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Mathematical modeling of a variable mass rocket's dynamics using the differential transform method

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Abstract

In this paper, the mathematical modelling of a rocket with varying mass is investigated to construct a function that can describe the velocity and position of the rocket as a function of time. This research is geared more towards small scale rockets where the nonlinear drag term is of great interest to the underlying dynamics of the rocket. A simple force balance on the rocket using Newton's second law of motion yields a Riccati differential equation for which the solution yields the velocity of the rocket at any given time. This solution can then be integrated with respect to time to get the position function. The Differential Transform Method (DTM) is applied to the Riccati differential equation while yields a polynomial series solution approximation. This solution is then compared to numerical solutions from existing commercial rocket flight simulators, and to experimental data from rocket flights. A parametric study is also performed to survey the effects of density, diameter of the rocket airframe, drag coefficient, mass flow rate, and thrust on the overall motion of the rocket. The comparisons of the DTM solution to existing data showed almost a perfect match and the parametric study provides an insight into the various effects of the variables listed above. The goal of this research is to aid rocket design teams, especially in university rocketry competitions, to use as an additional tool with the flight simulators. While the flight simulators yield outstanding results, it is difficult for the user to study the fundamental physics of the flight from the simulator alone, and therefore the DTM solution and its results can be enlightening and helpful.

Acknowledgements

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My appreciation also extends to the Office for Undergraduate Research and Creative Endeavor for funding all my research projects including this one. Their support with URaCE and the SEARCH grant made it possible to financially support all my research goals at UTC while also training and providing me with the means to become a better scientist.

Finally, I am extremely grateful to my family whose value to me only grows with time.

Nomenclature

ρ	$=$ Density
A_p	$=$ Projected area
C_{total}	$=$ Total drag coefficient
$m, m(t)$	$=$ Mass
g	$=$ Graviational acceleration
$v, v(t)$	$=$ Velocity
D	$=$ Drag Force
T	$=$ Thrust
v_e	$=$ Propellant exhaust velocity
P_e	$=$ Exit pressure
P_a	$=$ Atmospheric pressure
A_e	$=$ Exit area
G	$=$ Weight of rocket
m_0	$=$ Initial mass
s	$=$ Stage seperation time

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I. Introduction

This research was originally inspired by the University of Tennessee at Chattanooga's (UTC) rocket team that competed in the 2018 National Students for the Exploration and Development of Space (SEDS) University Rocketry Challenge (URC). The goal of the competition was to build a multistage rocket to achieve maximal altitude. One of the main problems in analysis of the design was to optimize the stage separation event. The team had difficulty in determining the point that exists between the booster burnout time and the time at booster apogee, to separate the two stages to achieve maximum altitude. This analysis first required an accurate model of the rocket's flight which was challenging as most large-scale rocket's equations like in Martinez-Sanchez's example ignores the dynamic drag resistance by evaluating a drag to weight ratio [2]. For large scale rocket's, the drag force is negligible compared to the weight and thrust and therefore it is dropped to simplify the calculations. However, for small scale rockets such as the one used by UTC's rocket team [1],

$$\rho = 0.25\frac{kg}{m^3} \quad A_p = 0.009m^2 \quad C_{total} = 0.26 \quad v = 210\frac{m}{s} \quad m = 2.063\ kg \tag{1}$$

$$\frac{0.5\rho A_p C_{total} v^2}{mg} = 0.429 \tag{2}$$

It is evident that 42.9% of the dynamic resistance on the rocket is contributed by the drag force and therefore it cannot be ignored in the calculations. The introduction of the drag term in the analysis creates a non-linear first order Riccati differential equation that must be solved to gain a solution for the velocity of the rocket as a function of time. This solution can then be integrated with respect to time to yield the position of the rocket as a function of time. The subject of this study is to compare this analytical solution to numerical solutions from existing commercial rocket flight simulators, and to experimental data from small scale rocket launches. This solution can aid in many parametric studies which will also be explored in the paper along with applications that can be further investigated.

II. Theoretical Framework

In this section, the forces acting on the rocket are considered. Shown below is a free body diagram illustrating the different forces acting on the rocket.

