5-2023

The Hilbert sequence and its associated Jacobi matrix

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Departmental Honors Thesis
The University of Tennessee at Chattanooga
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Examination Date: April 12, 2023

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ABSTRACT

In this project, we investigate positive definite sequences and their associated Jacobi matrices in Hilbert space. We set out to determine the Jacobi matrix associated to the Hilbert sequence by methods described in Akhiezer's book *The Classical Moment Problem.* Using methods in Teschl's book *Jacobi Operators and Completely Integrable Nonlinear Lattice*, we determine the essential spectrum of the corresponding Jacobi matrix.

DEDICATION

To my Savior Jesus Christ, my parents, Mr. Krue Brock, and Dr. Roger Nichols.

ACKNOWLEDGEMENTS

My thanks to the University of Tennessee at Chattanooga's Department of Mathematics and Honors College for allowing me to do this project. Thank you Dr. Roger Nichols who guided me on this project.

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CHAPTER 1

INTRODUCTION

The investigation of positive definite sequences often leads to interesting results. By a result due to Hilbert in 1894 [**2**] the sequence $\mathfrak{h} = \{\frac{1}{k+1}\}_{k=0}^{\infty}$ is positive definite. In the book *The Classical Moment Problem*, Akhiezer establishes a one-to-one correspondence between positive definite sequences and Jacobi matrices. The goal of this thesis is to find the Jacobi matrix associated to $\mathfrak{h}$ and to locate its essential spectrum.

Akhiezer [**1**, Sections 1.2–1.4] describes in detail the construction by which one may calculate the Jacobi matrix associated to a given positive definite sequence. A Jacobi matrix is a semi-infinite, tridiagonal matrix of the form:

$$\mathscr{J} = \begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & \cdots \\ b_0 & a_1 & b_1 & 0 & 0 & \cdots \\ 0 & b_1 & a_2 & b_2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

where $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$ are sequences of real numbers with $b_k > 0$ for all $k \in \mathbb{N}_0$. Here $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is an abbreviation for the natural numbers including zero. Akhiezer derives general expressions for the terms a_k and b_k. The expression for b_k is simpler in the sense that b_k is directly expressed in terms of the entries of the given positive definite sequence. However, Akhiezer's expressions for a_k involve a functional and a three-term recurrence relation. Therefore, these expressions for a_k are difficult to work with in practice. To calculate the b_k corresponding to $\mathfrak{h}$, we use Akhiezer's expressions for b_k and a known formula for the determinant of a Cauchy matrix. To calculate the a_k corresponding to $\mathfrak{h}$, we apply our formula for b_k and exploit the arbitrary nature of a parameter in Akhiezer's three-term recurrence relation.

Finally, using an abstract result on the essential spectrum of a Jacobi matrix from [**6**], we apply our expressions for a_k and b_k to locate the essential spectrum of the Jacobi matrix associated to $\mathfrak{h}$. We find that the essential spectrum is the interval [0,1].

CHAPTER 2

Mathematical Preliminaries

2.1. Positive Sequences and Jacobi Matrices

In *The Moment Problem*, Schmüdgen defines a sequence of real numbers $\mathfrak{s} = \{s_k\}_{k=0}^{\infty}$ to be *positive definite* if for every $n \in \mathbb{N}_0$,

$$\sum_{j,k=0}^{n} s_{j+k} x_j x_k > 0 \text{ for every } (x_j)_{j=0}^{n} \in \mathbb{R}^{n+1} \tag{2.1.1}$$

(see [**4**, Section 3.2]). Equivalently, by [**4**, Proposition 3.11], $\mathfrak{s}$ is positive definite if and only if the following determinant condition holds:

$$D_k = \begin{vmatrix} s_0 & s_1 & \dots & s_k \\ s_1 & s_2 & \dots & s_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_k & s_{k+1} & \dots & s_{2k} \end{vmatrix} > 0, \quad k \in \mathbb{N}_0, \tag{2.1.2}$$

where $|M|$ denotes the determinant of a square matrix M. The determinant D_k in (2.1.2) is called a *Hankel subdeterminant* corresponding to $\mathfrak{s}$ since the matrix under the determinant sign in (2.1.2) is a Hankel matrix.

