), (2.15), and (2.16). The effective potential will give the "turning points" for the orbits of the test particles. These turning points will give a maximum and minimum value which r can attain during a closed orbit. If (2.11) is divided by ds^2, the result is

$$1=-(1-2m/r)t'^2+(1-2m/r)^{-1}r'^2+r^2\emptyset'^2 \qquad (2.35)$$

When (2.15) and (2.16) are substituted into (2.35) the result is (after solving for dr/ds)

$$(dr/ds)^2=E^2-[(1-2m/r)(1+h^2/r^2)] \qquad (2.36)$$

where the effective potential is given by

$$V^2(r)=[(1-2m/r)(1+h^2/r^2)] \qquad (2.37)$$

If one graphs r verses V^2 one can then determine the turning points of the orbit.

The purpose of this section is to describe orbits about a central mass, given equation (2.11) as the line element. Mathematically, this has been accomplished. Physically, how does (2.34) differ from the classical Kepler problem? Remember that in Kepler's problem, one orbital solution was that of a closed ellipse. The relativistic solutions given in (2.33) and (in integral form) in equation (2.27) will cause a perhelion shift in the orbits . This shifting occurs in one plane ($\emptyset=\pi/2$ in this derivation). Relativistic effects have been observed, and the theory has been successful in explaining the perhelion shift of Mercury.[5]

Another interesting feature of the spherical line element as given in equation (2.11) needs to be discussed here. The Schwarzschild line element (2.11) exhibits singular behavior at $r=2m$ in that the g_{tt} component of the metric tensor vanishes, while the g_{rr} component is infinite at $r=2m$. The radius $r=2m$ has traditionally been called the Schwarzschild radius, or event horizon. When $r<2m$ the geometry of the line element changes dramatically in that g_{tt} becomes negative, while g_{rr} becomes positive. Thus, the dt^2 element becomes spatial, while the dr^2 element behaves like a time element. The r coordinate no longer represents a spatial coordinate, rather r behaves like time. This implies that r can no longer change direction after the boundary $r=2m$ has been crossed, i.e, r must continuously decrease after $r=2m$ has been crossed. Just as time cannot be stopped or reversed for $r>2m$, so the decrease in r cannot be reversed or stopped after a particle has crossed into $r<2m$. This is precisely why black-holes are "black": once any particle – whether it is a space ship or a photon – crosses the Schwarzschild radius $r=2m$ into the region $r<2m$, that particle will never pass back into the region $r>2m$. The particle is doomed to fall into the black-hole.

The infinity of g_{rr} and the vanishing of g_{tt} at $r=2m$ was considered to be a real, physical singularity for a number of years; however, in 1933 it was shown by Lemaitre[6] that the singularity at $r=2m$ is only a coordinate singularity. In 1960 Kruskal[7] and Szederes[8] independently found a coordinate system which lends greatest insight into Schwarzschild geometry. The Kruskal-Szekeres coordinate system allows a particle to cross $r=2m$, whereas the coordinate geometry as determined by (2.11) would not allow a particle to cross $r=2m$. The behavior of the radial coordinate once the particle has crossed $r=2m$ is unchanged. Thus, even with the Kruskal-Szekeres coordinate system a particle which crosses r=2m cannot escape the gravitational clutches of the mass within.

Thus, a description of the relativistic effects for spherical Einstein space has been discussed. In the remainder of this paper the line mass will be considered. Where possible the same techniques will be used to find the line element and geodesic equations. Chapter 3 is devoted to the classical orbits around a line mass, while chapter 4 gives the relativistic orbits. In both the classical and relativistic cases one must resort to numerical computations via computer to solve the geodesic equations.

CHAPTER 3 : The Classical Line Mass[1]

Thus far, the general principles of relativity have been discussed. Orbits in spherical Einstein space have also been presented. It was found that in the relativistic Keplerian problem the orbits exhibit perhelion shifts which have been observed in the real physical universe. Before discussing the relativistic line mass, the classical line mass (CLM) needs to be described, and certain orbits calculated.

There seems to be very little information on the classical line mass (and orbits about such a configuration), but the study of such a case does contain useful information. For example, any planet which is flattened at the poles should exibit some harmonics of orbital motion about a cylindrical line mass, since the planet could be taken as a deformed cylinder rather than a deformed sphere. The mathematics involved in describing orbital paths about the CLM is basically the same type of mathematics which is used to find orbits about a spherical mass distribution. The geometry of the cylindrical line mass is shown in figure 3.1. The line mass is situated along the z-axis and is 2l units in length.

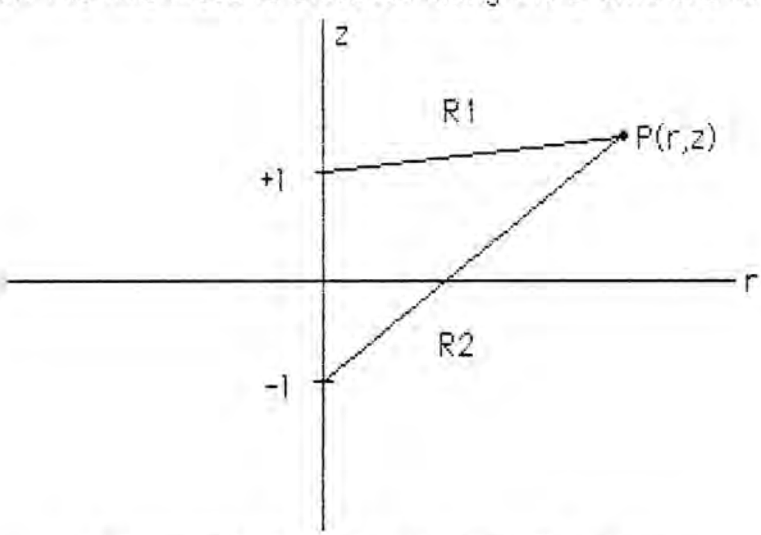

Figure 3.1: the axisymmetric line mass

For our purposes the line mass is infinitely thin, i.e. the line mass is that part of the z-axis which is 2l units long.

The element of potential at a point $P(r,\varnothing,z)$ due to the line mass with an element of mass dM is given by

$$dV=-GdM/R \qquad (3.1)$$

$$=-G\rho dz'/[r^2+(z-z')^2]^{1/2} \qquad (3.2)$$

where ρ= density. Upon integration, the potential of the CLM is found to be

$$V=(-GM/2l)\ln\{(R_1+R_2+2l)/(R_1+R_2-2l)\} \qquad (3.3)$$

where

$$R_1{}^2=(z-l)^2+r^2$$

$$R_2{}^2=(z+l)^2+r^2 \qquad (3.4)$$

It should be noted that the potential of (3.3) is infinite when R1+R2=2l. R1+R2 can only equal 2l when the point $P(r,\varnothing,z)$ is located somewhere on the CLM itself. This follows from the triangle inequality. R1+R2<2l gives rise to an undefined potential which is reasonable since r is taken to be non-negative. The infinite potential on the CLM is analogous to the infinite potential of a particle at r=0 in the spherically symmetric case.

The lagrangian of the system is given by

$$L=T-V \qquad (3.5)$$

where T is the kinetic energy of the system and V is the potential energy of the system. For a cylindrical system the line element (in Euclidean space is) given by

$$dq^2=dr^2+r^2d\varnothing^2+dz^2 \qquad (3.6)$$

The kinetic energy of a test particle can be found by dividing (3.6) by dq^2 (assuming that the mass of the test particle is given by m=1). Thus

$$T=(1/2)[r'^2+r^2\varnothing'^2+z'^2] \qquad (3.7)$$

Hence, the lagrangian for the CLM is given by

$$L=(1/2)[r'^2+r^2\o'^2+z'^2]+w\ln\{[R_1+R_2+2l]/[R_1+R_2-2l]\} \quad (3.8)$$

where $w=GM/2l$.

From (3.8) the various equations of motion for a test particle can be obtained using the Lagrange-Euler equation

$$(d/dt)(\partial L/\partial k')=\partial L/\partial k \quad (3.9)$$

Thus, for $\o$

$$d/dt(r2\o')=0 \quad (3.10)$$

which implies that

$$r^2\o'=h \quad (3.11)$$

where h is an arbitrary constant. Since the angular term $\o$ is being considered, h is taken to be the angular momentum constant: and hence the system exibits conservation of angular momentum. Equation (3.11) is a first integral for the system. During numerical computations, one way to check that the equations are really being solved is to check that the value of the first integral is actually conserved.

