ON THE MINIMUM RANK OF CERTAIN GRAPHS
WITH PATH COVER NUMBER 2

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ABSTRACT

The minimum rank problem is an interesting and ongoing problem in spectral graph theory which seeks to answer the question "Given a simple graph G what is the minimum rank of a matrix whose off-diagonal zero/nonzero pattern is described by G?" In recent years, the minimum rank of trees, unicyclic graphs, and cases of extreme minimum rank have been completely characterized. However, little is known about other families of graphs. Recent work in zero-forcing parameters, minimum semidefinite rank, and ranks of outerplanar graphs have given more ways to calculate upper and lower bounds for the minimum rank of a graph. We define a family of graphs with path cover number two and consider restrictions on the structure and minimum rank of these types of graphs. We consider a sub-family of these graphs and calculate the zero-forcing number and the positive semidefinite minimum rank. We also conjecture toward the minimum rank of these graphs.

DEDICATION

This work is dedicated to my beautiful wife.

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This work would not exist without the help of many people. I would like to thank the Graduate School and the Department of Mathematics for not only allowing me to study at UTC, but also allowing me to work at UTC as a graduate assistant. I would also like to thank the members of my committee, Drs. Ledoan, Smith, and Van der Merwe, for their involvement and advice. The greatest thanks go to my advisor and committee chair, Dr. Barioli. Without his advice and guidance, this thesis would never have been written.

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CHAPTER 1

INTRODUCTION

The Minimum Rank Problem

The minimum rank problem is an intriguing and ongoing problem in spectral graph theory. It seeks to answer the question, "For a given class of matrices, what is the minimum rank of matrices in such a class?" Of particular interest is the class of real symmetric matrices whose off-diagonal zero-nonzero pattern is described by a graph. So we ask the question: "For a given simple graph G, what is the minimum rank of a real symmetric matrix whose off diagonal zero/nonzero pattern is described by G?"

The minimum rank problem originates from the Inverse Eigenvalue Problem which asks a closely related question: "Given a set of eigenvalues, what kinds of matrices achieve this spectrum?" A closely related question come out of research into which matrices can achieve the given spectrum: "Given a class of matrices, what is the maximum multiplicity of an eigenvalue of a matrix within the class?" This is the core question of the Maximum Multiplicity Problem. Since the spectrum of a matrix can be perturbed so that 0 is the eigenvalue with maximum multiplicity, the solution to the maximum multiplicity question is equivalent to the solution to the question: "What is the maximum nullity of a real symmetric matrix with the given zero/nonzero pattern?" The connection to minimum rank is easy to see at this point: for a given $n \times n$ matrix A,

$$N(A) + rank(A) = n.$$

Equivalently:

$$rank(A) = n - N(A)$$

where $N(A)$ is the nullity of the matrix A. Clearly, as nullity is maximized, rank is minimized. So, the solution to the Minimum Rank Problem is equivalent to the solution to the Maximum Multiplicity Problem.

The study of minimum rank of symmetric matrices described by a graph was initiated by Nylen in 1996 in a paper describing the minimum rank of matrices described by trees [15]. With improvements from others, the minimum rank of matrices described by trees is fully known. Some progress has been made for graphs that are not trees, mostly in the last decade.

Preliminary Definitions and Examples

Unless otherwise noted "graph" will indicate a simple graph, and all matrices are real. A *graph* is a pair of sets $G = (V, E)$ where V is the set of vertices (herein numbered $1, 2, ..., n$) and E is the set of edges where each edge is a two element subset of V.

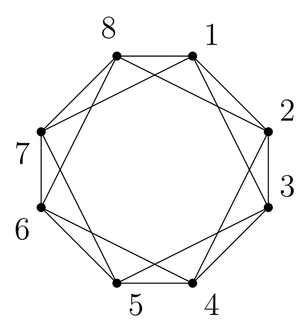

Figure 1.1: A graph on 8 vertices

The *order* of a graph, $|G|$, is the number of vertices in G. A graph $H = (V', E')$ is a *subgraph* of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. A graph $H = (V', E')$ is an *induced* *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $\{i, j\}$ is an edge in E' if and only if $i, j \in V'$ and $\{i, j\} \in E$.

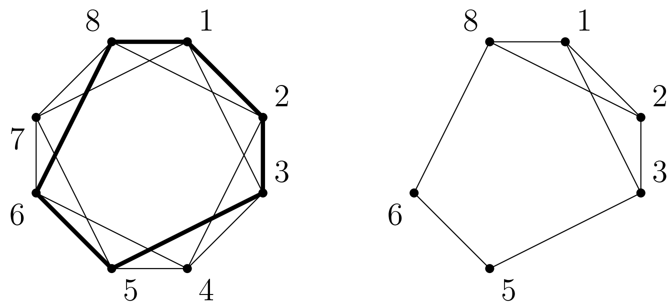

Figure 1.2: The cycle C_6 is a subgraph, but not an induced subgraph

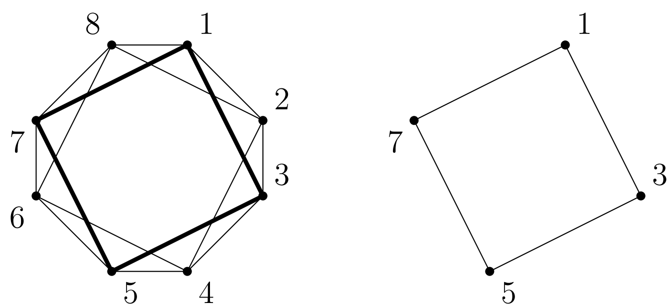

Figure 1.3: The graph C_4 is an induced subgraph

A graph is *connected* if there is a path subgraph between any two vertices. The *connected components*, or *components*, of a graph G are the subgraphs G_i such that each G_i subgraph is connected but none of the G_i's are connected to each other in G.

A graph P_n is a *path* if $P_n = (\{1, 2, \ldots, n\}, E)$ where $E = \{\{i, i+1\} : i = 1, 2, \ldots n-1\}$. The length of a path is one less than the number of vertices in the path. A graph C_n is a *cycle* if $C_n = (\{1, 2, \ldots n\}, E)$ where $E = \{\{i, i+1\} : i = 1, 2, ...n-1\} \cup \{1, n\}$.

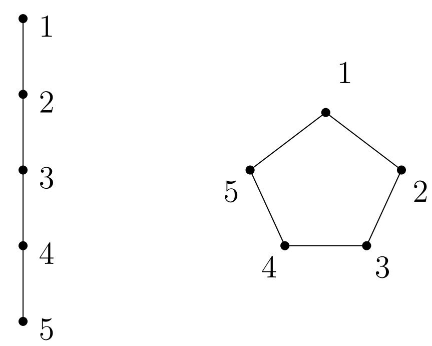

Figure 1.4: The path P_4 and the cycle C_5

A *leaf* or *pendant* is a vertex with only one neighbor. A *pendant path* is a path P_n that is an induced subgraph of a graph G such that one endpoint in P_n is a pendant, and the deletion of this pendant leaves the new endpoint of P_{n-1} as a pendant.

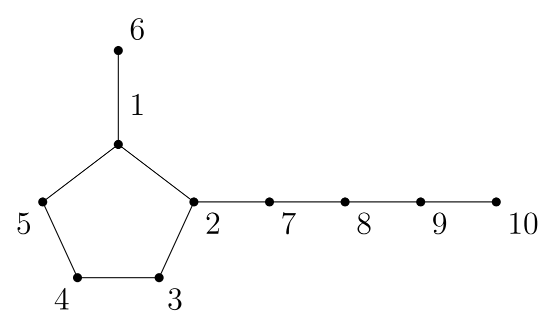

Figure 1.5: Vertex 6 is a leaf, and vertices $7, 8, 9$, and 10 form a pendant path

A graph T is a *tree* if it is a connected graph that contains no cycles as subgraphs.

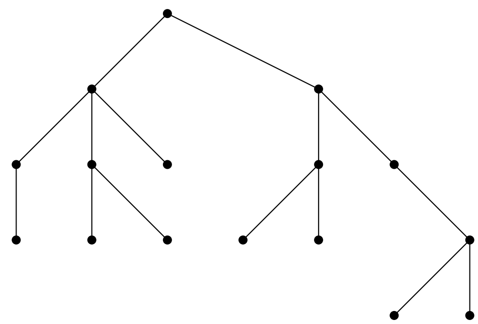

Figure 1.6: A tree on 16 vertices

A graph K_n is a *complete graph*, or a *clique*, if $K_n = (\{1, 2, \ldots n\}, E)$ where $E = \{\{i, j\} : i \neq j; i, j = 1, 2, \ldots, n\}$.

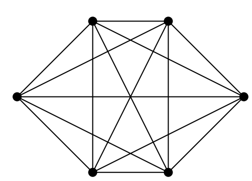

Figure 1.7: The complete graph K_6

A graph $G = (V, E)$ is *bipartite* if V can be partitioned into two subsets U and W so that no edge in E has both endpoints in U or both endpoints in W.

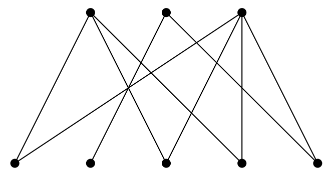

Figure 1.8: A bipartite graph

A graph $K_{p,q}$ is a *complete bipartite* graph if $K_{p,q}$ is bipartite, $|U| = p$, $|W| = q$, and $E = \{\{u, w\} : u \in U, w \in W\}$.

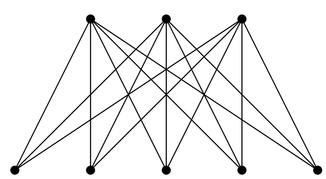

Figure 1.9: The complete bipartite graph $K_{3,5}$

A graph is *unicyclic* if it contains only one cycle as a subgraph.