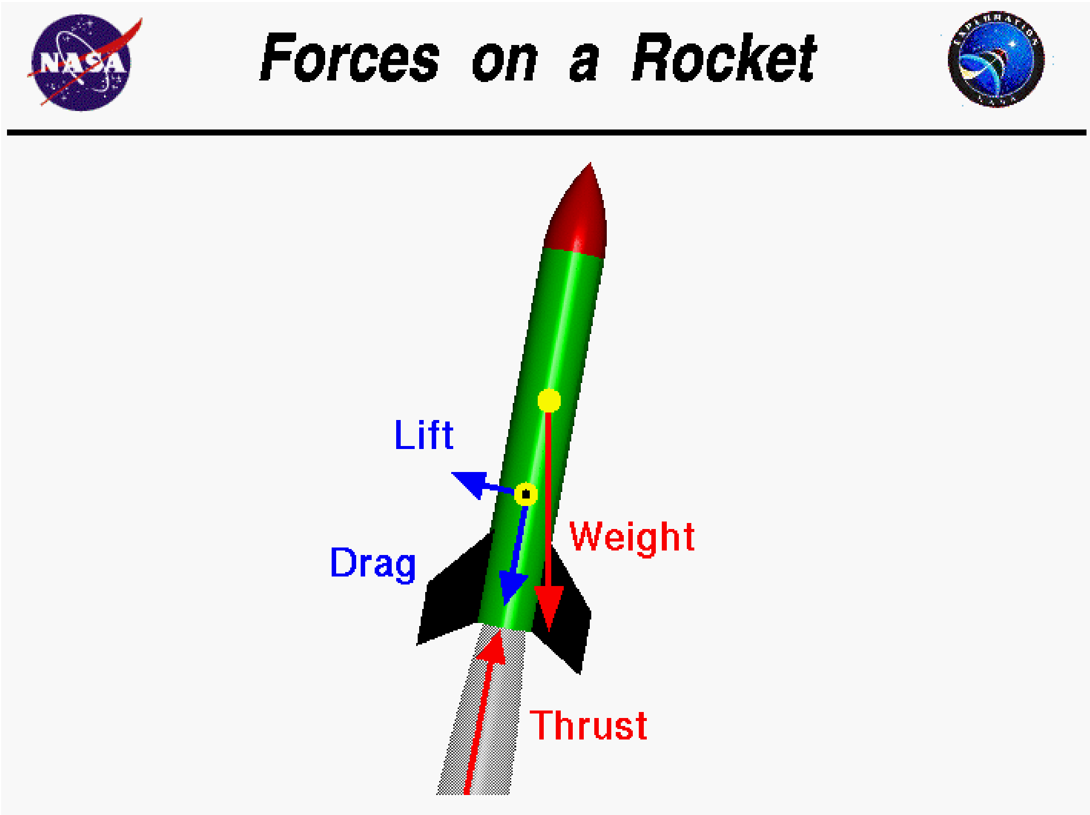

Figure 1: Free Body Diagram of Rocket [14]

For this study, a few assumptions are made to simplify the model. The flight is considered purely vertical and therefore, the lift force is neglected and only the weight, drag and thrust forces are considered. The weight and drag forces are variable and change with time, however an average thrust value is maintained through the calculations. The thrust is also assumed to be optimal which makes it independent of the pressure of the surrounding fluid. Various aspects affect the drag and it acts opposite and parallel to the velocity [3]. For a rocket that has a cylindrical geometry, the drag force is defined as [4]:

$$D = \frac{1}{2} C_{total} A_p \rho v^2 \tag{3}$$

Thrust is defined as [5]:

$$T = \frac{dm}{dt} v_e + (P_e - P_a) A_e \tag{4}$$

However, optimal thrust simplifies to [5]:

$$T = \frac{dm}{dt} v_e \tag{5}$$

Here the mass flow rate and exhaust gas velocity are assumed to be constant which is used to calculate the constant average thrust. Finally, the weight of the rocket is:

$$G = mg \tag{6}$$

Given the above three forces, Newton's second law can be applied:

$$\sum F_y = T - D - G \tag{7}$$

$$\frac{d(m(t)v(t))}{dt} = T(t) - D(t) - G(t) \tag{8}$$

$$m(t)\frac{dv(t)}{dt} + v(t)\frac{dm(t)}{dt} = T(t) - 0.5\rho A_p C_{total} v(t)^2 - m(t)g \tag{9}$$

$$\frac{dv(t)}{dt} + m(t)^{-1}\frac{dm(t)}{dt}v(t) = m(t)^{-1}T(t) - m(t)^{-1}0.5\rho A_p C_{total}v(t)^2 - g \quad (10)$$

$$\frac{dv(t)}{dt} = -m(t)^{-1}0.5\rho A_p C_{total}v(t)^2 - m(t)^{-1}\frac{dm(t)}{dt}v(t) + \frac{T(t)}{m(t)} - g \quad (11)$$

Assuming that the mass flow rate is constant and substituting equation (5) for thrust, equation (11) can be rearranged and written as,

$$\left(m_0 - \frac{dm(t)}{dt}t\right)\frac{dv(t)}{dt} + 0.5\rho A_p C_{total}v(t)^2 - \frac{dm(t)}{dt}v(t) - \frac{dm(t)}{dt}v_e + \left(m_0 - \frac{dm(t)}{dt}t\right)g = 0 \quad (12)$$

Equation (12) is the Riccati differential equation of interest that needs to be solved analytically. To do this, the Differential Transform Method (DTM) is employed.

III. Differential Transform Method

The Differential transform method is a unique analytical method that is based on the Taylor series expansion. It differs from the standard high order Taylor series expansion because it does not need any symbolic calculation of the derivatives of the function of interest [6]. It was originally applied by Zhou [7] and since then many researchers have attempted to apply this method to various nonlinear equations. For instance, Aruna and Kanth used DTM to solve the Klein-Gordon equation [8], Ghafoori et al. utilized the DTM to solve an important oscillation equation that yielded more accurate results than both HPM and VIM [7]. Eslami and Biazar used DTM for solve a Riccati differential equation [9] which was similar in anatomy to (12) and is where the inspiration for this paper was found. Evidently, many authors used DTM to solve various nonlinear equations and yielded promising results and therefore the credibility of the method has been established. This paper intends to employ the analytical method described above to solve, at least in approximation, the differential equation presented in (12). Much like any other transform method, the DTM transforms a function from a domain *D*, to a domain *K* where appropriate techniques are applied and then the inverse transform is taken to revert to the domain *D*. A unique advantage of DTM is the ability to directly apply it to nonlinear ODEs without utilizing discretization, or perturbation [1].

For an arbitrary function *y(x)* in C^k, the transformation of the *k*th derivative of *y(x)* is defined as [10]:

$$Y(k) = \frac{1}{k!}\frac{d^k y}{dx^k}(x)\Big|_{x=x_0}, \tag{13}$$

And the inverse transformation of *Y(k)* is given by [10],

$$y(x) = \sum_{k=0}^{\infty} Y(k)(x-x_0)^k, \tag{14}$$

Table 1 lists some of the common DTM transformations.

Table 1 – Common DTM transformation [1]

Original Function	Transformed Function
$x(t) = \alpha f(x) \pm \beta g(t)$	$X(k) = \alpha F(k) \pm \beta G(k)$
$x(t) = \dfrac{d^m f(t)}{dt^m}$	$X(k) = \dfrac{(k+m)!F(k+m)}{k!}$
$x(t) = f(t)g(t)$	$X(k) = \sum_{l=0}^{k} F(l)G(k-l)$
$x(t) = t^m$	$X(k) = \delta(k-m) = \begin{cases} 1, & \text{if } k = m \\ 0, & \text{if } k \neq m \end{cases}$
$x(t) = e(t)$	$X(k) = \dfrac{1}{k!}$
$x(t) = \sin(\omega t + \alpha)$	$X(k) = \dfrac{\omega^k}{k!} \sin\left(\dfrac{k\pi}{2} + \alpha\right)$
$x(t) = \cos(\omega t + \alpha)$	$X(k) = \dfrac{\omega^k}{k!} \cos\left(\dfrac{k\pi}{2} + \alpha\right)$

To illustrate how DTM is used, a simple example is presented below.