Akhiezer [**1**, Sections 1.2–1.4] constructs a bijection between the set of positive definite sequences and the set of Jacobi matrices. The one-to-one nature of this correspondence is also described in Schmüdgen [**4**, Sections 5.1–5.3], and a statement of the result is summarized in [**4**, Theorem 5.14].

A Jacobi matrix is a semi-infinite matrix of the form

$$\mathscr{J} = \begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & \cdots \\ b_0 & a_1 & b_1 & 0 & 0 & \cdots \\ 0 & b_1 & a_2 & b_2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{2.1.3}$$

where $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$ are sequences of real numbers and

$$b_k > 0, \quad k \in \mathbb{N}_0. \tag{2.1.4}$$

The Jacobi matrix $\mathscr{J}$ in (2.1.3) is symmetric in the sense that $\mathscr{J}$ is equal to its transpose, $\mathscr{J} = \mathscr{J}^T$. Jacobi matrices play a special role in operator theory where, by a theorem due to Stone [**5**, Theorem 7.13], they arise as a model for operators with a *simple spectrum* ([**5**, Definition 7.2]).

Example 2.1. *An explicit example of a positive definite sequence is Hilbert's sequence,*

$$\mathfrak{h} = \left\{ \frac{1}{k+1} \right\}_{k=0}^{\infty} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}. \tag{2.1.5}$$

In fact, Hilbert calculated the Hankel subdeterminant (2.1.2) *for* $\mathfrak{h}$ *in* [**2**] *and found*

$$D_k = \begin{vmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{k+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{k+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{k+1} & \frac{1}{k+2} & \cdots & \frac{1}{2k+1} \end{vmatrix} = \frac{\{1^k 2^{k-1} \cdots (k-1)^2 k^1\}^4}{1^{2k+1} 2^{2k} \cdots (2k)^2 (2k+1)^1}, \quad k \in \mathbb{N}_0. \tag{2.1.6}$$

Since the right-hand side of (2.1.6) *is a positive real number for all* $k \in \mathbb{N}_0$*, the sequence* $\mathfrak{h}$ *is positive definite. Thus, according to the construction provided by Akhiezer in* [**1**, Sections 1.2–1.3]*, there is a unique Jacobi matrix* $\mathscr{J}_{\mathfrak{h}}$ *corresponding to the Hilbert sequence* $\mathfrak{h}$*. The matrix under the determinant sign in* (2.1.6) *is a Cauchy matrix. Therefore, the identity in* (2.1.6) *may also be deduced from a known closed-form expression for the determinant of a Cauchy matrix. The expression for the determinant of a Cauchy matrix will be used later in a different context, so we have included the formula and its proof in Appendix A.*

2.2. The Associated Jacobi Matrix

In order to find the Jacobi matrix $\mathscr{J}_{\mathfrak{h}}$ corresponding to $\mathfrak{h}$, it is necessary to find the two sequences which make up the Jacobi matrix: $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$. According to Akhiezer's construction, the equation for the b_k is

$$b_k = \frac{\sqrt{D_{k-1}D_{k+1}}}{D_k}, \quad k \in \mathbb{N}_0, \tag{2.2.1}$$

where D_k is the determinant in (2.1.6); that is,

$$D_k = \frac{\{1^k 2^{k-1} \cdots (k-1)^2 k^1\}^4}{1^{2k+1}2^{2k} \cdots (2k)^2(2k+1)^1} = \frac{[\prod_{j=1}^{k} j^{k-j+1}]^4}{\prod_{j=1}^{2k+1} j^{2k-j+2}}, \quad k \in \mathbb{N}_0. \tag{2.2.2}$$