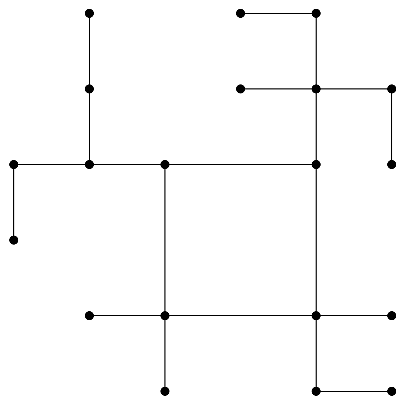

Figure 1.10: A unicyclic graph

A unicyclic graph is an *n-sun* if it is constructed by appending one leaf to each vertex in C_n. A *partial n-sun* is a unicyclic graph constructed by appending a leaf to at least one of the vertices in C_n.

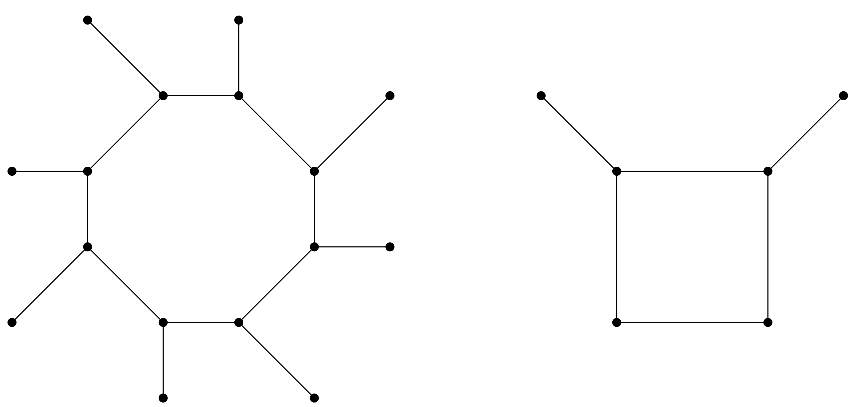

Figure 1.11: An 8-sun and a partial 4-sun

A *contraction* of an edge $\{u, v\}$ in a graph G is performed by deleting the edge $\{u, v\}$ and the vertices u and v, adding a new vertex w and new edges $\{i, w\}$ for all vertices i such that i was a neighbor of u or a neighbor of v.

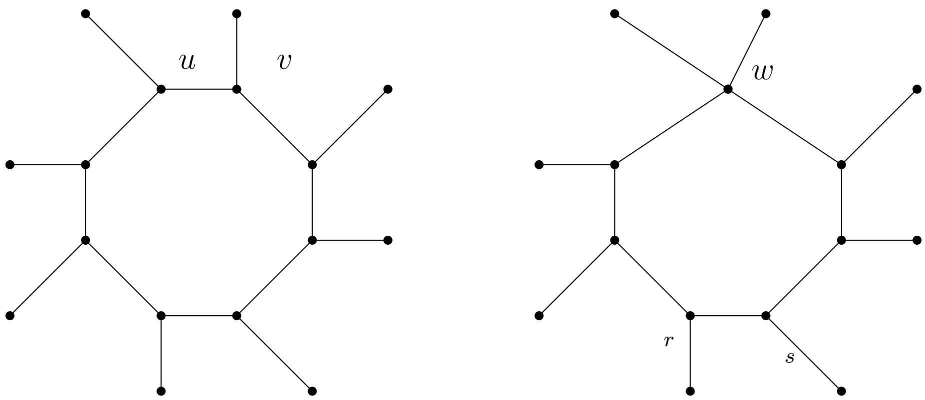

Figure 1.12: Contraction of the edge $\{u, v\}$ into the new vertex w

A graph H is a *minor* of a graph G if H can be obtained from G by a series of vertex and edge deletions and edge contractions.

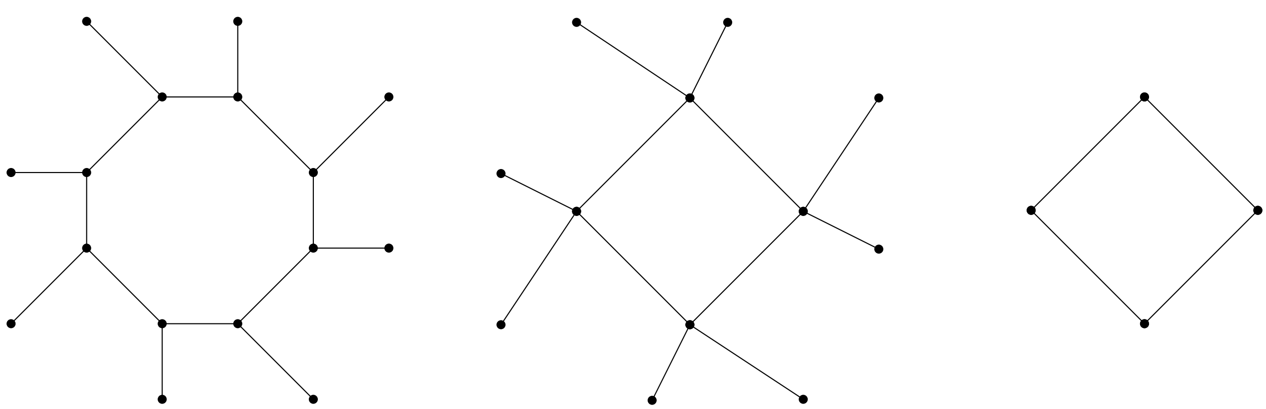

Figure 1.13: The graph from Figure 1.11 has C_4 as a minor

A *subdivision* of an edge $e = \{u, v\}$ in a graph G is obtained by deleting the edge $\{u, v\}$ in G, adding vertex w, and adding edges $\{v, w\}$ and $\{u, w\}$. The graph obtained from G by subdividing the edge e is denoted G_e.

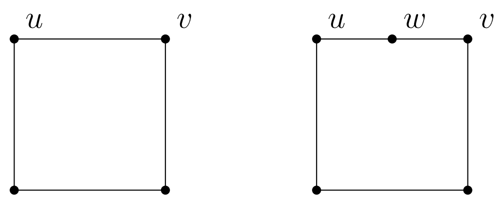

Figure 1.14: The subdivision of the edge $\{u, v\}$

A graph G is *planar*, or a *planar graph*, if G can be drawn in the plane such that no edges cross. A drawing of a planar graph G such that the edges do not cross is called a *planar embedding* of G.

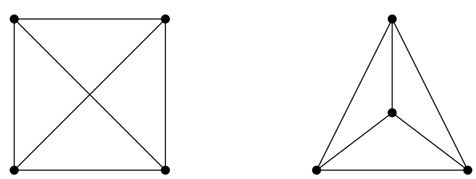

Figure 1.15: The graph K_4 is planar

A planar embedding of a graph divides the plane into a finite number of bounded *faces* and one unbounded face.

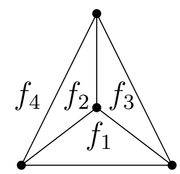

Figure 1.16: A planar embedding of K_4 with the faces marked

A planar graph G is *outerplanar* if, in a planar embedding of G, each vertex is adjacent to the unbounded face of the graph.

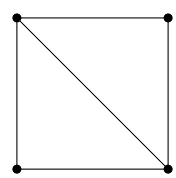

Figure 1.17: An outerplanar graph

The *dual* of a planar graph G is the graph obtained by assigning a vertex to each face of G such that two vertices in the dual are adjacent if the corresponding faces are separated by an edge.

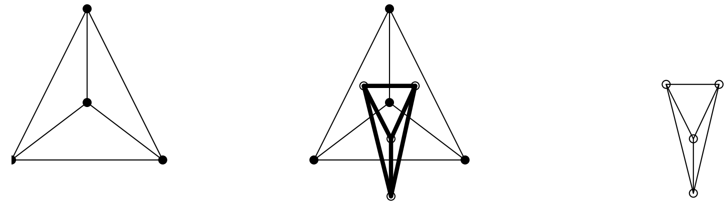

Figure 1.18: A planar graph and its dual

A *polygonal path* is a graph whose dual (minus the vertex corresponding to the unbounded face) is a path. A k*-tree* is a graph formed inductively by starting with K_{k+1} and connecting each new vertex to the vertices of an existing clique on k vertices. A *linear* k*-tree* is a k tree that is also a polygonal path.

The *path cover number*, $P(G)$, of a graph G is the minimum number of vertex disjoint paths (that is, paths that do not share any vertices), which occur as induced subgraphs of G such that every vertex v of G belongs to one of the paths.

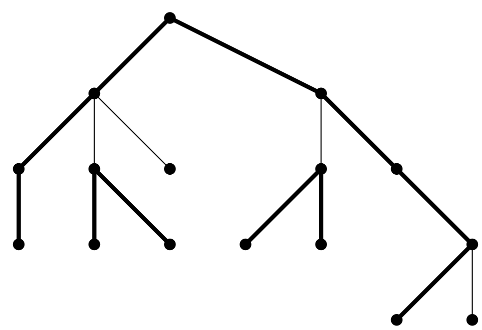

Figure 1.19: A tree T with $P(T) = 5$

The *clique cover number*, $cc(G)$, of a graph G is the minimum number of vertex disjoint cliques (or complete graphs) occurring as induced subgraphs of G such that every vertex v of G is in one of the cliques. The *join* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \vee G_2$ formed by adding the edge $\{v_i, v_j\}$ to the graph for each $v_i \in V_1$ and each $v_j \in V_2$ for $i = 1, 2, \ldots, |G_1|$ and $j = 1, 2, \ldots, |G_2|$. The *independence number* of a graph G, denoted $\alpha(G)$, is the order of the order of the largest induced subgraph of G consisting of isolated vertices.