For r the equation of motion is given by

$$r''-r\o'^2+[4rlw(R_1+R_2)]/[(R_1+R_2)^2-4l^2]=0 \quad (3.12)$$

For z, the equation of motion is given by

$$z''+[4wl/((R_1+R_2)^2-4l^2)][(z-l)/R_1+(z+l)/R_2]=0 \quad (3.13)$$

Substitution of (3.10) into (3.11) yields two equations of motion, equation (3.13) and

$$r''-h^2/r^3+4rlw(R_1+R_2)/[(R_1+R_2)^2-4l^2]=0 \quad (3.14)$$

For the purpose of numerical calculations equations (3.13) and (3.14) should be scaled. Scaling involves assigning various quantities, such as length and time, natural units. The obvious choice for a natural unit of length in this problem is the length of the CLM. When the system is properly scaled, the end result is a system of differential equations (two in this case) which can be computed easily and effectively, with little roundoff error.

Before considering some orbits of the CLM, the effective potential of the system should be considered. The effective potential can be found by using the concept of total energy which is called the Hamiltonian of the system, given by

$$H=T+V \tag{3.15}$$

The total energy of the system is a conserved quantity, and can be used as a furter check on the validity of computer solutions. For the CLM, H is given by

$$E=H=(1/2)[r'^2+r2\phi'^2+z'^2]-w\ln[(R_1+R_2+2l)/(R_1+R_2-2l)] \tag{3.16}$$

where E is the total energy of the system. The effective potential can be found directly from (3.16) by solving for $r'^2+z'^2$:

$$r'^2+z'^2=2\{E+w\ln[(R_1+R_2+2l)/(R_1+R_2-2l)]\}-r^2\phi'^2 \tag{3.17}$$

Substitution of (3.11) into (3.17) yields the effective potential, V,

$$V=2\{E+w\ln[(R_1+R_2+2l)/(R_1+R_2-2l)]\}-h^2/r^2 \tag{3.18}$$

Equation (3.18) implies that the effective potential for the CLM is a surface rather than a functional curve since $V=V(r,z)$.

Graphs of various classical orbits are given in the diagrams which follow. Note that in the classical case the orbits are nicely shaped smooth paths. This will not necessarily be the case for orbits about the relativistic line mass. The initial conditions are given in each graph. The graphs are paired in radial plots and angular plots. In the angular plots the line mass is being looked upon from the top.

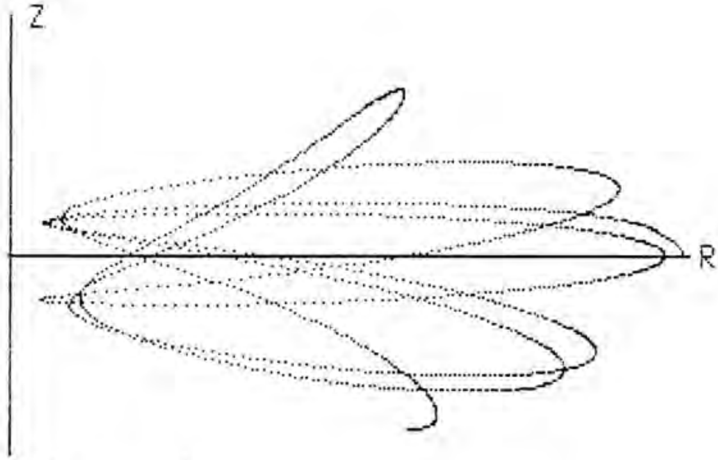

Figure 3. 2a
Initial Conditions:
H=0.2, R=2.0, R'=0.0, Z=0.2, Z'=0.0

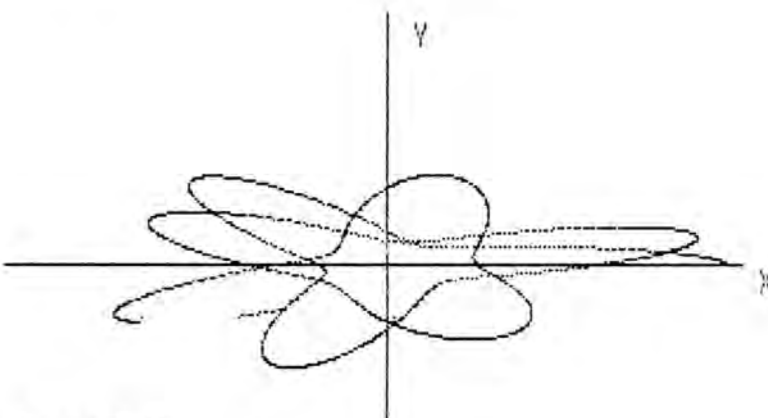

Figure 3.2b
Initial Conditions
R=2.0; R'=0.2; Z=0.2; Z'=0.0

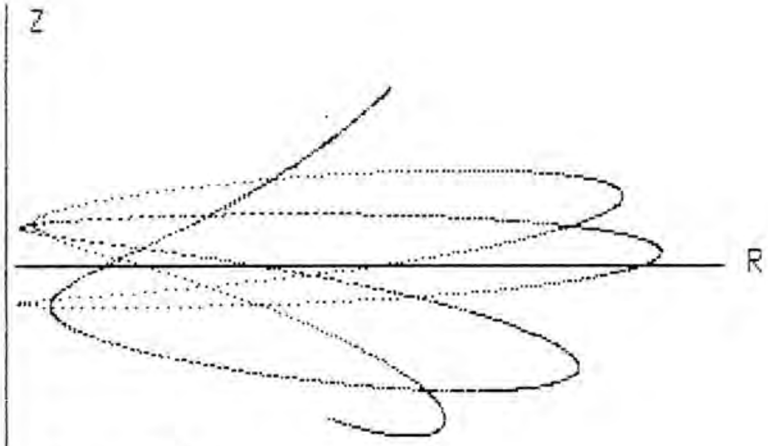

Figure 3. 3a
Initial Conditions:
H=0.2, R=2.0, R′=0.05, Z=0.2, Z′=0.0

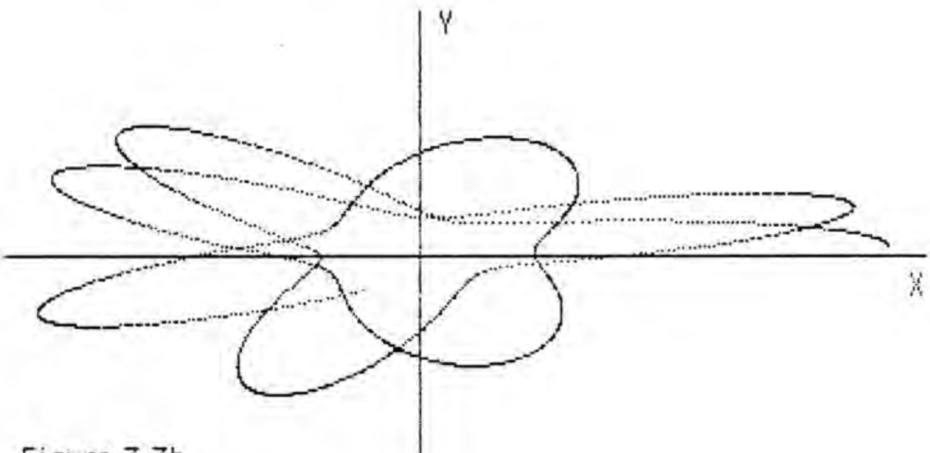

Figure 3.3b
Initial Conditions:
R=2.0; R′=0.05; Z=0.2; Z′=0.0; ø=0.0

CHAPTER 4: The Relativistic Line Mass

It is now time to investigate properties of the relativistic line mass (RLM), and the orbits of test particles about the RLM. In section 1 of this chapter the metric for the RLM will be derived and discussed. In section 2 the equations of motion will be derived.