Example:

Let *y(x)* be a solution to the initial-value problem:

$$y'(x) - y(x) = 0 \quad ; \quad y(0) = 1 \tag{15}$$

Clearly the solution to this problem is *y(x)* = **exp***(x)* and followingly, the DTM is used check if the same solution can be attained.

Using the definitions in Table 1 and applying it to (15) yields,

$$\frac{(k+1)!\,Y(k+1)}{k!} = Y(k) \tag{16}$$

Which simplifies to:

$$(k+1)Y(k+1) = Y(k) \tag{17}$$

Now iterating for integer values of *k* starting from 0 and utilizing the given initial condition gives the following. For *k* = 0,

$$Y(1) = Y(0) = 1 \tag{18}$$

For *k* = 1,

$$Y(2) = \frac{1}{2} \tag{19}$$

For *k* = 2,

$$Y(3) = \frac{1}{6} \tag{20}$$

For *k* = 3,

$$Y(4) = \frac{1}{24} \tag{21}$$

$$\vdots$$

Now applying the inverse transformation shown in (14) provides:

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \tag{22}$$

The last equality in equation (22) comes from the Taylor series expansion of the exponential solution [11]. This agrees with the solution presented above and thus, how the DTM method is applied has been demonstrated.

IV. Applying DTM

This section will employ the DTM to be used on equation (12). Since the rocket starts at rest the initial condition is

$$V(0) = 0 \tag{23}$$

Where *V(k)* is the transformed form of *v(t)*. Using the common transformations shown in Table 1, equation (12) can be transformed into:

$$\begin{aligned} m_0(k+1)V(k+1) - \dot{m}\sum_{k_1=0}^{k}\big(\delta(k_1-1)(k+1-k_1)V(k+1-k_1)\big) \\ + 0.5C_{total}\rho A_p\sum_{k_1=0}^{k}\big(V(k_1)V(k-k_1)\big) - \dot{m}V(k) + (m_0 g - \dot{m}v_e)\delta(k) \\ - \dot{m}g\delta(k-1) = 0 \end{aligned} \tag{24}$$

By using the initial condition in (23) and solving for terms of *V(k)* by iterating for integer values of *k* starting from 0,

$$V(1) = \frac{\dot{m}v_e - m_0 g}{m_0}$$

$$V(2) = \frac{mg + 2\dot{m}V(1)}{2m_0} = \frac{\dot{m}(\dot{m}v_e - \frac{1}{2}gm_0)}{m_0^2} \tag{25}$$

$$V(3) = \frac{3\dot{m}V(2) - \frac{1}{4}C_{total}\rho\pi d^2 V(1)^2}{3m_0}$$

$$= -\frac{\frac{1}{2}C_{total}\rho\pi d^2 g^2 \dot{m}^2 + 3gm_0\dot{m}^2 - C_{total}\rho\pi d^2 g m_0 \dot{m} v_e - 6\dot{m}^3 v_e + \frac{1}{2}C_{total}\rho\pi d^2 \dot{m}^2 v_e^2}{6m_0^3}$$

$$\vdots$$

V. Verification of DTM

By substituting the transformed terms *V(k)* from equation (25) into the definition of the inverse transform presented in equation (14), the truncated polynomial approximation can be constructed. For instance, UTC's NASA USLI competition rocket from 2019 had the following parameters: v_e = 1810.6 m/s, $\dot{m}$ = 0.289 kg/s , m_0 = 3.063 kg, d = 0.1016 m, C_{total} = 0.75, ρ = 1.0 kg/m^3 [1]. After substituting the aforementioned parameters into the equation (25), *v(t)* can be determined as,

$$v(t) = -2.33t^3 + 16.07t^2 + 161.03t \tag{26}$$

Higher order terms can be calculated for increased accuracy if desired but for most practical purposes that is discussed in this paper, cubic polynomials are sufficient. The exhaust velocity was determined by equation (5) and the mass flow rate was calculated by diving the final change in mass of the rocket by the specified motor burn time from the motor manufacturer. Equation (26) can be compared to an OpenRocket simulation designed for the same rocket. OpenRocket is an opensource rocket flight simulator that uses Runge-Kutta 4 (RK4) integration method as a numerical solver [12]. UTC's experience with OpenRocket has been resourceful and provided highly accurate results for their launches. The velocity profile generated by the DTM polynomial is compared to the OpenRocket simulation and the comparison is shown below.