The set S_n denotes the set of all real symmetric $n \times n$ matrices. For a matrix $B \in S_n$, the graph of B, $\mathcal{G}(B)$, is the graph with vertices $\{1, 2, ...n\}$ and with edges $\{\{i, j\} : b_{ij} \neq 0, i \neq j\}$. The diagonal of B is not used in determining $\mathcal{G}(B)$. Define the set $\mathcal{S}(G) = \{B \in S_n : \mathcal{G}(B) = G\}$.

Example 1.1. *For the matrix* $B = \begin{bmatrix} 0 & 1 & 0 & 0.5 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1.5 \\ 0.5 & 1 & 1.5 & 0 \end{bmatrix}$*,* $\mathcal{G}(B)$ *is shown in Figure 1.20*

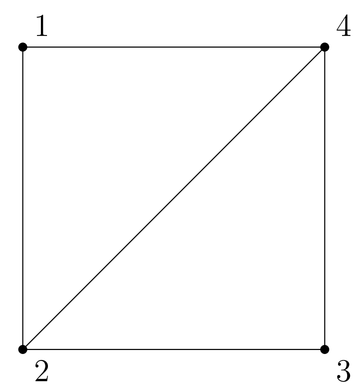

Figure 1.20: $\mathcal{G}(B)$

The *minimum rank* of G is defined as

$$mr(G) = \min\{rank(B) : \mathcal{G}(B) = G\}.$$

The *minimum positive semidefinite rank* of G is

$$\min\left\{rank(B) : B \in \mathcal{G}(B),\ x^T Bx \geq 0,\ \forall x \in \mathbb{R}^n\right\}.$$

The *maximum nullity* of G is defined as

$$M(G) = \max\{N(B) : B \in \mathcal{G}(B)\}$$

where $N(B)$ is the nullity of the matrix B.

CHAPTER 2

SURVEY OF LITURATURE

Over the past 20 years, research and interest in the minimum rank problem has grown steadily. Beginning with trees, now the minimum ranks of several families of graphs have been fully characterized.

Trees

The minimum rank of trees was first studied by Nylen in 1996 [15] and was fully characterized by Johnson and Leal Duarte in 1999 [14]. The minimum semidefinite rank of trees was fully characterized by Van der Holst in his 2003 paper [12].

Theorem 2.1. *[14] For any tree T, $mr(T) = |T| - P(T) = |T| - M(T)$.*

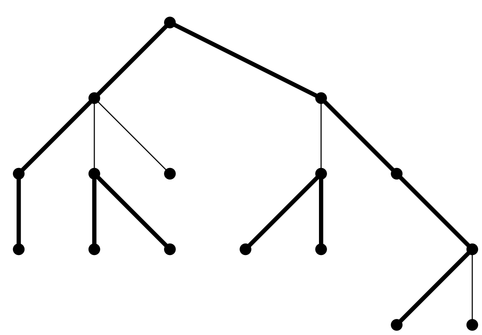

Figure 2.1: A tree with $mr(T) = 16 - 5 = 11$

Theorem 2.2. *[12]* $mr_+(G) = |G| - 1$ *if and only if* G *is a tree.*

Unicyclic Graphs

A graph G is a *unicyclic graph* if it is connected and contains exactly one cycle as a subgraph. A vertex v of G is an *appropriate vertex* if there are at least two pendant paths from G at v. The idea of the appropriateness of a vertex was developed to assist the study of these graphs. Several "trimming" operations were developed to remove certain vertices and edges whose effect on the minimum rank were well known. The *trimmed form* of a unicyclic graph is obtained by performing the following trimming procedures in any order until they can no longer be performed:

1. Deletion of an appropriate vertex.
2. Deletion of an isolated path.
3. Deletion of a leaf at the end of a pendant path.

Theorem 2.3. *[2] Let* G *be a unicyclic graph. Then,*

$$mr(G) = \begin{cases} |G| - P(G) + 1, & \textit{if the trimmed form of } G \textit{ is an } n\textit{-sun with } n > 3\textit{, and } n \textit{ is odd} \\ |G| - P(G), & \textit{otherwise} \end{cases}$$

Graphs with Extreme Minimum Rank

Clearly, $mr(G) \geq 1$ and $mr(G) \leq |G| - 1$. Recently, the types of graphs which obtain these extreme minimum ranks, as well as graphs which obtain minimum ranks of 2 or $|G| - 2$ have been completely characterized.

Theorem 2.4. *[10]For* $n \geq 2, mr(K_n) = 1$ *and if* G *is connected,* $mr(G) = 1$ *implies* $G = K_{|G|}$.

Theorem 2.5. *[10],[9]* $\forall G$, $mr(G) = |G| - 1$ *if and only if* $G = P_{|G|}$.

Theorem 2.6. *[5] A connected graph G has* $mr(G) \leq 2$ *if and only if G does not contain as an induced subgraph any of* P_4, $K_{3,3,3}$, *Dart or* $(P_2 \cup K_1 \cup K_1) \vee K_1$, *as shown in Figure 2.2.*

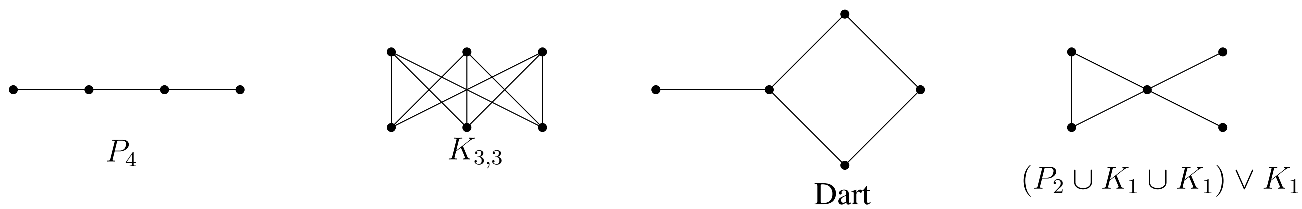

Figure 2.2: P_4, $K_{3,3,3}$, Dart or $(P_2 \cup K_1 \cup K_1) \vee K_1$

Theorem 2.7. *[10] A graph G has* $mr(G) \leq 2$ *if and only if the complement of G has the form* $(K_{s_1} \cup K_{s_2} \cup K_{p_1 q_1} \cup \ldots \cup K_{p_k q_k}) \vee K_r$ *for appropriate integers* $k, s_1, s_2, p_1, q_1, \ldots, p_k, q_k, r$ *with* $p_i + q_i > 0$ *for all* $i = 1, \ldots, k$.

Theorem 2.8. *[10] Let G be a 2-connected graph, then* $mr(G) = |G| - 2$ *if and only if G is a polygonal path.*

Cut-Vertices

A vertex is a *cut vertex* if $G - v$ has at least one more connected component than does G.

Example 2.9. *The vertex* v *is a cut vertex of the graph* G. *The components of* $G - v$ *are shown.*

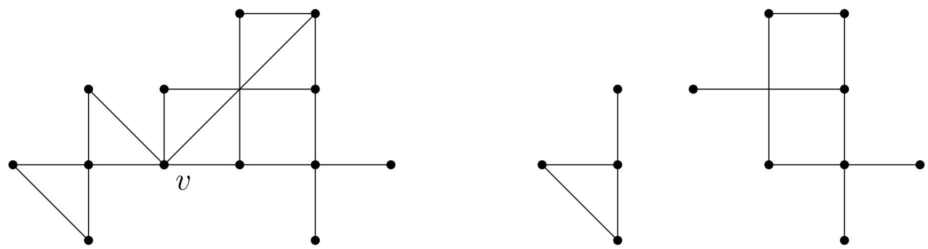

Figure 2.3: A graph with a cut-vertex removed

The effect of cut vertices on minimum rank has been well studied and their effect on the minimum rank of a graph is fully known. Before considering the minimum rank across cut vertices, we must first define the rank-spread of a graph. For a given graph G and vertex v of G. The *rank-spread* of G at v is $mr(G) - mr(G - v)$.

Theorem 2.10. *[15] For any vertex v of G, $0 \leq mr(G) - mr(G - v) \leq 2$. That is, $r_v(G) \leq 2$.*

Theorem 2.11. *[1] Let v be a cut vertex of a graph G. For $i = 1, \dots h$, let $W_i \subseteq V(G)$ be the vertics of the i^{th} component of $G - v$ and let G_i be the subgraph induced by $v \cup W_i$, then*

$$r_v(G) = \min\left\{\sum_{i=1}^{h} r_v(G_i), 2\right\}$$

and so,

$$mr(G) = \min\left\{\sum_{i=1}^{h} mr(G_i),\ \sum_{i=1}^{h} mr(G_i - v) + 2\right\}.$$

Theorem 2.12. *[10] If $r_v(G_i) = 0$ for all but at most one of the G_i, then $mr(G) = \sum_{i=1}^{h} mr(G_i)$.*

Theorem 2.13. *[13], [6] Let v be a cut vertex of G. For $i = 1, \dots, h$, let $W_i \subseteq V(G)$ be the vertices of the i^{th} component of $G - v$ and let G_i be the subgraph induced by $v \cup W_i$. Then,*

$$mr_+(G) = \sum_{i=1}^{h} mr_+(G_i).$$

Zero-Forcing Parameters

Let $G = (V, E)$ be a graph. A *coloring* of G is a subset $Z \subseteq V$ defining an initial set of vertices in G to be colored black. All vertices not in Z are colored white. For a black vertex u and a white vertex v, the *color change rule* is to change the color of v to black if v is the unique white neighbor of u. In this case, we say that u *forces* v and write $u \to v$. For a given coloring of G, the *derived set* or *final coloring* is the set of black vertices obtained by applying the color change rule to the coloring until no more changes are possible. A subset of vertices Z is called a *zero forcing set* for G if the vertices in Z are used as an initial coloring, then the derived set is V. The *zero forcing number* $Z(G)$ the minimum order of all zero forcing sets $Z \subseteq V$.