SECTION 1: The Metric for Axisymmetric Einstein Space

Cylindrical (or axial) symmetry is simply defined: if cylindrical coordinates $(r,\varnothing,z)$ are used with $r=0$ on the axis of symmetry, then the gravitational potential is independent of $\varnothing$. (This is implicit in the derivation of the potential for the CLM in chapter 3.) In employing this idea in general relativity, one must consider the metric tensor g_{ij} as independent of $\varnothing$ and thus dependent only on r and z. Further, restrictions must be placed on t and $\varnothing$, namely that t and $\varnothing$ are reversible in the sense that replacing t or $\varnothing$ by $-t$ or $-\varnothing$, respectively , does not change the form of the metric.[1] This restriction means physically that the mass is non-rotating. Mathematically, the restrictions imply that the metric contains only squared terms of $d\varnothing$ and dt. The line element is thus of the form

$$ds^2=g_{44}dt^2+(g_{11}(dx^1)^2+2g_{12}dx^1dx^2+g_{22}(dx^2)^2)+g_{33}(dx^3)^2 \quad (4.1)$$

where $x^1=r$, $x^2=z$, $x^3=\varnothing$, and $x^4=t$. By way of an isothermal transformation[2] (4.1) can be reduced to

$$ds^2=a^2[(dx^1)^2+(dx^2)^2]+b^2(dx^3)^2-c^2(dx^4)^2 \quad (4.2)$$

where a,b, and c are functions of x^1 and x^2.

By straightfoward evaluation of the Riemann and Ricci tensors, the Ricci tensor is found to have the following surviving components:

$$R_{11}=(a_1/a)_1+(a_2/a)_2+b_{11}/b+c_{11}/c$$

$$+(a_2/a)(b_2/b+c_2/c)-(a_1/a)(b_1/b+c_2/c)\,,$$

$$R_{22}=(a_1/a)_1+(a_2/a)_2+b_{22}/b+c_{22}/c$$

$$+(a_1/a)(b_1/b+c_1/c)-(a_1/a)(b_2/b-c_2/c)\,, \qquad (4.3)$$

$$R_{12}=b_{12}/b+c_{12}/c-(a_2/a)(b_1/b+c_2/c)-(a_1/a)(b_2/b+c_2/c)\,,$$

$$R_{33}=b/a^2\{\Delta b+(1/c)(b_1c_1+b_2c_2)\}\,,$$

$$R_{44}=-c/a^2\{\Delta c+(1/b)(b_1c_1+b_2c)\}.$$

Subscripts to the right indicate partial differentiation with respect to x_1 and x_2. Also,

$$\Delta b=b_{11}+b_{22}\quad,\quad \Delta c=c_{11}+c_{22} \qquad (4.4)$$

Note that

$$R_{33}+R_{44}=(1/a^2bc)\Delta(bc) \qquad (4.5)$$

Equation (4.5) is true whether matter is present or not. Now, by the application of the Einstein field equations (1. 21), equation (4.5) gives rise to

$$\Delta(bc)=0 \qquad (4.6)$$

Equation (4.6) is Laplace's equation[3], and thus

$$bc=r(x^1,x^2) \qquad (4.7)$$

Now, using material as found in R. V. Churchill (see endnote 3), there exists a conjugate harmonic function such that

$$r(x^1,x^2)+iz(x^1,x^2)=f(x^1+ix^2) \qquad (4.8)$$

It follows that x^1 and x^2 can be mapped into r and z.

$$(x^1,x^2) \dashrightarrow (r,z) \qquad (4.9)$$

The isothermal nature of (4.2) is preserved by the transformation (see

Churchill, page 205-207). Thus

$$a^2[(dx^1)^2+dx^2)^2]=A(dr^2+dz^2) \tag{4.10}$$

where A is a function of (r,z). By equation (4.7)

$$b=r/c \tag{4.11}$$

and this means that (4.2) has not three, but two arbitrary functions which must be found.

Now, to restate the results, forget the meanings of x^1,x^2, a, b, and c. Let the new meanings be

$$x^1=r \quad , \quad x^2=z \quad , \quad x^3=\text{ø} \quad , \quad x^4=t \tag{4.12}$$

Thus far it has been shown that where t and ø are reversable, and where $R^3{}_3+R^4{}_4=0$, axial symmetry of the system naturally follows. It also follows from the above discussion that the metric now has the form

$$ds^2=a^2(dr^2+dz^2)+r^2c^{-2}d\text{ø}^2-c^2dt^2 \tag{4.13}$$

Using the more convient substitutions

$$a=e^{v-u} \quad , \quad b=re^{-u} \quad , \quad c=e^u \tag{4.14}$$

the metric becomes

$$ds^2=e^{2(v-u)}(dr^2+dz^2)+r^2e^{-2u}d\text{ø}^2-e^{2u}dt^2 \tag{4.15}$$

Now, from equation (4.3) it can be shown that

$$(1/2)(R_{11}+R_{22})=\Delta v-(\Delta u+u_1/r)+u_{12}+u_{22}$$

$$(1/2)(R_{11}-R_{22})=u_{12}-u_{22}-v_1/r$$

$$R_{12}=2u_1u_2-v_2/r \tag{4.16}$$

$$R^3{}_3-R^4{}_4=-(2/a^2)(\Delta u+u_1/r)$$

$$R^3{}_3+R^4{}_4=0$$

Applying $R_{ij}=0$ to equations (4.16) it can be shown that

$$\Delta u+u_1/r=0 \tag{4.17}$$

$$v_1=r(u_1{}^2-u_2{}^2) \qquad , \qquad v_2=2ru_1u_2 \tag{4.18}$$

$$\Delta v+u_1{}^2+u_2{}^2=0 \tag{4.19}$$

If (4.17) is satisfied then (4.18) are integrable, and thus (4.19) is implied. It follows that

$$v=\int r\{(u_1{}^2-u_2{}^2)dr+2u_1u_2dz)\} \tag{4.20}$$

Equation (4.20) follows from (4.18). It is implied from $R_{ij}=0$ that the field discribed by (4.20) lies outside any cylindrical mass, particularly, outside of the line mass.

Something else follows from (4.17) which makes the search for a specific line element almost complete. In explicit form (4.17) is given by

$$\partial^2u/\partial r^2+(1/r)(\partial u/\partial r)+\partial^2u/\partial z^2=0 \tag{4.21}$$

Equation (4.21) is Laplace's equation in cylindrical coordinates. Thus all that is needed is to find a solution to (4.21). From the above work in chapter 3, it turns out that the potential as found in equation (3.3) satisfies (4.21). The only modification needed is to divide the Newtonian potential by c^2, since for each Newtonian potential, N, there exists a corresponding general relativistic solution u such that $u=N/c^2$.[5] However, in the system of units in which $G=c=1$, u is given by

$$u=(M/2l)\ln\{\,(R_1+R_2-2l)/(R_1+R_2+2l)\,\} \tag{4.22}$$

Thus, it remains to find v. The derivation of v is lengthy and will not be given here. It can be shown[5] that

$$v=(1/2)(M/l)^2\ln\{[(R_1+R_2)^2-4l^2]/[4R_1R_2]\}. \tag{4.23}$$

Section 4.2: Orbits About the Relativistic Line Mass

With expressions for u and v the various equations of motion (the geodesic equations) can be found by using equation (1.7). Thus, the equations of motion are given by

$$t''+2u_r t'r'+2u_z t'z'=0 \tag{4.24}$$

$$r''+e^{4u-2v}v_r t'^2+(v_r-u_r)r'^2+2(v_z-u_z)z'r'$$
$$+(u_r-v_r)z'^2+e^{-2v}(r^2u_r-r)\phi'^2=0 \tag{4.25}$$

$$z''+e^{4u-2v}u_z t'^2+(u_z-v_z)r'^2+2(v_r-u_r)r'z'$$
$$+(v_z-u_z)z'^2+e^{2v}r^2u_z\phi'^2=0 \tag{4.26}$$

$$\phi''+2(1/r-u_r)r'\phi'-2u_z z'\phi'=0 \tag{4.27}$$

From the Lagrange-Euler method for finding the equations of motion, (4.24) and (4.27) reduce to

$$t'=-Ee^{-2u}/2 \qquad \text{where } E=\text{const} \tag{4.28}$$
$$\phi'=he^{2u}/2r^2 \qquad \text{where } h=\text{const} \tag{4.29}$$

Hence, (4.25) and (4.26) can be expressed as

$$r''+e^{-2v}v_r(E^2/4)+(v_r-u_r)r'^2+2(v_z-u_z)z'r'$$
$$+(u_r-v_r)z'^2+e^{4u-2v}(u_r/r^2-1/r^3)(h^2/4)=0 \tag{4.30}$$

$$z''+e^{-2v}u_zE^2/4+(u_z-v_z)r'^2+2(v_r-u_r)r'z'$$
$$+(v_z-u_z)z'^2+e^{2v+4u}u_z(h^2/4r^2)=0 \tag{4.31}$$

The constant E is interpreted here as the constant for energy, while h is interpreted as the angular momentum constant. Equations (4.30) and (4.31) are the general equations of motion for the RLM.