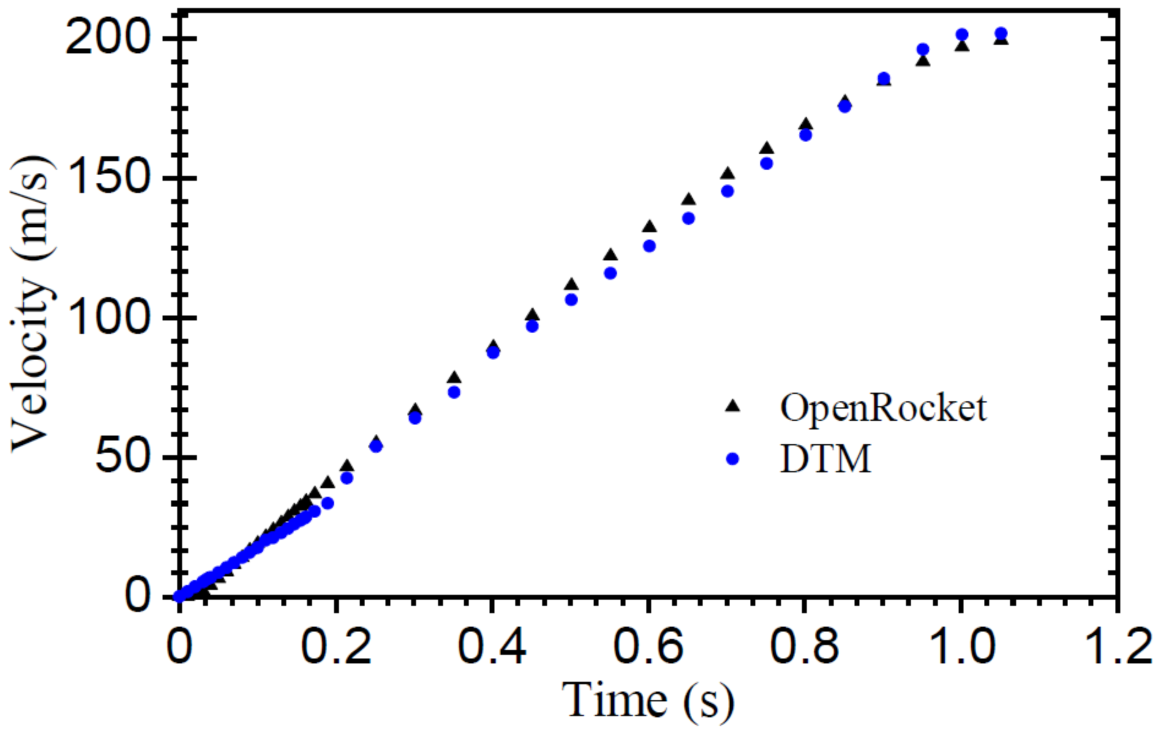

Figure 2: Comparison of Velocity of DTM vs. OpenRocket [1]

From Fig. 2, calculations yielded an average difference of 7% for DTM from OpenRocket [1]. DTM was also used to generate a velocity curve for UTC's 2019 Altitude Busters' rocket that broke the world record for highest apogee attained by a rocket with under 640 N-s of impulse. The team used a Raven-3 as an onboard altimeter to track their flight and collect data. Shown below is a 3D model of the rocket that was used in Kansas:

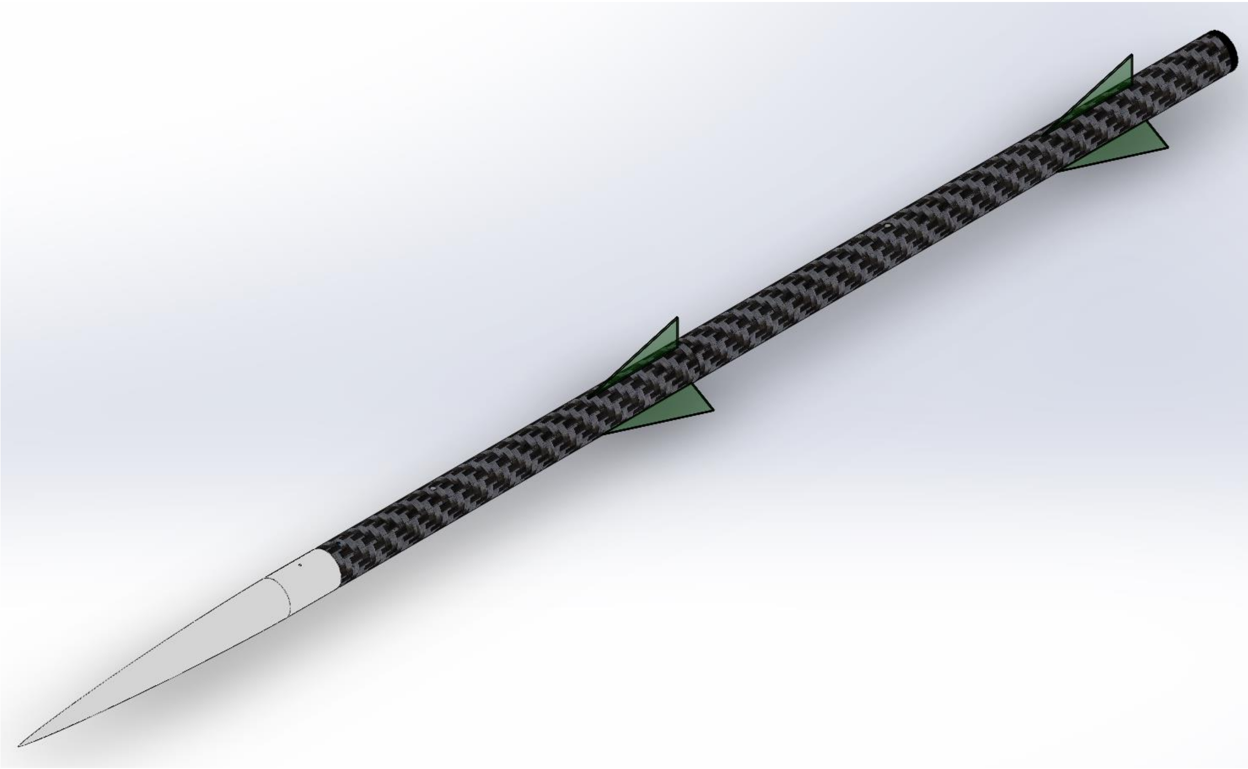

Figure 3: Computer generated 3D model of 2019 Altitude Busters' rocket

The parameters of this rocket with the motors it flew on were: v_e = 2054.81 m/s, $\dot{m}$ = 0.124 kg/s , m_0 = 1.19 kg, d = 0.032 m, C_{total} = 0.50, ρ = 1.0 kg/m^3. Using these parameters and applying DTM generates the following polynomial.