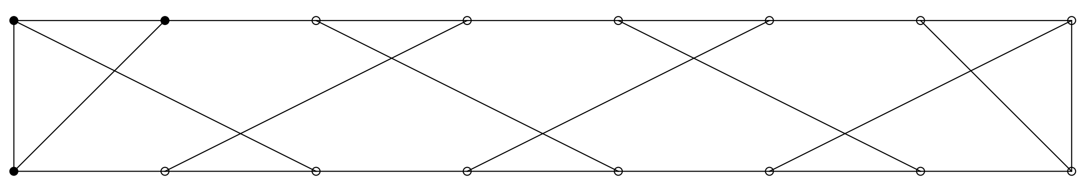

Figure 2.4: An initial coloring of a graph

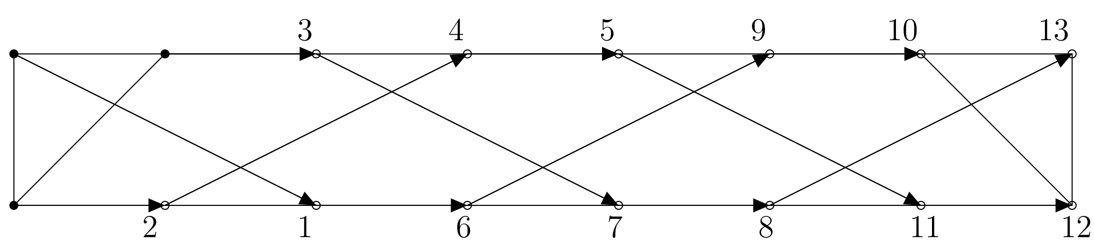

Figure 2.5: The forcing chain on the graph in Figure 2.4

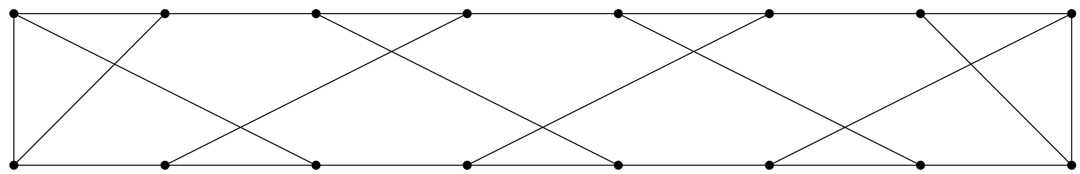

Figure 2.6: The final coloring of the graph in Figure 2.4

Let B be the set consisting of all the black vertices in G. Let $W_1, \ldots, W_k$ be the sets of vertices of the $k \geq 1$ components of $G - B$. Let $w \in W_i$. The *positive semidefinite color change rule* is: If $u \in B$ and w is the only white neighbor of u in $G[W_i \cup B]$, then the color of w is changed to black. In this case, we say u *forces* w and write $u \to w$. A subset B of vertices in G such that the vertices in B are initially colored black and all other vertices in G are colored white is a *positive semidefinite zero forcing set* if the set of black vertices obtained after applying the positive semidefinite color change rule until no more changes are possible is V. The minimum of $|B|$ over all positive semidefinite zero forcing sets $B \subseteq V$ is the *positive semidefinite zero forcing number* $Z_+(G)$.

Theorem 2.14. *[10] For any graph G, $1 \leq Z(G) \leq |G|$ and if G has at least one edge, then $1 \leq Z(G) \leq |G| - 1$.*

Theorem 2.15. *[18] For any graph G, $mr(G) \geq |G| - Z(G)$.*

Theorem 2.16. *[3] If G is a connected graph of order greater than 1, then G does not have a unique minimum zero forcing set and no single vertex is a member of every minimum zero forcing set for G.*

Theorem 2.17. *[7] For any vertex v and edge e of G, $-1 \leq Z(G) - Z(G - v) \leq 1$, and $-1 \leq Z(G) - Z(G - e) \leq 1$.*

Theorem 2.18. *[10] $Z(G) = 1$ if and only if G is a path.*

Theorem 2.19. *[16] $Z(G) = 2$ if and only if G is a graph on two parallel paths.*

Theorem 2.20. *[8] If G is a partial 2-tree, then $mr_+(G) = |G| - Z_+(G)$.*

Theorem 2.21. *[10] Any zero forcing set is a positive semidefinite zero forcing set. Therefore, $Z_+(G) \leq Z(G)$.*

Theorem 2.22. *[10] For any G, $1 \leq Z_+(G) \leq |G|$ and if G has at least one edge, $1 \leq Z_+(G) \leq |G| - 1$.*

CHAPTER 3

GRAPHS WITH PATH COVER NUMBER 2

We are interested in studying the minimum rank of graphs with path cover number 2. However, this description encompasses a large family of graphs. Therefore, we restrict our study to graphs with a certain structure that still have the desired path cover number.

Graph Structure

Unless otherwise indicated, all graphs G are simple graphs constructed in the following way:

- Construct the paths P_m and P_n where the vertices of P_m are numbered from 1 to m and the vertices of P_n are numbered from $m+1$ to $m+n$.
- Add the edges $\{1, m+n\}$ and $\{m, m+1\}$ so that the graph is now C_{m+n}.
- Add more edges as needed such that each additional edge has a vertex in P_m and a vertex in P_n.

The graphs described above are similar in construction to the "graphs on two parallel paths" described in [16]. The difference being that crossings were not allowed in [16]. Because of the similarity in graph construction, we will be referring to graphs constructed in the above way as *graphs between two parallel paths*.

Example 3.1. *The graph G shown in Figure 3.1 is built on P_8 and P_8.*

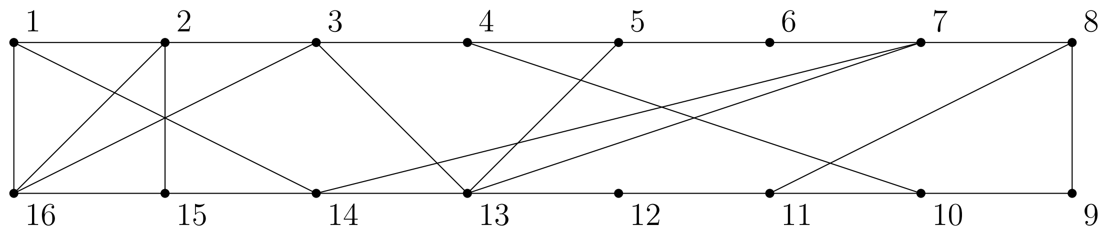

Figure 3.1: A graph between two parallel paths of equal length

Example 3.2. *The graph H shown in Figure 3.2 is built on P_6 and P_3.*

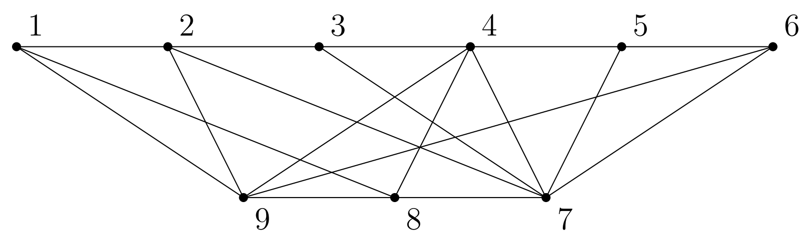

Figure 3.2: A path between two parallel paths of different lengths

To make discussion easier, we introduce the following definitions and assumptions: Unless otherwise noted, graphs are assumed to have the same vertex labeling as in Examples 3.1 and 3.2. This labeling will be referred to as the *standard numbering* of a graph.

Definition 3.3. *For a given graph G, an edge $e = \{i, j\}$, except for the edges $e_1 = \{1, m+n\}$ and $e_2 = \{m, m+1\}$ is a* connecting edge *if one vertex is in P_m and the other vertex is in P_n.*

In Figure 3.2, the edge $\{2, 7\}$ is a connecting edge. The edge $\{3, 4\}$ is not a connecting edge.

Definition 3.4. *For a given graph G, connecting edges $e_1 = \{i, j\}$ and $e_2 = \{p, q\}$* cross *(resulting in a* crossing*) in G if $p < i$ and $q < j$ or if $p > i$ and $q > j$ in the standard numbering of G.*

In Figure 3.2, the edge $\{1, 8\}$ crosses $\{2, 9\}$. The edge $\{6, 9\}$ has crossings with $\{1, 8\}$, $\{2, 7\}$, $\{3, 7\}$, $\{4, 7\}$, $\{4, 8\}$, and $\{5, 7\}$.

Immediately, we can make the following observations about the minimum rank of graphs constructed in this way:

Proposition 3.5. $mr(G) \leq |G| - 2$.