For motion in the meridian plane (4.30) and (4.31) will reduce to one equation. Suppose that a test particle is given initial conditions of $z(0)=0$ and $z'(0)=0$, and some initial values of $r(0)$ and $r'(0)$. Since no z momentum is given to the particle, it will not leave the meridian plane -

that plane perpendicular to the RLM which cuts the RLM in half, and where z=0. This follows from (4.31) where it can be seen that for the given initial conditions $z''(0)=0$, which implies that $z'(s)=0$ and $z(s)=0$ for all other s after s=0. The restriction to the meridian plane can also be seen from the physics of the RLM. If a particle is given the above initial conditions, then the gravitational attraction of the RLM from one side is balanced by the gravitational attraction of the other side. Thus, the test particle is trapped in the meridian plane and cannot escape that plane. The equations of motion reduce to

$$r''+e^{-2v}v_r(E^2/4)+(v_r-u_r)r'^2+e^{4u-2v}(u_r/r^2-1/r^3)(h^2/4)=0 \qquad (4.32)$$

It was shown in chapter 2 that for Schwarzschild geometry there is a radius which, if crossed by a test particle, cannot be recrossed. The particle is trapped- doomed to fall into the black-hole. The same behavior seems to occur with the RLM, and in this section the behavior of a surface similiar to the Schwarzschild radius of chapter 2 will be discussed.

In order to find a Schwarzschild-like radius (henceforth called an event horizon) it is necessary to investigate the line-element for the RLM. In its general form, the line element is given by equation (4.15)

$$ds^2=e^{2(v-u)}(dr^2+dz^2)+r^2e^{-2u}d\theta^2-e^{2u}dt^2$$

where u and v are given by (4.22) and (4.23)

$$u=(M/2l)\ln\{(R_1+R_2-2l)/(R_1+R_2+2l)\}$$

$$v=(1/2)(M/l)^2\ln\{[(R_1+R_2)2-4l^2]/(4R_1R_2)\}$$

In its expanded form, including the expressions for R_1 and R_2 as given in (3.4), the line element becomes

$$ds^2=\{\{[(z-1)^2+r^2]^{1/2}+[(z+1)^2+r^2]^{1/2}-21\}$$
$$/\{[(z-1)^2+r^2]^{1/2}+[(z+1)^2+r^2]^{1/2}+21\}\}^{M/1}(dt^2)$$
$$-\{\{[((z-1)^2+r^2)^{1/2}+((z+1)^2+r^2)^{1/2}]^2-41^2\}$$
$$/\{4[(z-1)^2+r^2]^{1/2}[(z+1)^2+r^2]^{1/2}\}\}^{(M/1)2}$$
$$*\{\{[(z-1)^2+r^2]^{1/2}+[(z+1)^2+r^2]^{1/2}-21\}$$
$$/\{[(z-1)^2+r^2]^{1/2}+[(z+1)^2+r^2]^{1/2}+21\}\}^{-(M/1)}(dr^2+dz^2)$$
$$-r^2\{\{[(z-1)^2+r^2]^{1/2}+[(z+1)^2+r^2]^{1/2}-21\}$$
$$/\{[(z-1)^2+r^2]^{1/2}+[(z+1)^2+r^2]^{1/2}+21\}\}^{-(M/1)}d\phi^2 \qquad (4.33)$$

A further topic of research would be to classify event horizons for various values of M/1. The case where M/1=1 then the coefficient for the (dr^2+dz^2) term reduces to

$$-\{\{[(z-1)^2+r^2]^{1/2}+[(z+1)^2+r^2]^{1/2}+21\}^2$$
$$/\{4[(z-1)^2+r^2]^{1/2}[(z+1)^2+r^2]^{1/2}\}\} \qquad (4.34)$$

Hence ds^2 becomes

$$ds^2=\{\{[(z-1)^2+r^2]^{1/2}+[(z+1)^2+r^2]^{1/2}-21\}$$
$$/\{[(z-1)^2+r^2]^{1/2}+[(z+1)^2+r^2]^{1/2}+21\}\}dt^2$$
$$-\{\{\{[(z-1)^2+r^2]^{1/2}+[(z+1)^2+r^2]^{1/2}+21\}^2$$
$$/\{4[(z-1)^2+r^2]^{1/2}[(z+1)^2+r^2]^{1/2}\}\}(dr^2+dz^2)$$
$$-r^2\{\{[(z-1)^2+r^2]^{1/2}+[(z+1)^2+r^2]^{1/2}-21\}$$
$$/\{[(z-1)^2+r^2]^{1/2}+[(z+1)^2+r^2]^{1/2}+21\}\}^{-1}d\phi^2 \qquad (4.35)$$

When

$$[(z-1)^2+r^2]^{1/2}+[(z+1)^2+r^2]^{1/2}=21 \qquad (4.36)$$

the singularity of the axisymmetric line mass (M/1=1) occurs: g_{tt} vanishes, while $g_{\phi\phi}$ becomes infinite. Where

$$[(z-1)^2+r^2]^{1/2}+[(z+1)^2+r^2]^{1/2}<21 \qquad (4.37)$$

g_{tt} becomes negative, while $g_{\phi\phi}$ becomes positive. As with the spherical mass, this behavior indicates that t is transformed into a spatial coordinate. It makes perfectly good sense that (4.34) is always negative: otherwise, two time-like coordinates would appear in the region where (4.37) is valid. This would not be physically possible. This leaves only the ø-coordinate. Indeed, as (4.35) and (4.37) indicate, the coefficient of $dø^2$ changes sign when (4.37) is valid. This behavior indicates that ø becomes the time-like coordinate, and that t becomes a spatial coordinate.

Equation (4.33) also gives insight into the Newtonian limit. Recall that each line-element in the relativistic case must reduce to the classical Newtonian line-element when one is far from the gravitating mass. Upon investigation of the limits of the metric tensor as given in (4.33), it turns out that as $(r,z)\rightarrow\infty$, each component of the metric tensor approaches its Newtonian limit. Thus, the line element is valid.

The effective potential as introduced in chapters 2 and 3 can also be helpful in determining the region in which an orbit of given initial conditions can occur. As with the CLM, solving for $r'^2+z'^2$ will give the effective potential for the RLM. It turns out that

$$V=r'^2+z'^2=e^{2(u-v)}-e^{(4u-2v)}[h^2/4r^2]+e^{-2v}[E^2/4] \qquad (4.38)$$

Again, as with the CLM, the RLM has an effective potential which is described by a surface, since $V=V(r,z)$. The surface will give a region in three-dimensional space in which the test- particle can move. The only region around the RLM in which a particle will move in a plane is the meridian plane as described above.

Specific orbits of a test particle about the RLM are given below. These orbits are given the same initial conditions as those in the classical case, but the results are somewhat different. This is due to the relativistic perturbations of the system. The initial conditions given to the various systems are listed with each pair of graphs. For the first graph the angular momentum, H=0.2; R=2.0; R′=0.0; Z=0.2; Z′=0.0. R is taken to be the radial distance from the origin; R′ is the initial velocity in the radial direction; Z and Z′ are defined similiarly. The half-length of the radius in each plot is l=1. The plots of the first system is given below.