$$v(t) = -2.39t^3 + 21.70t^2 + 203.80t \tag{27}$$

This compares with the flight data collected from the Raven3 as,

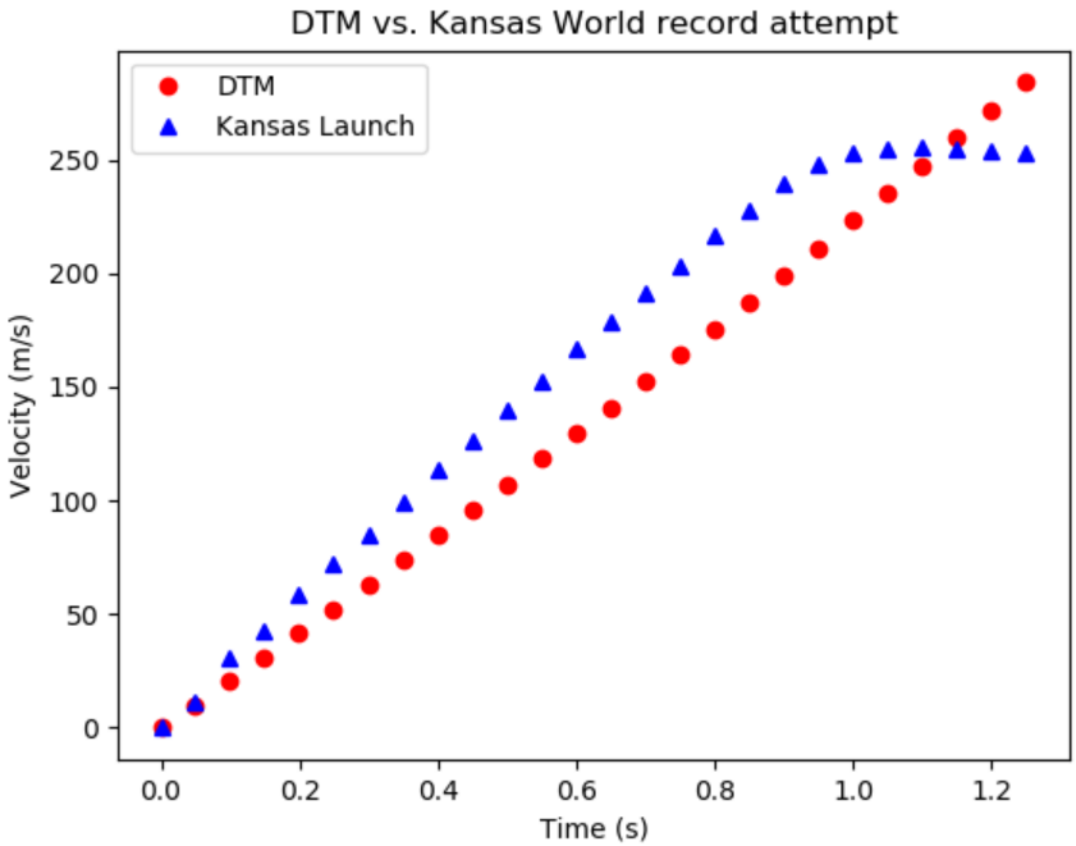

Figure 4: Comparison of Velocity of DTM vs. Flight Data

The data in Figure 4 yielded an average error of 18% but this is mainly due to assuming a constant thrust value. There is a large change in velocity in the neighborhood around the motor burn out time but this change is not captured due to the model not accounting for the decreasing thrust. A constant mass flow rate could also be a factor in the differences. Obviously, there can never be a perfect match with the actual flight given all the assumptions and this is evident with OpenRocket as well but how sensitive is the model to the various input parameters? This following section does a parametric study to evaluate this.

VI. Parametric Study

The effects of density, diameter, drag coefficient, mass flow rate, and thrust on the velocity are investigated in this section. For the following study, the base parameters are established off the Kansas launch presented above. The parameter of interest is varied while all other parameters are held constant. Estimating the overall drag on the rocket is a daunting task due to variable density (with altitude and classification of flow), a changing cross-sectional area, and estimating the drag coefficient.

Density

The density of the fluid in the boundary layer is of interest here [13] and it is difficult to perfectly model. This density obviously decreases with increasing altitude, but it also changes across a shock when transitioning to supersonic flight. However, an average density of 1.0 kg/m^3 was used here for simplification of the math. Demonstrated in the following image is the sensitivity of the overall solution to the density input:

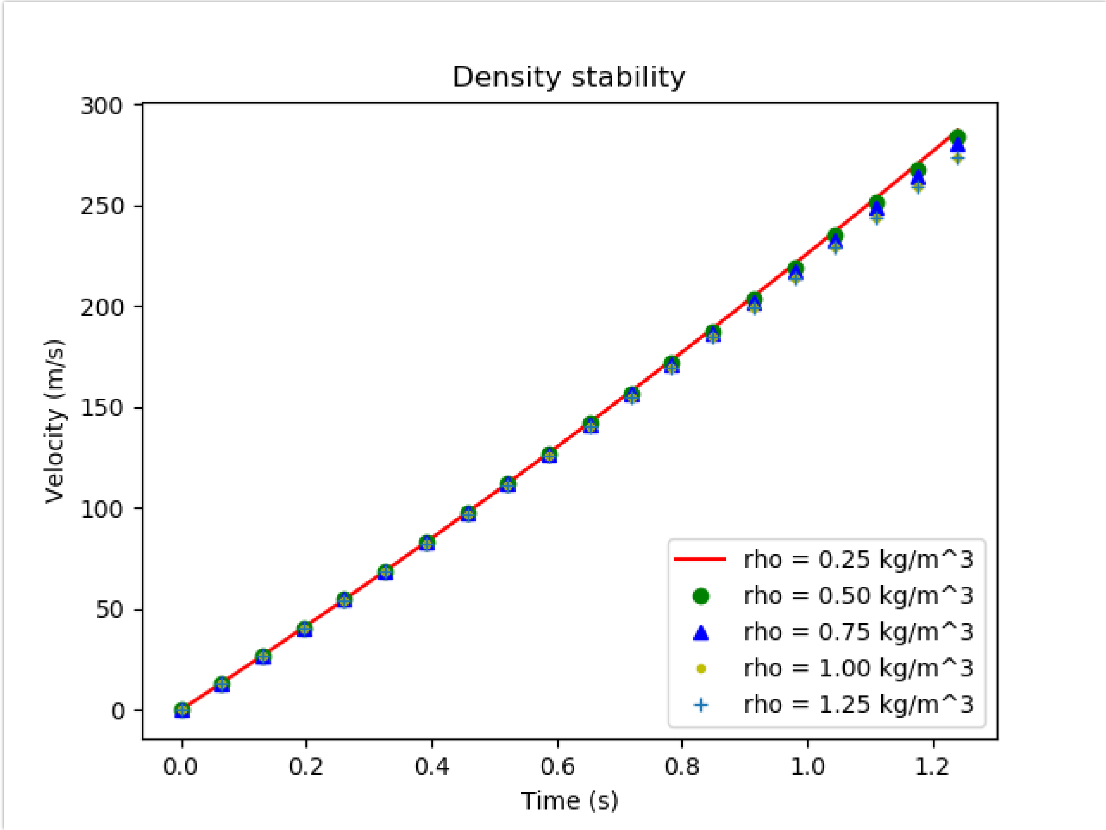

Figure 5: Density Parametric Study

Cleary, for this model the solution is not very sensitive to the density with less than 2% variation in the solution. Therefore, most differences in Figure 4 are not from the density.