Proof If G contains no crossings, then G is a polygonal path and $mr(G) = |G| - 2$ [11]. It is well known that adding an edge to a graph does not increase the minimum rank and decreases the minimum rank by at most 1. Therefore, $mr(G) \leq |G| - 2$. $\square$

We can further restrict this bound, but we must first classify how crossing affect the characteristics of the graph:

Proposition 3.6. *G is outerplanar if and only if G contains no crossings.*

Proof For the sufficiency, suppose G has at least one crossing. Then G has the following structure:

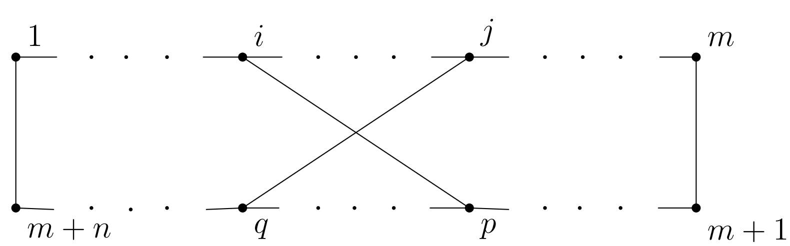

Figure 3.3: A graph with at least one crossing

In any order, contract all edges between 1 and i, between i and $j - 1$, between j and m, between $m + 1$ and p, and between $p + 1$ and $m + n$ as shown in Figure 3.4.

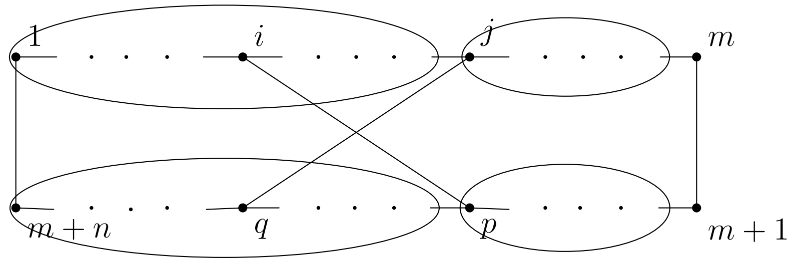

Figure 3.4: Edges to be contracted in G

The graph K_4 is achieved by deleting any double edges that were introduced by the edge contractions. Therefore, K_4 is a minor of G. Since K_4 is a minor of G, G is not outerplanar.

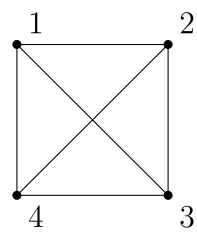

Figure 3.5: K_4 is the result of the edge contractions and deletions

For the necessity, if G has no crossings, then G is a polygonal path and is planar. Clearly, each vertex is open to the outer face. Therefore, G is outerplanar. $\square$

Corollary 3.7. *$mr(G) = |G| - 2$ if and only if G contains no crossings. Otherwise, $mr(G) \leq |G| - 3$.*

Proof For the sufficiency, if G has a crossing, then, by proposition 3.6, G is not outerplanar. Therefore, $mr(G) \leq |G| - 3$ [17]. For the necessity, if G has no crossings, then G is a polygonal path. Therefore, by Theorem 2.9, $mr(G) = |G| - 2$. $\square$

To facilitate our study of the structure of G and of the minimum rank of G we introduce the following:

Definition 3.8. *For G, a graph between two parallel paths, the graph G' is constructed in the following way:*

- *If e_k is a connecting edge in G then v'_k is a vertex in G'.*
- *If e_k and e_h cross in G, then (v'_k, v'_h) is an edge in G'.*

Example 3.9. *The derived graph G' for the graph G in Example* 3.1.

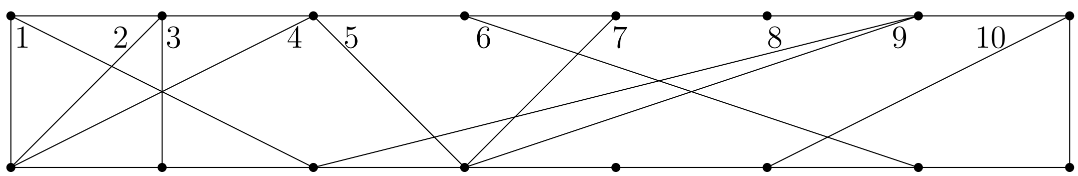

Figure 3.6: The graph G from Example 3.1 with the connecting edges numbered

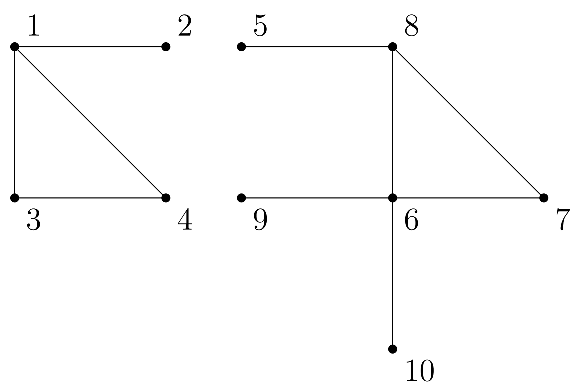

Figure 3.7: The derived graph G' of G

Definition 3.10. *The minimum number of connecting edges in G that need to be removed to obtain a polygonal path is called the* polygonal path distance *and is represented by $pd(G)$.*

We can now calculate an exact value for $pd(G)$ and use it to find a lower bound on $mr(G)$.

Proposition 3.11. *$pd(G) = |G'| - \alpha(G')$, where $\alpha(G)$ denotes the independence number of G.*

Proof Deleting a minimum number of connecting edges such that no remaining connecting edges cross, corresponds to deleting a minimum number of vertices in G' such that the remaining vertices are isolated. Minimizing the number of deleted vertices maximizes the number of isolated vertices in G'. The maximum cardinality of a set of isolated vertices in G' is given by $\alpha(G)$. Therefore, the appropriate number of vertices to delete from G', and corresponding connecting edges to delete from G is given by $|G| - \alpha(G')$. $\square$

Example 3.12. *The graph G in Figure 3.8 has the derived graph G' in Figure 3.9 and $pd(G) = 5$.*

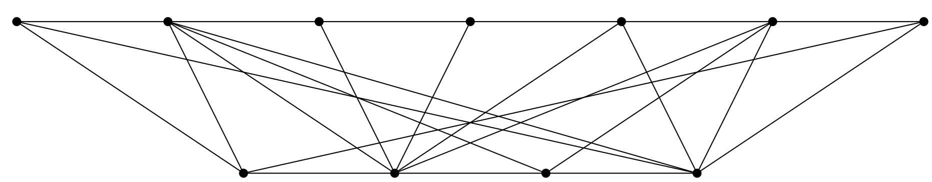

Figure 3.8: A graph G between two parallel paths of length 6 and 3

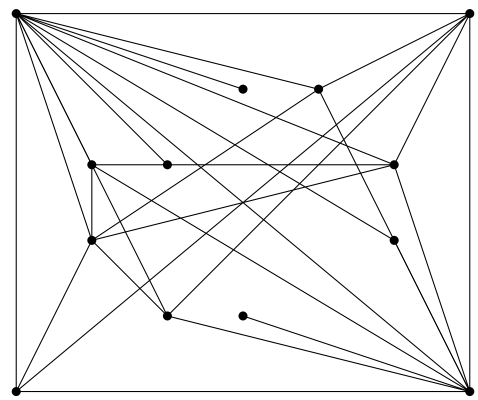

Figure 3.9: The derived graph G' of the graph G in Figure 3.9

We can establish the following lower bound on $mr(G)$ using $pd(G)$:

Proposition 3.13. $mr(G) \geq |G| - 2 - pd(G)$.

Proof It is well known that the addition of an edge to a graph decreases the minimum rank by at most one. Starting from a polygonal path, adding edges that cross existing edges will lower the minimum rank by at most $pd(G)$. $\square$

Limitations on the Structure of G and G'

The construction of G' provides a convenient tool for organizing our study of the minimum rank of G. By looking at various families of graphs for G' (e.g., trees, cycles, cliques, bipartite graphs, etc.) we can build toward a complete description of the minimum rank of graphs of this nature. This, however, introduces the question, "What limitations, if any, are imposed on the structure of G' because of the construction of G?" We make the following observations:

Proposition 3.14. *If G' is a tree, then G is a caterpillar.*

Proof Suppose G' is a tree that is not a caterpillar. Then the following structure must be a subgraph of G':

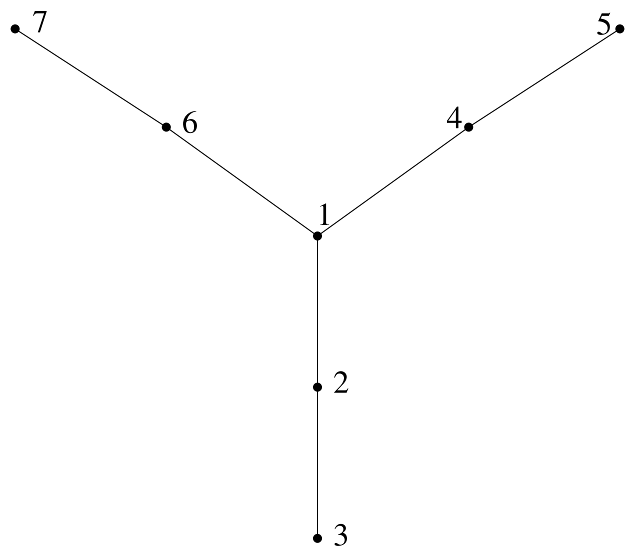

Figure 3.10: Basic structure of a tree that is not a caterpillar

In G, let v'_1 from G' correspond to edge $e_1 = \{i, j\}$. Without loss of generality, let vertex v'_2 correspond to edge $e_2 = \{i + a, j + r\}$, vertex v'_4 correspond to edge $e_4 = \{i + b, j + q\}$ and vertex v'_6 correspond to edge $e_6 = \{i + c, j + p\}$ with $0 < a \leq b \leq c$ and $0 < p \leq q \leq r$ as shown in Figure 3.11. Clearly, e_1 crosses $e_2, e_4,$ and e_6. For vertex v'_3 to have a corresponding edge e_3 in G, e_3 must be between two vertices v^+ and v^{++} where v^+ comes before i and v^{++} is between $j + q$ and $j + r$ in the standard numbering of G so that it crosses e_2 but no other edges as shown in Figure 3.11. Similarly, e_7, the edge in G corresponding to v'_7 must be between vertices v^* and v^{**} where v^* is between $i + b$ and $i + c$ and v^{**} is before j in the standard numbering of G as shown in Figure 3.4. Clearly, there is no way to construct a connecting edge e_5 corresponding to vertex v'_5 so that it only crosses e_4. Any connecting edge crossing e_4 must also cross e_1, e_2, e_6 or some combination of those. Therefore, T is not a valid structure for G'. $\square$