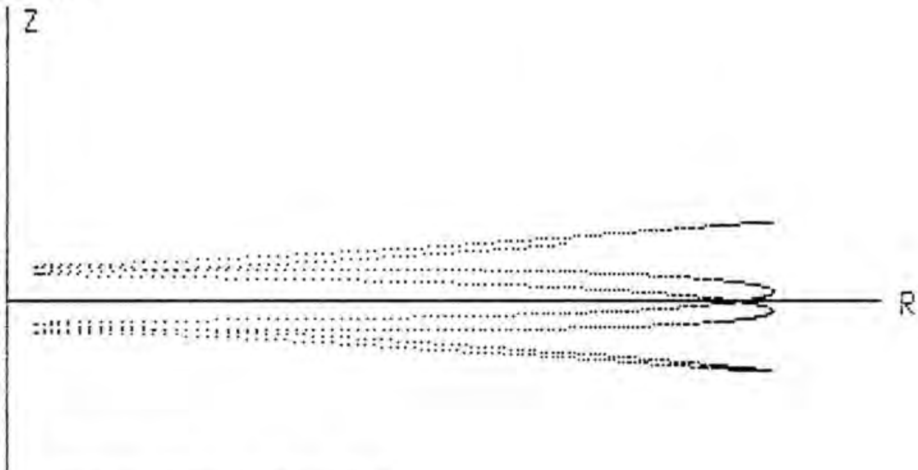

Figure 4.2a: radial plot
Initial Conditions:
R=2.0, R′=0.0, Z=0.2, Z′=0.0

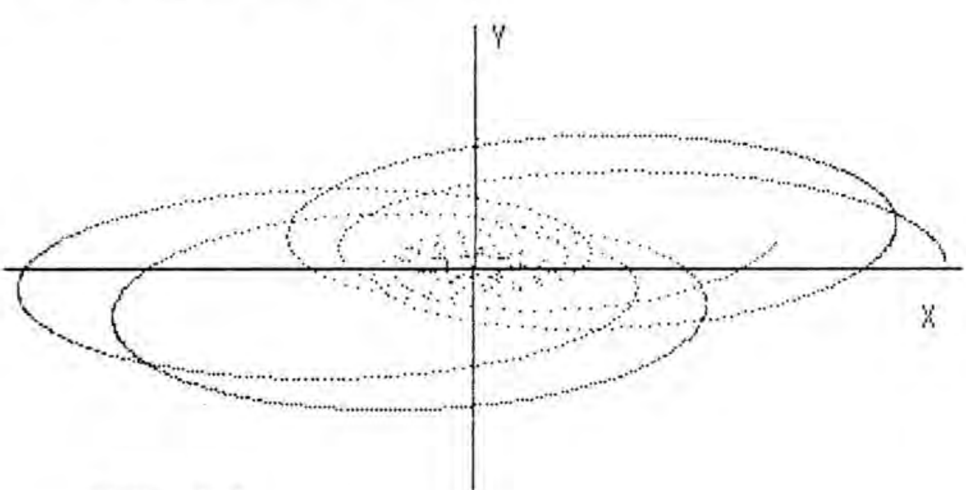

Figure 4.2b
Initial Conditions
H=0.2, R=2.0, R′=0.0, Z=0.2, Z′=0.0, ø=0.0

The reason the center of figure 4.2b is hard to follow is because of the high velocity of the test particle in this region. This is due to what can be described as a slingshot effect. As the test particle approaches the z-axis, its speed increases quickly, causing distances between two consecutive orbital points to be largely separated near the center of the graph. One can actually see the increase in speed as the points are being plotted on a computer terminal.

The next two sets of figures are orbits about the RLM given different initial conditions.