Diameter

The rocket airframe has a changing diameter for the nose cone, and two different diameters for the body and airframe as seen in Figure 3. The airframe diameter is used in this model, but it is shown below that velocity as predicted by the model is not very sensitive to the diameter used with also less than 2% variation. Therefore, most differences in Figure 4 are not from the diameter.

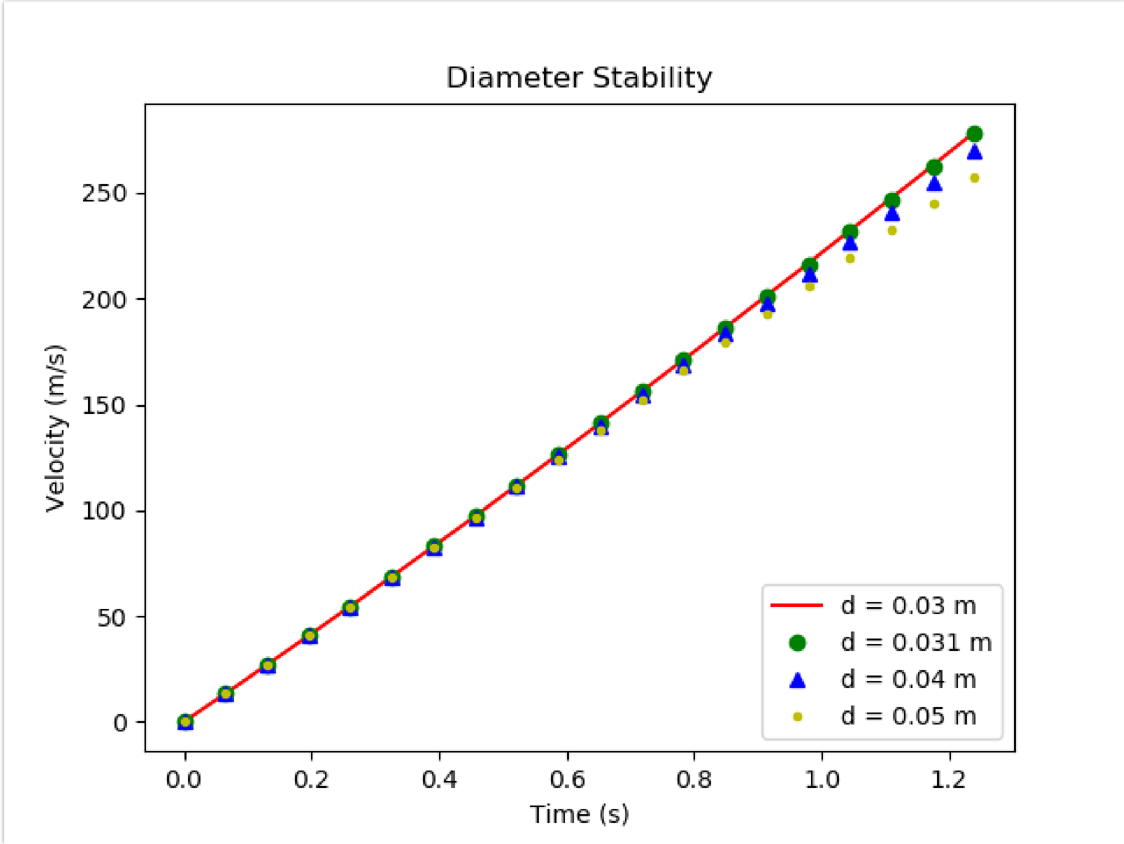

Figure 6: Diameter Parametric Study

Drag Coefficient

The drag coefficient, like density, changes with classification of the flow around the rocket and with other parameters. A constant drag coefficient is used in this model. This drag coefficient was estimated by running a CFD simulation in ANSYS, but it is shown below with less than 1% variation that the velocity as predicted by the model is not very sensitive to the drag coefficient used either. Therefore, most differences in Figure 4 are not from the drag coefficient.