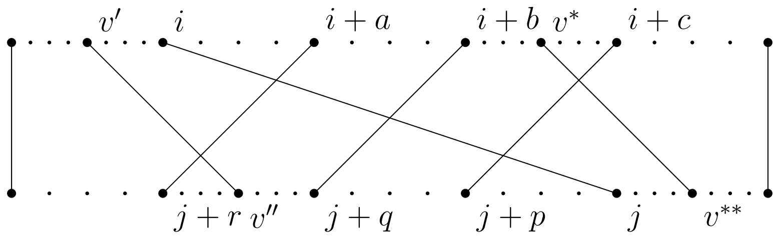

Figure 3.11: A graph G where G' is a tree

Proposition 3.15. *If C_n is an induced subgraph of G', then $n = 3$ or $n = 4$.*

Proof Figures 3.12 and 3.13 show the structure that graphs must have so that C_3 or C_4 are induced subgraphs of G'. Suppose that C_n is an induced subgraph of G' with $n \geq 5$. The structure shown in Figure 3.14 is the structure G must have so that P_n is an induced subgraph of G'. Clearly, there is no way for an edge to cross e_{n-1} and e_1 without crossing other edges. $\square$

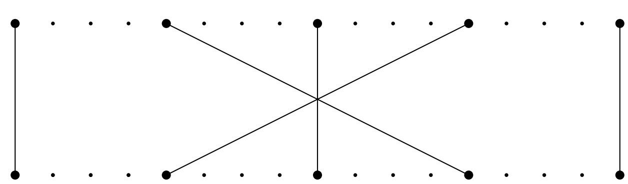

Figure 3.12: A graph which has C_3 as an induced subgraph of G'

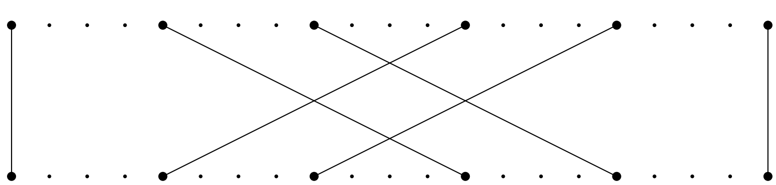

Figure 3.13: A graph with C_4 as an induced subgraph of G'

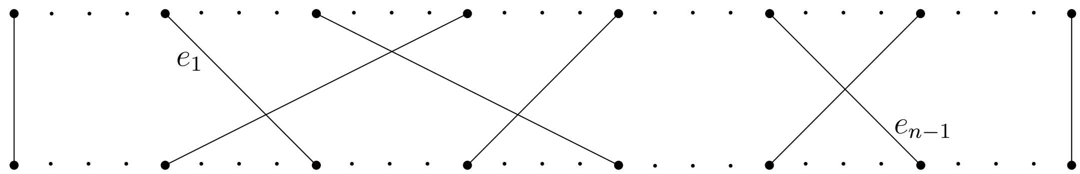

Figure 3.14: A graph with P_{n-1} as an induced subgraph of G'

Proposition 3.16. *If K_r, $r \geq 3$ is a subgraph of G', then G is not planar.*

Proof Suppose K_r is a subgraph of G' with $r \geq 3$. Then, the edges in G corresponding to K_r in G' must be of the form $\{i_k, j_k\}$ for $i = 1, 2, \ldots r$ where $i_p < i_q$ and $j_p < j_q$ in the standard numbering of G for each $1 \leq p < q \leq r$. That is, each edge must cross every other edge, as shown in Figure 3.15.

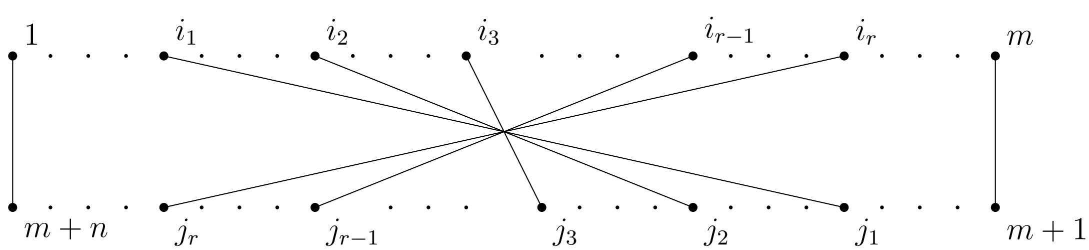

Figure 3.15: A graph G which has K_r as a subgraph of G'

Contract the edges between 1 and i_1, between $i_1 + 1$ and i_2, between $i_2 + 1$ and m in the standard numbering of G. Also, contract the edges between $m + 1$ and j_1, between $j_1 + 1$ and j_2, and between $j_2 + 1$ and $m + n$ in the standard numbering of G as shown in Figure 3.16.

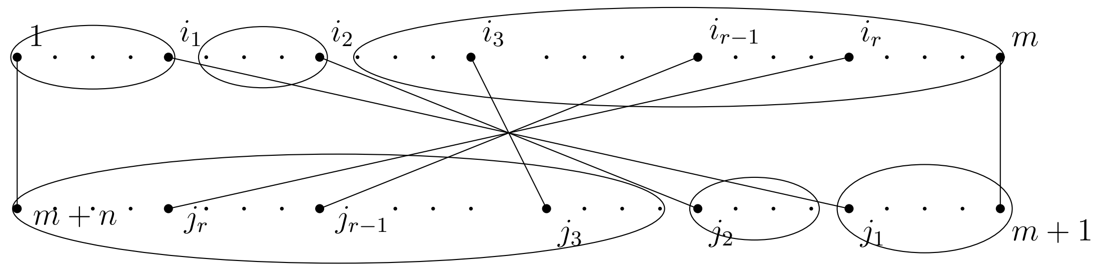

Figure 3.16: Edges to be contracted in G

After deleting any extra edges that were the result of the edge contractions, the resulting graph is shown in Figure 3.17.

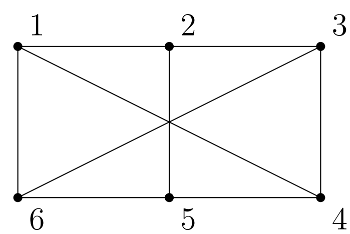

Figure 3.17: The resulting graph after edge contractions and edge deletions

We can redraw the graph by switching the positions of vertices 2 and 5, as shown, to see that the resulting graph is $K_{3,3}$.

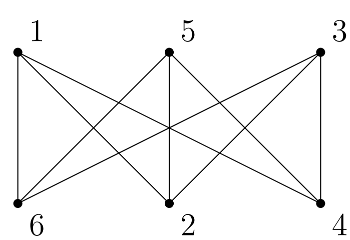

Figure 3.18: The resulting graph redrawn to show $K_{3,3}$

Therefore, $K_{3,3}$ is a minor of G, so that G is not planar. $\square$

Proposition 3.17. *If G' is a path, then* $\deg(v) = 2, 3,$ *or* 4*.*

Proof Let G' be a path. Figure 3.19 shows a case where vertices of degree 2, 3, and 4 are achieved.

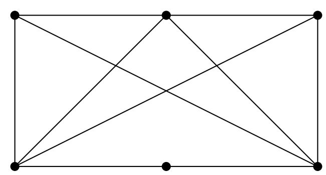

Figure 3.19: A graph with vertices of degree $2, 3,$ and 4 where G' is a path

Suppose $v \in G$ is has degree 5 or more. Two of the edges from v are edges on the perimeter of G. Then at least two edges from v correspond to vertices of degree 2 in G'. Therefore, these edges must cross at most two edges each. Any edges crossing the edges from v must cross at most two edges from v. Also, none of the connecting edges crossing the edges from v may cross each other (or K_3 would be a subgraph of G'). This is not possible. Any edges other than the first and last connecting edges edges from v force G' to be a caterpillar that is not a path by forcing one of the connecting edges not from v to cross three or more edges from v, or they force G' to be at least two disjointed paths. Therefore, if G' is a path, $\deg(v) =$ 2,3, or 4 for all $v \in G$. $\square$

CHAPTER 4

G' IS A PATH

We now consider the minimum rank of G in the specific case where G' is a path.