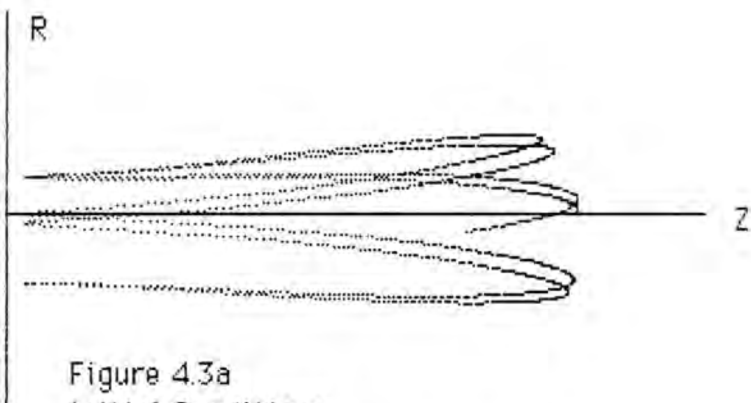

Figure 4.3a
Initial Conditions
H=0.2, R=2.0, R′=0.0, Z=0.2, Z′=0.1

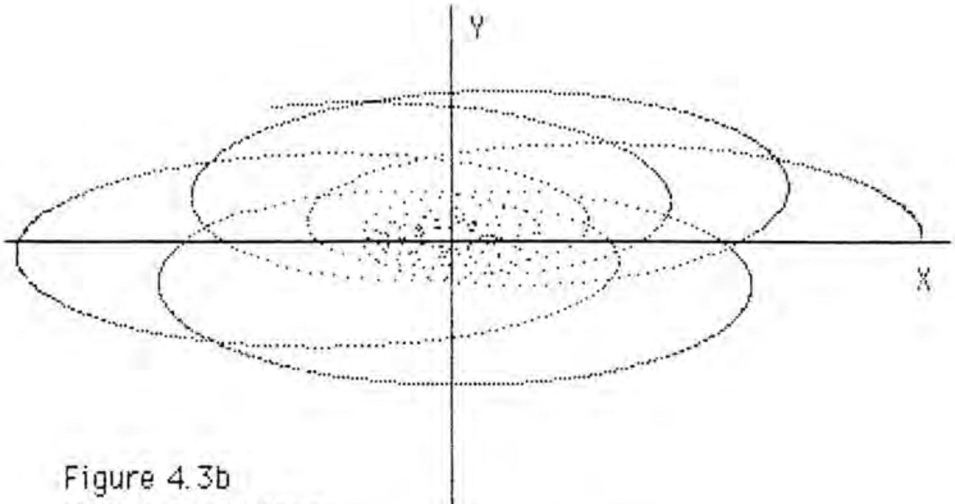

Figure 4.3b
Initial Conditions:
H=0.2, R=2.0, R′=0.0, Z=0.2, Z′=0.1, ø=0.0

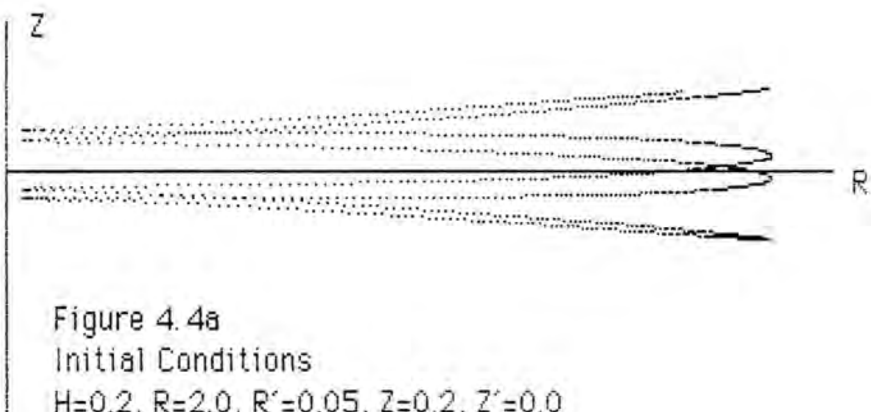

Figure 4.4a
Initial Conditions
H=0.2, R=2.0, R′=0.05, Z=0.2, Z′=0.0

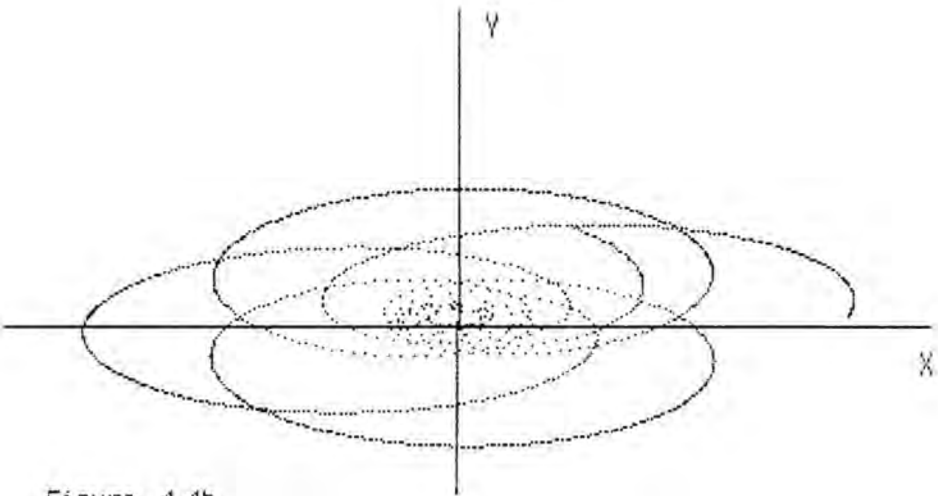

Figure 4.4b
Initial Conditions
H=0.2, R=2.0, R´=0.05, Z=0.2, Z´=0.0, ø=0.0

When each pair of figures is studied one can see how the test particle orbits the RLM. It must be remembered that the coordinates of r and z do not necessarily mean the same thing as the coordinates r and z in the classical case. What is seen in figures 4.2a to 4.4b is the mapping of r and z from the side, or x and y from the top, in flat 2-dimensional Euclidean space. The topology of the space about the RLM may not (and is not expected to) look like the flat Euclidean space about which we are familiar. It remains as further research then to carefully study the geodesics about the RLM and in some fasion chacacterize the space about the axisymmetric mass distribution.

Appendix A: Program Listing for the Relativistic Orbital Paths

Below is a listing of the program used to generate the orbital paths of the relativistic line mass. The program is written in HP PASCAL and was run on the HP-3000 SERIES 44. The listing includes various plotting commands used specifically with the TEKTRONICS 4025 terminal. For this paper and the relativistic graphs found in chapter 4, it was necessary to copy the numerical output of this program into a data file, which was later used to plot the orbits on the Apple LISA micro-computer.

```
$USLINIT$
PROGRAM AXISYM(INPUT,OUTPUT);

(* THIS PROGRAM INTEGRATES THE GENERAL RELATIVISTIC LINE
MASS *)

TYPE
 ONEDIM=ARRAY[0..4] OF REAL;

VAR
  Y,YO:ONEDIM; (* YO[0] =TIME; YO[1]=R; YO[2]=RDOT; YO[3]=Z;
YO[4]=ZDOT *)
 H,ANGMOM,K,L,PSI,HPSI,HPSI2,L2,LSQ,PENERGY:REAL;
 COUNT,NUMSTEPS:INTEGER;

PROCEDURE INIT;
(* INITIALIZES GRAPHICS ON THE TEKTRONICS 4025 TERMINAL *)
 BEGIN
 WRITELN('!WOR 32');
 WRITELN('!GRA 1,32');
 END; (* INIT *)

PROCEDURE AXES;
(* DRAWS AXIS ON THE TEKTRONICS 4025 TERMINAL *)
 BEGIN
 WRITELN('!VEC 0,0,0,447');
 WRITELN('!VEC 0,223,639,223');
 END; (*AXES*)
```

```
PROCEDURE PLOT(T,Y:REAL);
(* PLOTS POINTS ON THE TEKTRONICS 4025 TERMINAL *)
VAR
 IX0,IY0,IX1,IY1:INTEGER;
 BEGIN
IX0:=TRUNC(639.0*T/25.0);
IY0:=TRUNC(223.0*(Y+1.0));
IY1:=IY0;
IX1:=IX0;
WRITELN('!VEC',IX0,',',IY0,',',IX1,',',IY1);
 END; (*PLOT*)

(*------------------------***---------------------*)

PROCEDURE NUGAMMA(AR,ZEE:REAL;YES:BOOLEAN;Var
NU,GAMMA,NUR,NUZ, GAMMAR,GAMMAZ,EPOT:REAL);
 (* THIS PROCEDURE SETS UP THE VALUES OF R1, R2, NU AND GAMMA
FOR THE RELATIVISTIC LINE MASS *)

VAR
 R1,R2,SUMR,SUMR2,PR,ZL1,ZL2:REAL;

BEGIN
 R1:=SQRT((L-ZEE)*(L-ZEE)+AR*AR);
 R2:=SQRT((L+ZEE)*(L+ZEE)+AR*AR);
 SUMR:=R1+R2;
 SUMR2:=SUMR*SUMR;
 PR:=R1*R2;
 ZL1:=(ZEE-L)/R1;
 ZL2:=(ZEE+L)/R2;
 NU:=HPSI*LN((SUMR-L)/(SUMR+L));
 GAMMA:=HPSI2*LN((SUMR2-LSQ)/(4.0*PR));
 NUR:=(SUMR/PR)*L*PSI*AR/(SUMR2-LSQ);
 NUZ:=(L*PSI/(SUMR2-LSQ))*(ZL1+ZL2);
 GAMMAZ:=HPSI2*2.0*SUMR*(ZL1+ZL2)/(SUMR2-LSQ)-
 HPSI2*(R1*ZL2+R2*ZL1)/PR;
 GAMMAR:=HPSI2*2.0*AR*SUMR2/((SUMR2-LSQ)*PR)-
 HPSI2*AR*(R1*R1+R2*R2)/(PR*PR);
 IF YES THEN
```

```
 EPOT:=EXP(2.0*NU-2.0*GAMMA)-K*EXP(-2.0*GAMMA)+
     ANGMOM*EXP(4.0*NU-2.0*GAMMA)/(AR*AR);
END; (*NUGAMMA*)

(*-----------------------***------------------------*)

FUNCTION LAGRANGIAN(POT,RDOT,ZDOT:REAL):REAL;
BEGIN
LAGRANGIAN:=RDOT*RDOT+ZDOT*ZDOT+POT;
END;(*LAGRANGIAN*)
(* LAGRANGIAN SHOULD BE ZERO (OR CLOSE TO IT) DURING THE
PROGRAM RUN *)

(*-----------------***---------------------*)


PROCEDURE FOFY(YIN:ONEDIM;LAGRANGE:BOOLEAN;VAR FOUT:ONEDIM;
          VAR EPOT:REAL);
 (* FOFY GIVES VALUES FOR THE SECOND DERIVATIVES OF THE LINE
ELEMENT *)

VAR
 EG2,ENG,N,G,GR,GZ,NR,NZ:REAL;
BEGIN
NUGAMMA(YIN[1],YIN[3],LAGRANGE,N,G,NR,NZ,GR,GZ,EPOT);
ENG:=EXP(4.0*N-2.0*G);
EG2:=EXP(-2.0*G);
 FOUT[1]:=YIN[2];
 FOUT[2]:=(NR-GR)*YIN[2]*YIN[2]+(GR-NR)*YIN[4]*YIN[4]
+((1.0-YIN[1]*NR)/(YIN[1]*YIN[1]*YIN[1]))*ANGMOM*ENG-K*NR*EG2
+2.0*(NZ-GZ)*YIN[2]*YIN[4];
 FOUT[3]:=YIN[4];
 FOUT[4]:=(GZ-NZ)*YIN[2]*YIN[2]+(NZ-GZ)*YIN[4]*YIN[4]
-ANGMOM*NZ*ENG/(YIN[1]*YIN[1])+2.0*(NR-GR)*YIN[2]*YIN[4]
-K*NZ*EG2;
 END; (* FOFY *)

PROCEDURE ONESTEP(YIN:ONEDIM;VAR YOUT:ONEDIM;VAR PE:REAL);
 (* RUNGE-KUTTA METHOD FOR FINDING THE NEXT VALUES OF R,RDOT,
Z, ZDOT, T*)
```

```
TYPE
 RKARRAY=ARRAY[1..4,1..4] OF REAL;

VAR
 YLOCALA,YLOCALB:ONEDIM;
 COUNT:INTEGER;
 K:RKARRAY;
 POTENTIAL:REAL;

BEGIN
 YLOCALA:=YIN;
 FOFY(YLOCALA,FALSE,YLOCALB,POTENTIAL);
 FOR COUNT:=1 TO 4 DO BEGIN
 K[COUNT,1]:=H*YLOCALB[COUNT];
 YLOCALA[COUNT]:=YIN[COUNT]+0.5*K[COUNT,1];
 END;
 FOFY(YLOCALA,FALSE,YLOCALB,POTENTIAL);
 FOR COUNT:=1 TO 4 DO BEGIN
 K[COUNT,2]:=H*YLOCALB[COUNT];
 YLOCALA[COUNT]:=YIN[COUNT]+0.5*K[COUNT,2];
 END;
 FOFY(YLOCALA,FALSE,YLOCALB,POTENTIAL);
 FOR COUNT:=1 TO 4 DO BEGIN
 K[COUNT,3]:=H*YLOCALB[COUNT];
 YLOCALA[COUNT]:=YIN[COUNT]+K[COUNT,3];
 END;
 FOFY(YLOCALA,FALSE,YLOCALB,POTENTIAL);
 FOR COUNT:=1 TO 4 DO
 K[COUNT,4]:=H*YLOCALB[COUNT];
FOR COUNT:=1 TO 4 DO
YOUT[COUNT]:=YIN[COUNT]+(K[COUNT,1]+2.0*K[COUNT,2]
 +2.0*K[COUNT,3]+K[COUNT,4])/6.0;
FOFY(YOUT,TRUE,YLOCALB,POTENTIAL);
 YOUT[0]:=YIN[0]+H;
 PE:=POTENTIAL;
 END; (* ONE STEP *)

(*------------------------***------------------------*)
FUNCTION EPARM(H,R,RDOT,Z,ZDOT:REAL):REAL;
```

```
(* SETS INITIAL ENERGY PARAMETER FROM GIVEN INITIAL
CONDITIONS. THIS QUANTITY SHOULD BE CONSERVED DURING THE RUN
OF THE PROGRAM *)

VAR
 NU0,GAMMA0,NUR0,NUZ0,GAMMAR0,GAMMAZ0,K0,EPOT0:REAL;
BEGIN
NUGAMMA(R,Z,TRUE,NU0,GAMMA0,NUR0,NUZ0,GAMMAR0,GAMMAZ0,EPO
T0);
K0:=RDOT*RDOT+ZDOT*ZDOT;
K0:=K0+EXP(2.0*NU0-2.0*GAMMA0);
K0:=K0+H*EXP(4.0*NU0-2.0*GAMMA0)/(R*R);
K0:=K0*EXP(2.0*GAMMA0);
EPARM:=K0;
END;
(* END EPARM *)

(*-----------------------***-----------------------*)
(*              MAIN              *)


BEGIN
WRITELN('ENTER STEP SIZE');
READLN(H);
WRITELN('ENTER NUMBER OF STEPS');
READLN(NUMSTEPS);
WRITELN('ENTER DIMENSIONLESS ANGULAR MOMENTUM');
READLN(ANGMOM);
ANGMOM:=ANGMOM*ANGMOM;
WRITELN('ENTER HALF-LENGTHIN UNITS OF MU ');
READLN(L);
PSI:=1.0/L;
HPSI:=0.5*PSI;
HPSI2:=0.5*PSI*PSI;
LSQ:=L*L;
WRITELN('ENTER INITIAL VALUES');
WRITELN('R? ');
READLN(Y0[1]);
WRITELN('RDOT? ');
READLN(Y0[2]);
```

```
WRITELN('Z? ');
READLN(Y0[3]);
WRITELN('ZDOT? ');
READLN(Y0[4]);
K:=EPARM(ANGMOM,Y0[1],Y0[2],Y0[3],Y0[4]);
WRITELN('INTIAL ENERGY PARAMTER IS PLUS OR MINUS ',SQRT(K));
Y0[0]:=0.0;
INIT;
AXES;
FOR COUNT:=1 TO NUMSTEPS DO BEGIN
ONESTEP(Y0,Y,PENERGY);
PLOT(Y[1],Y[3]);
WRITELN(Y[0],'  ',Y[1],'  ',Y[2],' ',LAGRANGIAN(PENERGY,Y[2],Y[4]));
Y0:=Y;
END;
END.
```