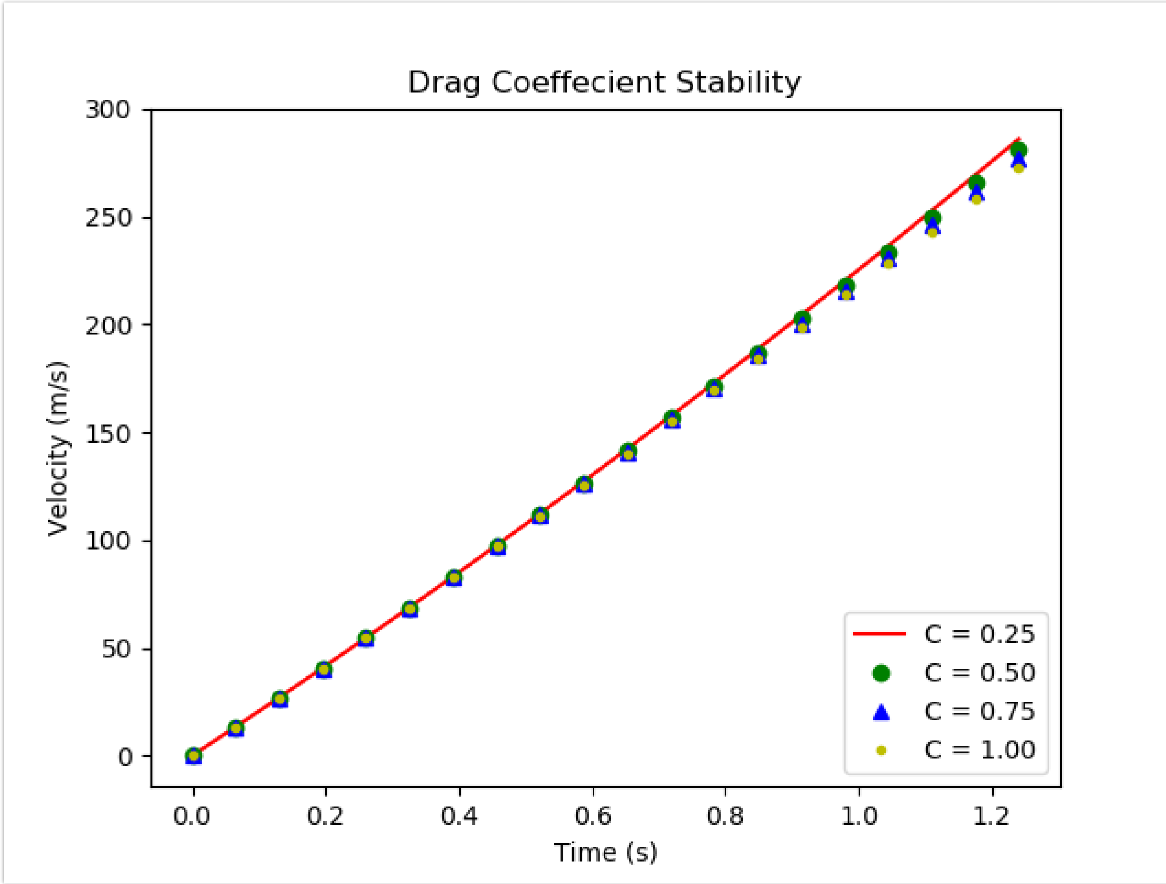

Figure 7: Drag Coefficient Parametric Study

Mass Flow Rate

The mass flow rate is assumed to be linear and therefore calculated by dividing the mass of the propellant that is specified on the rocket motor manufacturer's page by the burn time that is also specified on the page. This assumption coupled with the assumption of an optimal motor yields the propellant exhaust velocity given by equation (5). This assumption simplifies the math but assuming a variable mass flow rate can provide more accurate results. The next figure shows the effect that varying the mass flow rate has on the overall solution.

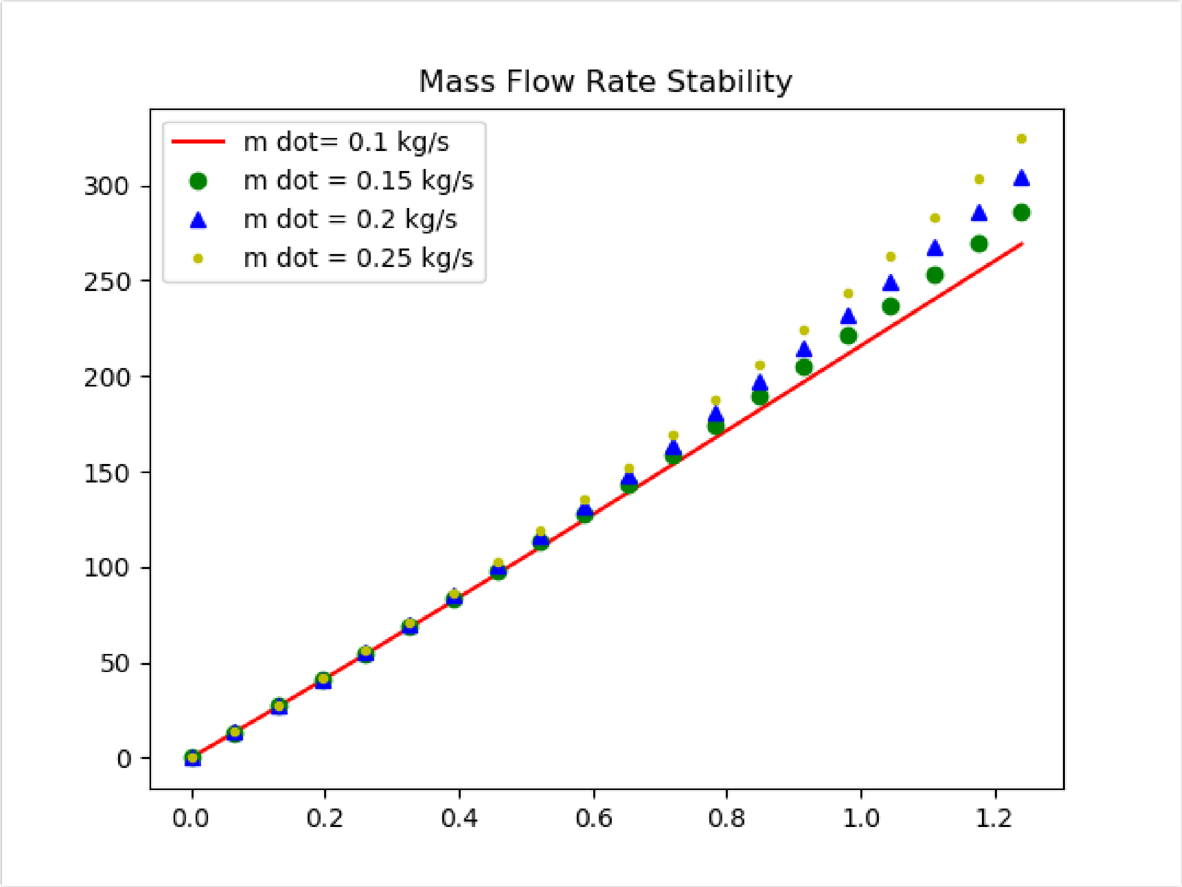

Figure 8: Mass Flow Rate Parametric Study

Clearly, unlike the other parameters mentioned above, the mass flow rate solution starts to diverge at values in the neighborhood of the burn time. Therefore, more careful measures need to be taken in estimating mass flow rate.