Basic and Condensed Paths

Definition 4.1. *G is a* basic path *if G' is a path, and every vertex in G has degree 3.*

Example 4.2. *The graph G in Figure 4.1 has derived graph G', shown in Figure 4.2 which is a path. It is clear that if G is a basic path, then the component paths that make up G are of the same length, and the derived graph G' is a path of the same length also.*

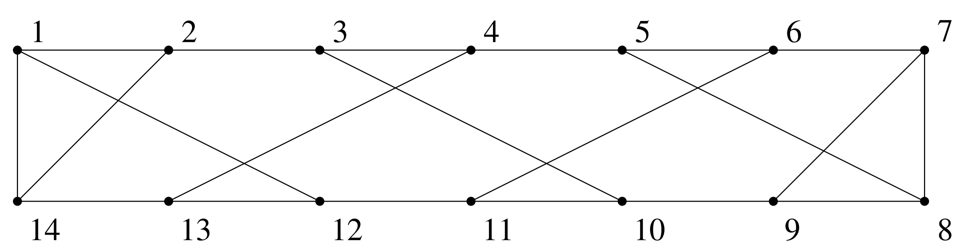

Figure 4.1: A basic path

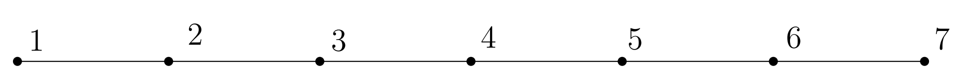

Figure 4.2: The derived graph G' for G in Figure 4.1

Definition 4.3. *G is a* condensed path *if G' is a path, there is at least one vertex of degree 4 and all other vertices are of degree 3.*

Example 4.4. *The graph G in Figure 4.3 is a condensed path whose derived graph G' (shown in Figure 4.4) is a path of length 6.*

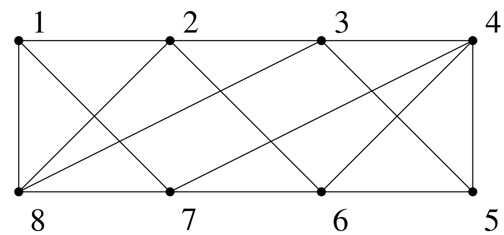

Figure 4.3: A condensed path

Figure 4.4: The derived graph for the graph in Figure 4.3

Proposition 4.5. *If G is a basic or condensed path, $Z(G) = 3$.*

Proof Select, as an initial coloring, the vertices 1, 2, and $m + n$ in the standard numbering of G as shown in Figures 4.5 and 4.6.

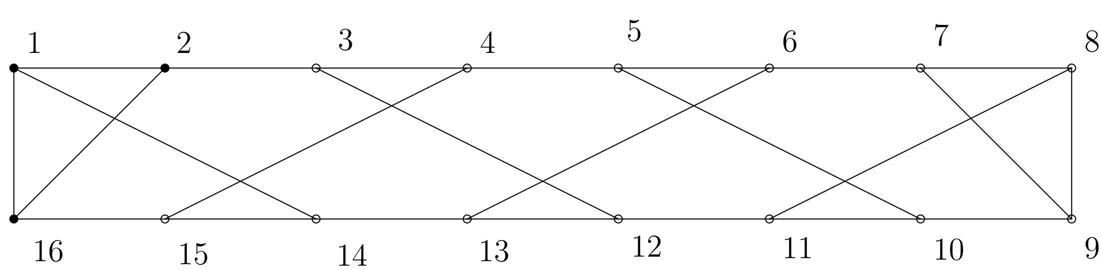

Figure 4.5: An initial coloring of a basic path

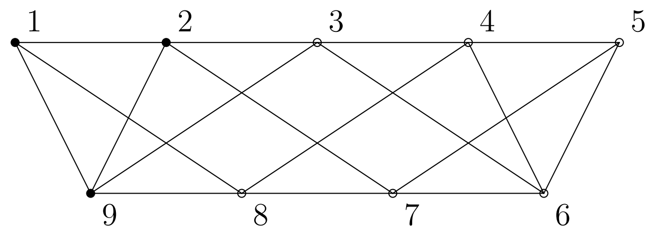

Figure 4.6: An initial coloring of a condensed path

Then, in the basic path, 2 forces $m+n$, $m+n$ forces $m+n-1$, 1 forces $m+n-2$, etc. From Figure 4.7, the forcing chain clearly alternates down both paths at the same rate until the whole graph is colored.

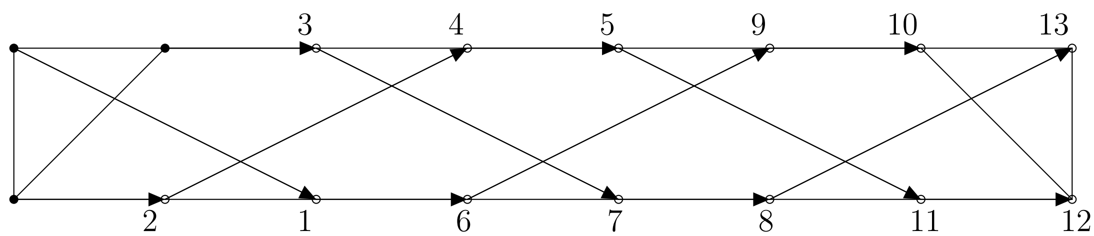

Figure 4.7: The forcing chain on a basic path

In the condensed path, 1 forces $m+n-1$, $m+n$ forces 3, 2 forces $m+n-2$, etc. Clearly the forcing chain alternates down the two paths of G.

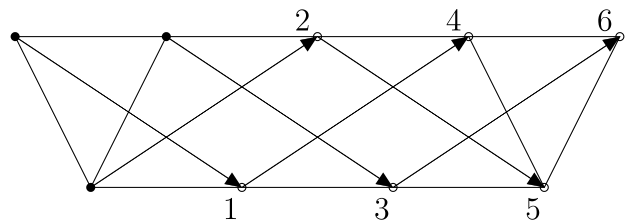

Figure 4.8: The forcing chain for a condensed path

Therefore, $Z(G) \leq 3$. By inspection, an initial coloring of two vertices is not sufficient, since each vertex has degree at least three. Each vertex in the initial coloring will have at least two white neighbors. Therefore, no additional vertices can be colored by the initial coloring. Therefore, $Z(G) \geq 3$. Thus, $Z(G) = 3$. $\square$

Corollary 4.6. *If G is a basic or condensed path, $mr(G) = |G| - 3$.*

Proof G is a basic or condensed path, by Proposition 4.5, $Z(G) = 3$. Therefore, $mr(G) \geq |G| - 3$ by Theorem 2.5. Since G' is a path, G has at least one crossing. By Proposition 3.16, G is outerplanar. So $mr(G) \leq |G| - 3$. Therefore, $mr(G) = |G| - 3$. $\square$

Subdivided paths

We now consider the case where G' is a path, and some of the vertices have degree other than 3 or 4. By Proposition 3.17 other vertices must have degree two. We introduce the following definition to classify these types of graphs:

Definition 4.7. *G is a* subdivided path *if G' is a path and G can be obtained from a basic or condensed path by subdividing certain edges in P_m or P_n an appropriate number of times.*

Definition 4.8. *If G is a basic or condensed path, a* zero-increasing edge *is an edge in P_m or P_n whose endpoints are also endpoints of distinct connecting edges whose corresponding vertices in*

G' are adjacent and of degree two. An edge in P_m or P_n that is not a zero-increasing edge is a non-increasing edge.

Definition 4.9. *For a graph G between two parallel paths where G' is a path, a* zero-increasing split *is a path P_k that is an induced subgraph of G, whose endpoints are adjacent to endpoints of two connecting edges whose corresponding vertices in G' are adjacent and of degree two. In other words, a* zero-increasing split *is an induced path subgraph created by subdividing a zero-increasing edge. A* non-increasing split *is an induced path subgraph created by subdividing a non-increasing edge.*

Example 4.10. *The edge e_1, in Figure 4.9 is a zero-increasing edge. The edge e_2 is a non-increasing edge.*

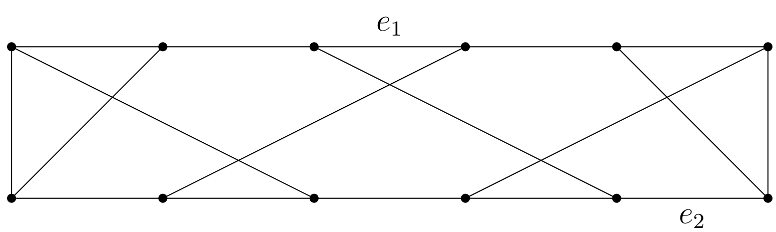

Figure 4.9: A graph G with zero-increasing and non-increasing edges

In Figure 4.10, e_1 and e_2 have been subdivided to form zero- and non-increasing splits s_1 and s_2 respectively.

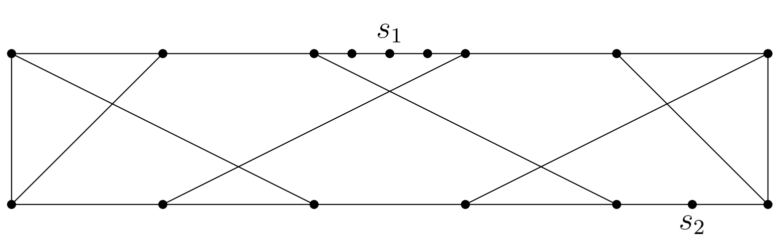

Figure 4.10: A graph G with zero-increasing and non-increasing splits

For convenience, zero-increasing and non-increasing splits will be represented by a single vertex, regardless of actual split length. This is an acceptable simplification because of the following result:

Proposition 4.11. *[4] For a graph G and an edge e in G, $mr(G_e) = mr(G) + 1$ if at least one endpoint of e is of degree 1 or 2.*

It is clear then, that after the first subdivision of a zero- or non-increasing edge, subsequent subdivisions simply increase the minimum rank by one each time. It is much more difficult to determine what happens to the minimum rank when the first subdivision occurs.