Apendix B: Program Listing for the Classical Orbital Paths

The program listed below was used to calculate the classical orbits of a classical line mass. As with the previous program, this program was run on the HP-3000 series 44. Data files were used to transfer data to the Apple LISA so that plots could be made.

```
$USLINIT$
PROGRAM AXISYM(INPUT,OUTPUT);
 (* THIS PROGRAM INTEGRATES THE CLASSICAL LINE MASS *)
TYPE
 ONEDIM=ARRAY[0..4] OF REAL;

VAR
  Y,YO:ONEDIM;  (*  YO[0]=TIME;  YO[1]=R;  YO[2]=RDOT;  YO[3]=Z;
YO[4]=ZDOTN *)
 H,BETA,K,L,L2,LSQ2,PENERGY:REAL;
 COUNT,NUMSTEPS:INTEGER;
PROCEDURE INIT;  (* INITIALIZES GRAPHICS ON THE TEKTRONICS
4025 TERMINAL *)
 BEGIN
 WRITELN('!WOR 32');
 WRITELN('!GRA 1,32');
 END; (* INIT *)
PROCEDURE AXES; (* PLACES AXIS ON SCREEN OF TEKTRONICS 4025*)
 BEGIN
 WRITELN('!VEC 0,0,0,447');
 WRITELN('!VEC 0,223,639,223');
 END; (*AXES*)


PROCEDURE PLOT(T,Y:REAL); (* PLOTS POINTS ON THE TEKTRONICS
4025 TERMINAL *)

VAR
 IXO,IYO,IX1,IY1:INTEGER;
 BEGIN
IXO:=TRUNC(639.0*T/15.0);
```

```
IY0:=TRUNC(223.0*(Y+1.0));
IY1:=IY0;
IX1:=IX0;
WRITELN('!VEC',IX0,',',IY0,',',IX1,',',IY1);
 END; (*PLOT*)


(*------------------------***---------------------*)
(* FUNCTIONS  FUNCTIONS  FUNCTIONS FUNCTIONS   *)


FUNCTION RBAR1(R,Z:REAL):REAL;
BEGIN
RBAR1:=SQRT(R*R+(Z-1.0)*(Z-1.0));
END; (* END RBAR1 *)

FUNCTION RBAR2(R,Z:REAL):REAL;
BEGIN
RBAR2:=SQRT(R*R+(Z+1.0)*(Z+1.0))
END;(* END RBAR2*)

FUNCTION HAMILTONIAN(X:ONEDIM):REAL;
(* COMPUTES THE HAMILTONIAN OF THE LINE MASS, WHICH SOULD BE
A CONSERVED QUANTITY DURING THE RUN OF THE PROGRAM *)
VAR
RH1,RH2, T,V:REAL;
BEGIN
RH1:=RBAR1(X[1],X[3]);
RH2:=RBAR2(X[1],X[3]);
T:=0.5*(X[2]*X[2]+X[4]*X[4]);
V:=0.5*BETA/(X[1]*X[1])-0.25*LN((RH1+RH2+2.0)/(RH1+RH2-2.0));
HAMILTONIAN:=T+V;
END;

(*   END OF FUNCTIONS   *)


(*-----------------------***-------------------------*)
(*   VECTOR FORCE FUNCTION                    *)
```

```
PROCEDURE FOFY(YIN:ONEDIM;VAR FOUT:ONEDIM);
(* FINDS FUNCTIONAL VALUES OF THE SECOND DERIVATIVES OF THE
LINE MASS *)

VAR
 R1,R2,SUMR,SUMRSQ,DEN:REAL;
BEGIN
 R1:=RBAR1(YIN[1],YIN[3]);
 R2:=RBAR2(YIN[1],YIN[3]);
 SUMR:=R1+R2;
 SUMRSQ:=SUMR*SUMR;
 DEN:=1.0/(SUMRSQ-4.0);
 FOUT[1]:=YIN[2];
 FOUT[2]:=BETA/(YIN[1]*YIN[1]*YIN[1])-YIN[1]*SUMR*DEN/(R1*R2);
 FOUT[3]:=YIN[4];
 FOUT[4]:=-((YIN[3]-1.0)/R1+(YIN[3]+1.0)/R2)*DEN;
 END; (* FOFY *)

PROCEDURE ONESTEP(YIN:ONEDIM;VAR YOUT:ONEDIM);
(* RUNGE-KUTTA PROCEDURE FOR FINDING NEXT VALUES OF YO *)

TYPE
 RKARRAY=ARRAY[1..4,1..4] OF REAL;

VAR
 YLOCALA,YLOCALB:ONEDIM;
 COUNT:INTEGER;
 K:RKARRAY;

BEGIN
 YLOCALA:=YIN;
 FOFY(YLOCALA,YLOCALB);
 FOR COUNT:=1 TO 4 DO BEGIN
 K[COUNT,1]:=H*YLOCALB[COUNT];
 YLOCALA[COUNT]:=YIN[COUNT]+0.5*K[COUNT,1];
 END;
 FOFY(YLOCALA,YLOCALB);
 FOR COUNT:=1 TO 4 DO BEGIN
```

```
 K[COUNT,2]:=H*YLOCALB[COUNT];
 YLOCALA[COUNT]:=YIN[COUNT]+0.5*K[COUNT,2];
 END;
 FOFY(YLOCALA,YLOCALB);
 FOR COUNT:=1 TO 4 DO BEGIN
 K[COUNT,3]:=H*YLOCALB[COUNT];
 YLOCALA[COUNT]:=YIN[COUNT]+K[COUNT,3];
 END;
 FOFY(YLOCALA,YLOCALB);
 FOR COUNT:=1 TO 4 DO
 K[COUNT,4]:=H*YLOCALB[COUNT];
FOR COUNT:=1 TO 4 DO
YOUT[COUNT]:=YIN[COUNT]+(K[COUNT,1]+2.0*K[COUNT,2]
 +2.0*K[COUNT,3]+K[COUNT,4])/6.0;
 YOUT[0]:=YIN[0]+H;
 END; (* ONE STEP *)

(*-----------------------***----------------------*)
(*MAIN MAIN MAIN MAIN MAIN MAIN MAIN  MAIN MAIN *)
(*-----------------------***-----------------------*)

BEGIN
WRITELN('ENTER STEP SIZE');
READLN(H);
WRITELN('ENTER NUMBER OF STEPS');
READLN(NUMSTEPS);
WRITELN('ENTER     DIMENSIONLESS     ANGULAR     MOMENTUM
PARAMETER-BETA');
READLN(BETA);
WRITELN('ENTER INITIAL VALUES');
WRITELN('R AND RDOT');
READLN(YO[1],YO[2]);
WRITELN('Z AND ZDOT');
READLN(YO[3],YO[4]);
YO[0]:=0.0;
INIT;
AXES;
FOR COUNT:=1 TO NUMSTEPS DO BEGIN
ONESTEP(YO,Y);
```

```
PLOT(Y[1],Y[3]);
WRITELN(Y[0],' ',HAMILTONIAN(Y));
Y0:=Y;
END;
```