Thus, applying (2.2.2) in (2.2.1), the full representation of b_k is,

$$\begin{aligned} b_k &= \frac{[\prod_{j=1}^{k-1} j^{k-j}]^2}{[\prod_{j=1}^{2k-1} j^{2k-j}]^{1/2}} \frac{[\prod_{j=1}^{k+1} j^{k-j+2}]^2}{[\prod_{j=1}^{2k+3} j^{2k-j+4}]^{1/2}} \frac{\prod_{j=1}^{2k+1} j^{2k-j+2}}{[\prod_{j=1}^{k} j^{k-j+1}]^4} \\ &= \frac{(k+1)^2[\prod_{j=1}^{2k-1} j^{2k-j+2}](2k)^2(2k+1)}{[\prod_{j=1}^{2k-1} j^{k-\frac{j}{2}}][\prod_{j=1}^{2k-1} j^{k-\frac{j}{2}+2}](2k+1)^{1/2}(2k+2)(2k+3)^{1/2}} \\ &= \frac{k+1}{2\sqrt{2k+1}\sqrt{2k+3}}, \quad k \in \mathbb{N}_0. \end{aligned} \tag{2.2.3}$$

Also, Akhiezer describes the process of determining the a_k in [**1**, Sections 1.2–1.3]. To recall the main elements of Akhiezer's construction, let us first define the set of all polynomials on $\mathbb{R}$ with real coefficients as $\mathcal{P}(\mathbb{R})$. Here, each $p \in \mathcal{P}(\mathbb{R})$ may be written as

$$p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n, \quad \lambda \in \mathbb{R}, \tag{2.2.4}$$

for some $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\{a_k\}_{k=1}^{n} \subset \mathbb{R}$.

If $\mathfrak{s} = \{s_k\}_{k=0}^{\infty}$ is a positive definite sequence, then we define the functional $\mathfrak{S} : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ associated to $\mathfrak{s}$ by

$$\mathfrak{S}\{p\} = s_0a_0 + s_1a_1 + s_2a_2 + \cdots + s_na_n, \quad p \in \mathcal{P}(\mathbb{R}). \tag{2.2.5}$$

The sequence $\mathfrak{s}$ gives rise to a sequence of polynomials $\{P_k(\lambda)\}_{k=0}^{\infty}$ defined by,

$$P_0(\lambda) = 1, \quad P_k(\lambda) = \frac{1}{\sqrt{D_{k-1}D_k}} \begin{vmatrix} s_0 & s_1 & \cdots & s_k \\ s_1 & s_2 & \cdots & s_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{k-1} & s_k & \cdots & s_{2k-1} \\ 1 & \lambda & \cdots & \lambda^k \end{vmatrix}, \quad \lambda \in \mathbb{R},\ k \in \mathbb{N}. \tag{2.2.6}$$

The polynomial $P_k(\lambda)$, $k \in \mathbb{N}_0$, is a polynomial of degree k. Also, we may observe that the sequence of these polynomials are orthonormal in the sense that,

$$\mathfrak{S}\{P_j(\lambda)P_k(\lambda)\} = \delta_{j,k}, \quad j, k \in \mathbb{N}_0, \tag{2.2.7}$$

where $\delta_{j,k}$ denotes the Kronecker delta function. Akhiezer shows that the polynomials $\{P_k(\lambda)\}_{k=0}^{\infty}$ satisfy the three-term recurrence relation:

$$\begin{gathered} a_0P_0(\lambda) + b_0P_1(\lambda) = \lambda P_0(\lambda), \\ b_{k-1}P_{k-1}(\lambda) + a_kP_k(\lambda) + b_kP_{k+1}(\lambda) = \lambda P_k(\lambda), \quad k \in \mathbb{N}, \end{gathered} \tag{2.2.8}$$

where $\{b_k\}_{k=0}^{\infty}$ is given by (2.2.1) and $\{a_k\}_{k=0}^{\infty}$ is given by

$$a_k = \mathfrak{S}\{\lambda P_k(\lambda)P_k(\lambda)\}, \quad k \in \mathbb{N}_0. \tag{2.2.9}$$