Proposition 4.12. *Subdividing a non-increasing edge does not increase the zero forcing number.*

Proof With an initial coloring of vertices 1, 2, and $m + n$ the forcing chain will proceed as in Figure 4.7 up to the non-increasing split. Since all prior vertices have been colored, and since the only white neighbor of v, the vertex adjacent to the endpoint of the split, is the endpoint of the split, the non-increasing split is colored without increasing the zero forcing number. $\square$

Clearly, zero-increasing edges are the only edges in P_m or P_n that can increase the zero-forcing number of G by being subdivided. However, this occurs only under certain conditions. To help clarify when the zero-forcing number is increased, we introduce the following:

Definition 4.13. *For G, a graph between two parallel paths, G^* is the graph constructed in the following way:*

- *A zero-increasing split in G corresponds to a vertex in G^*.*
- *If two zero-increasing splits in G have endpoints that are endpoints of the same connecting edge, then the corresponding vertices in G^* are adjacent.*

Proposition 4.14. *If G is a graph between two parallel paths with G' as a path, then $Z(G) = 3 + \alpha(G^*)$*

Proof Clearly, $Z(G) \geq 3$. To determine how many, if any, vertices should be added to the initial three discussed in the proof of Proposition 4.5, observe that the construction of G^* means that its structure is either one or more independent paths, isolated vertices, or a combination of paths and vertices, as shown in Figure 4.11.

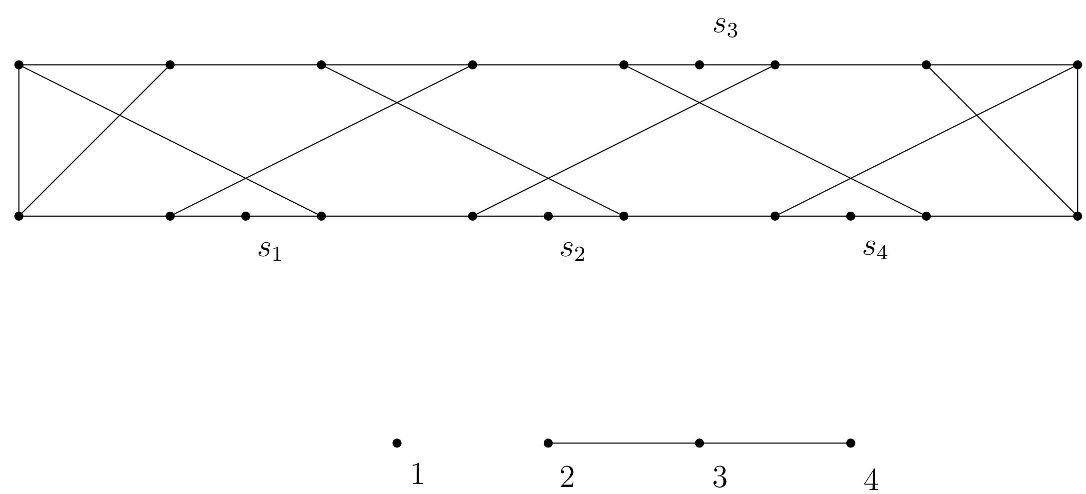

Figure 4.11: A graph G and its derived graph G^*

If G^* has an independent vertex, then, assuming an initial coloring that forces each vertex up to the zero-increasing split corresponding to the independent vertex in G^*, then the first (or only) vertex in the split needs to be included in the initial coloring in order for the forcing chain to move down the rest of G.

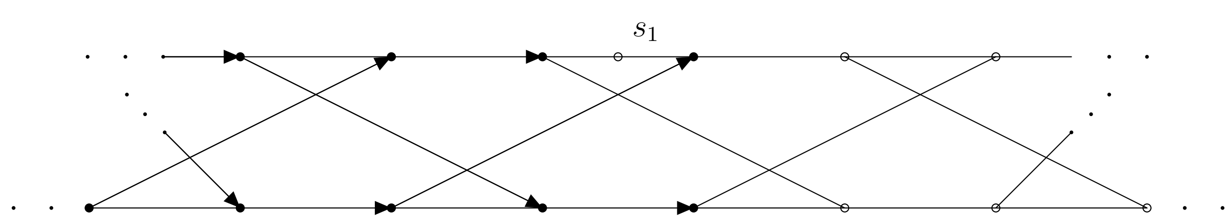

Figure 4.12: The first vertex of s_1 must be in the initial coloring for the forcing chain to proceed

If G^* has a path as a subgraph, then, in G, whichever of P_m or P_n contains the largest number of splits corresponding to vertices in the path in G^*, should have the first (or only) vertices in those splits included in an initial coloring. That is, if P_k is a subgraph of G^*, then the first (or only) vertices in $\lceil \frac{k}{2} \rceil$ splits corresponding to P_k must be part of the initial coloring of G.

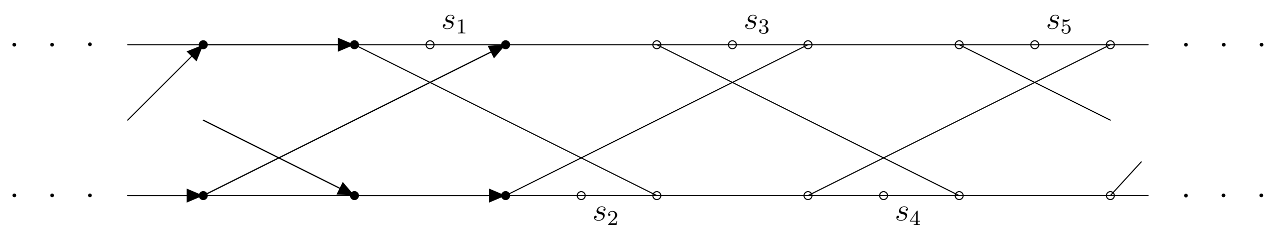

Figure 4.13: The forcing chain cannot proceed unless s_1, s_3, and s_5 are in the initial coloring

The independence number of G^*, $\alpha(G^*)$ is the number of isolated vertices in G^* plus $\lceil \frac{k_i}{2} \rceil$ for each P_{k_i} that is a subgraph of G^*. This is also the minimum number of vertices that must be added to the initial 3 vertices to form a zero forcing set. Therefore, $Z(G) = 3 + \alpha(G^*)$. $\square$

From Proposition 4.14 and Corollary 3.7, we can say that $|G| - Z(G) \leq mr(G) \leq |G| - 3$. Now that we have lower and upper bounds for $mr(G)$, we must attempt to find exactly what $mr(G)$ is for these graphs. Starting from $mr_+(G)$ does not offer much help:

Proposition 4.15. *[13] For a given graph G and edge e in G, $mr_+(G_e) = mr_+(G) + 1$.*

A straightforward consequence is the following

Corollary 4.16. *For a graph G between two parallel paths such that G is a basic or condensed path, $mr_+(G) = |G| - 3$.*

This does not narrow our bounds at all. We offer the following:

Conjecture 4.17. $mr(G) = |G| - Z(G)$.

A proof of this conjecture was not forthcoming, and is left for future research. However, several examples should suffice to offer evidence that the conjecture is true. We also offer a scheme for how to find a matrix achieving the desired rank.

Example 4.18. *Consider the graph, G_1, in Figure 4.14. According to Conjecture 4.17, $mr(G_1) = 9 - 4 = 5$. The matrix M_1 in Figure 4.15 achieves this rank.*

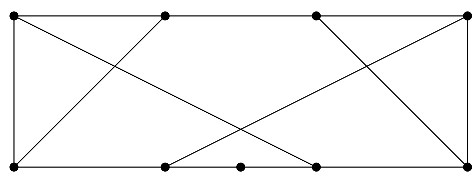

Figure 4.14: A graph G_1 of order 9 and rank 5

$$M_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

Figure 4.15: A 9×9 matrix which has G_1 as its graph

Example 4.19. *The graph* (G_2) *has order* 16 *and by Conjecture 4.17,* $mr(G) = 11$. *The matrix* M_2 *has the desired rank.*

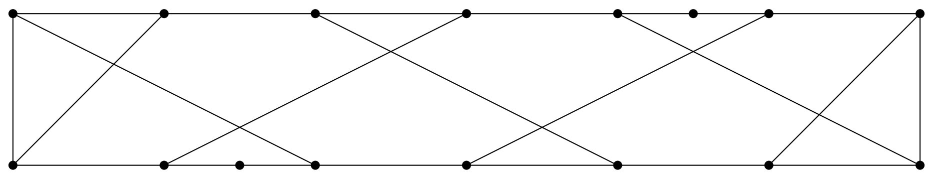

Figure 4.16: The graph G_2

$$
M_2 = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 3 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 6 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & -1 & 0 & 0 & 0 & 0 & 18 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 3 & 0 & 0 & 0 & 0 & 25 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0
\end{bmatrix}
$$

Figure 4.17: The matrix M_2 has graph G_2 and rank 11

To calculate a matrix of rank $|G| - Z(G)$ the following procedure can be used:

- Contract all zero-increasing splits in both of P_m or P_n into zero-increasing edges.
- Delete all zero-increasing edges from one of P_m or P_n.
- Using matrices of 1's and 0's (corresponding to the 2- and 3-cliques remaining in the graph), form the matrix A. $rank(A)$ is at most $|G|$ minus at most twice the number of zero-increasing edges removed.

- For each edge that was deleted:
  - Create the vector V that is in the column space of A so that the nonzero entries of V correspond to the neighbors of the removed edge.
  - Split V into two vectors v_1 and v_2 corresponding to the two vertices in the edge removed.
  - With the $2 \times n$ matrix $U = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$, create the matrix

$$B = \begin{bmatrix} A & U \\ U^T & U^T(A^+)U \end{bmatrix}$$

    where U^T is the transpose of U and A^+ is the Moore-Penrose pseudoinverse of A.
  - Repeat for the next edge with $A = B$
- Subdivide each zero-increasing edge that was originally a zero-increasing split. This corresponds to creating a vector whose nonzero elements correspond to the endpoints of the edge being subdivided, adding this vector to the matrix in the same way as U was added above, and then using elementary matrix operations to remove the unnecessary elements corresponding to the edge between the endpoints of the zero-increasing split.

A proof of Conjecture 4.17 was not found and is left for future research.

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