Appendix C: A Brief Introduction to Scaling

Scaling is a process whereby equations are made managable for numerical analysis. The equations of orbital motion would have been hard to solve on the computer if they had not at first been properly scaled. When one scales equations for the purpose of numerical analysis one must choose a scaling factor which is appropriate to the problem at hand. For the problem of orbits about a classical and relativistic line mass the one unit of length which can effectively be used as a scaling factor is the unit of length of the line mass. This unit of length is called L, and is used to scale all other lengths in the differential equations which govern the orbital paths about the line mass.

Here I will briefly derive the scaling of equation (4.30) in the text. First, it is necessary to set up dimensionless lengths using the scaling factor L. These lengths are defined by

$$z=ZL$$

$$r=RL \qquad (C.1)$$

Now L is defined to be 2μ , where μ is given by $\mu=GM/c^2$. Second, it is necessary to scale the time value by β, thereby obtaining

$$s=\beta T \qquad (C.2)$$

In terms of the scale factor for the length this implies that

$$\beta=L/c=2GM/c^2. \qquad (C.3)$$

Using these three equations of scaling in equation (4.30) yields

$$(L/\beta^2)(d^2R/dT^2)+(v_R-u_R)(1/L)(L^2/\beta^2)(dR/dT)^2$$

$$+(u_R-v_R)(1/L)(L^2/\beta^2)(dZ/dT)^2$$

$$+(c^4h^2/L^3R^3)e^{4u-2v}(LRu_R(1/L)-1)+K^2c^2(1/L)u_Re^{-2v}$$

$$+2(dR/dT)(dZ/dT)(L^2/\beta^2)(v_Z-u_Z)(1/L)=0 \qquad (C.4)$$

Upon slight manipulation equation (C.4) becomes

$$R''+(v_R-u_R)(R'^2)+(u_R-v_R)(Z'^2)+2R'Z'(v_Z-u_Z)$$
$$+(h^2c^2/L^2R^3)e^{4u-2v}(Ru_R-1)+k^2u_Re^{-2v}=0 \qquad (C.5)$$

where ′ now means d/dT.

The scaling is now complete. For the convience of computation, the value of h^2c^2/L^2R^3 is chosen to be H, the dimensionless angular momentum. Hence, if one so desires one may calculate the real lenghts(in meters), and the proper times (in seconds or meters) from the data evaluated by the computer programs given in appendix C.1 and C.2.

END NOTES

Chapter 1

[1]Isaac Newton, Mathematical Principles of Natural Philosophy, trans. A. Motte, rev. F. Cajori (Chicago:Britannica, 1952),p. 8.

[2]*ibid.*

[3]C. Moller, The Theory of Relativity, (Oxford:Clarendon, 1966), p. 220.

[4]Stephen Weinberg, Gravitation and Cosmology (New York:Wiley, 1972),p. 145.

[5]Richard Price, "General Relativity Primer," American Journal of Physics, 50(1982),p. 318,319.

[6]Wolfgang Pauli, Theory of Relativity (1958; rpt. New York: Dover,1981),p. 145.

[7]For additional information on tensor analysis see Mary Boaz, Mathematical Methods in the Physical Sciences (New York: wiley,1983), p. 431-443; C. Misner, K. Thorne, and J. Wheeler, Gravitation (San Fransisco: Freeman,1973), p. 203f,p.233f; Wolfgang Rindler, Essential Relativity (New York: Springer-Verlag, 1977), p. 126-133.

[8]Ronald Adler, Maurice Bazin, and Menahem Schiffer, Introduction to General Relativity (New York: McGraw-Hill, 1965), p. 9,10.

[9]Weinberg, p. 69.

[10]Misner,p. 334.

[11]Adler, p. 72

[12]A. Z. Petrov, Einstein Spaces (New York: Pergamon, 1969), p. 76.

[13]*ibid.*

Chapter 2

[1]Misner, p. 607.

2Wolfgang Rindler, p. 123-125.

[3]For more information on the classical Kepler problem see Jerry B. Marion, Classical Dynamics, (New York: Academic Press, 1965), p. 269-282; Ralph Baeirlein, Newtonian Dynamics, (New York: McGraw-Hill,1983), p. 156-175.

[4]J. L. Synge, Relativity, The General Theory, (Amsterdam: North-Holland, 1966), p. 293.

[5]Richard C. Tolman, Relativity, Thermodynamics, and Cosmology, (Oxford: Clarendon,1934), p. 208-209.

[6]G. Lemaitre, "Spherical Condensations in the Expanding Universe," Academy of Science, 196(1960)903-904.

[7]M. D. Kruskal, "Maximal Extension of Schwarzschild Metric," Physics Review, 119(1960)1743-1745.

[8]G. Szederes, "On the Singularities of a Riemannian Manifold," Publication of Mathematics Debrecen 7, (1960)285-301.

Chapter 3

[1]See H. P. Robertson, Thomas Noonan, Relativity and Cosmology, (Philidelphia: Saunders, 1968), p, 273 for the physical situation in the classical case.

Chapter 4

[1]Synge, p. 309.

[2]Synge, p. 310.

[3]Ruel Churchill, Complex Variables with Applications, 3^{rd} ed., (New York: McGraw-Hill, 1976), p. 201-205.

[4]Robertson, p. 273.

[5]Robertson, p. 275.

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Misner, Charles W., Kip S. Thorne, John A. Wheeler. Gravitation. San Franscisco: Freeman, 1973. QC178.M57

Møller, C. The Theory of Relativity. Oxford: Clarendon, 1966. QC6.M52

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