Thrust

A constant average thrust value is used for the model. This average value is also provided on the rocket motor manufacturer's page. This assumption is solely made to simplify the math as a variable thrust would inhibit the use of the DTM on equation (12). Therefore, like the mass flow rate, there is a tradeoff in accuracy for values in the neighborhood of the burn time as is demonstrated below.

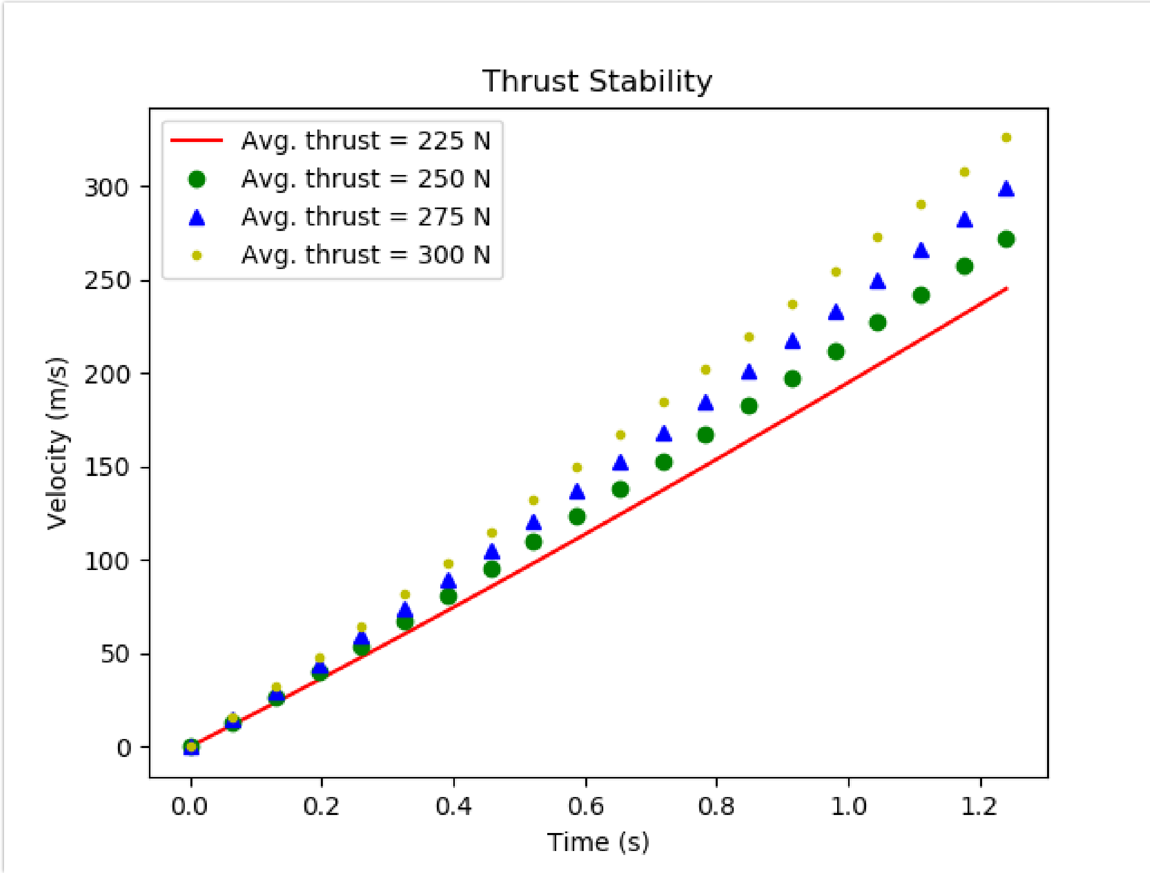

Figure 9: Thrust Parametric Study

VII. Conclusions

In this paper, the mathematical modelling of a variable mass rocket's dynamics is examined by utilizing the Differential Transform Method (DTM). Newton's second law was applied to a straight rocket in flight and DTM was used on the resulting differential equation for which the solution yielded the velocity of the rocket as a function of time. This solution was compared with numerical solutions from OpenRocket which showed a correlation with only an average difference of 7%. However, when compared with experimental data, the solution diverged for times about the burn time with an average error of 18%. As shown in the parametric study, this is mainly due the assumption of a constant mass flow rate and a thrust. These assumptions are to simplify the mathematics but the decrease in velocity near the burn time is not captured in the model and therefore results in a poor comparison. If the model is corrected to account for a variable mass flow rate and thrust while also having a dependency on its rotational dynamics, turbulence, etc. then more accurate results can be found.

In future work, it is expected that wider parametric studies will be implemented to ensure that adequate studies are conducted on the various dependencies and improvements mentioned above. Additionally, the original inspiration for this research as mentioned in the introduction was to optimize the stage separation event for a multistage rocket. The method illustrated in this paper can be used to do this for a two-stage rocket by allowing velocity to be written as a piecewise function of both time and stage separation time:

$$v(t,s) = \begin{cases} v_1(t)\,, & t_0 \le t \le t_1 \\ v_2(t)\,, & t_1 \le t \le s \\ v_3(t)\,, & s \le t \le t_3 \\ v_4(t)\,, & t_3 \le t \le t_4 \end{cases} \tag{28}$$

Where *v(t,s)* is the velocity of the rocket for the from launch to apogee and *s* is the time at which separation happens. v_n is the solution from DTM and *s* existst between booster burn out time (t_1) and the time to booster apogee. To maintain continuity, $v_n(t_n) = v_{n+1}(t_n)$, and the initial conditions for DTM will be applied appropriately. Depending on the choice of *s*, v_n will be determined uniquely for n = 2,3,4. Integrating equation (28) with respect to time from initial time to time at apogee will result in the altitude of the rocket. If we let *s* be written as t_2 then the apogee as a function of *s* can be written as:

$$H(s) = \int_{t_0}^{t_4} v(t,s)dt = \sum_{n=0}^{3} \int_{t_n}^{t_{n+1}} v_{n+1}(t)dt \qquad (29)$$

Where *H(s)* is the apogee as function of the separation time. This function can be optimized using basic calculus to find the optimal separation point, *s*.

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