The first equation in (2.2.8) is called the "initial condition." Formally, the recurrence relation (2.2.8) may be expressed as the semi-infinite matrix equation,

$$\underbrace{\begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & \cdots \\ b_0 & a_1 & b_1 & 0 & 0 & \cdots \\ 0 & b_1 & a_2 & b_2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}}_{\mathscr{J}_{\mathfrak{s}}} \begin{bmatrix} P_0(\lambda) \\ P_1(\lambda) \\ P_2(\lambda) \\ \vdots \end{bmatrix} = \lambda \begin{bmatrix} P_0(\lambda) \\ P_1(\lambda) \\ P_2(\lambda) \\ \vdots \end{bmatrix}, \quad \lambda \in \mathbb{R}, \tag{2.2.10}$$

where $\mathscr{J}_{\mathfrak{s}}$ is the Jacobi matrix associated to $\mathfrak{h}$. Algebraically manipulating this recurrence relation yields both the initial condition and the recurrence relation in terms of a_k,

$$\begin{aligned} a_0 &= \lambda P_0(\lambda) - b_0P_1(\lambda), \\ a_k &= \frac{\lambda P_k(\lambda) - b_{k-1}P_{k-1}(\lambda) - b_kP_{k+1}(\lambda)}{P_k(\lambda)} \text{ if } P_k(\lambda) \neq 0, \quad \lambda \in \mathbb{R},\ k \in \mathbb{N}. \end{aligned} \tag{2.2.11}$$

Seemingly, this does us no service, as we are still reliant upon the P_k, but the equations (2.2.11) are valid for all values of λ for which $P_k(\lambda) \neq 0$. Thus, we choose the most convenient value of $\lambda = 0$. With this value of λ, the recurrence relation now yields,

$$\begin{aligned} a_0 &= -b_0 P_1(0), \\ a_k &= \frac{-b_{k-1}P_{k-1}(0) - b_k P_{k+1}(0)}{P_k(0)}, \quad k \in \mathbb{N}, \end{aligned} \tag{2.2.12}$$

provided that $P_k(0) \neq 0$. The fixed value $P_k(0)$ is simpler to compute since by (2.2.6):

$$P_0(0) = 1, \quad P_k(0) = \frac{1}{\sqrt{D_{k-1}D_k}} \begin{vmatrix} s_0 & s_1 & \cdots & s_k \\ s_1 & s_2 & \cdots & s_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{k-1} & s_k & \cdots & s_{2k-1} \\ 1 & 0 & \cdots & 0 \end{vmatrix}, \quad k \in \mathbb{N}. \tag{2.2.13}$$

For $k \in \mathbb{N}$, the determinant in (2.2.13) may be reduced by expanding about the last row. The result is:

$$P_k(0) = \frac{(-1)^k}{\sqrt{D_{k-1}D_k}} \begin{vmatrix} s_1 & s_2 & \cdots & s_k \\ s_2 & s_3 & \cdots & s_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_k & s_{k+1} & \cdots & s_{2k-1} \end{vmatrix}, \quad k \in \mathbb{N}. \tag{2.2.14}$$

In the particular case when $\mathfrak{s} = \mathfrak{h}$, for each $k \in \mathbb{N}$, the matrix under the determinant sign in (2.2.14) is a Cauchy matrix of the form

$$S_k = \begin{bmatrix} s_1 & s_2 & \cdots & s_k \\ s_2 & s_3 & \cdots & s_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_k & s_{k+1} & \cdots & s_{2k-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{k+1} \\ \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{k+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{k+1} & \frac{1}{k+2} & \cdots & \frac{1}{2k} \end{bmatrix}. \tag{2.2.15}$$

We recall that an $n \times n$ matrix $C_n \in \mathbb{R}^{n\times n}$, where $n \in \mathbb{N}$ is fixed, is a *Cauchy matrix* if C_n can be written in the form

$$C_n = \begin{bmatrix} \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \cdots & \frac{1}{x_1+y_n} \\ \frac{1}{x_2+y_1} & \frac{1}{x_2+y_2} & \cdots & \frac{1}{x_2+y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_n+y_1} & \frac{1}{x_n+y_2} & \cdots & \frac{1}{x_n+y_n} \end{bmatrix} \tag{2.2.16}$$

for some pair of finite sequences of real numbers, $\{x_k\}_{k=1}^n, \{y_k\}_{k=1}^n$ with the property that $x_j + y_k \neq 0$ for all $1 \leq j, k \leq n$. The determinant of C_n may be computed in closed form, according to the following lemma.

Lemma 2.2. *If $n \in \mathbb{N}$ and C_n is the Cauchy matrix given by* (2.2.16)*, then the determinant of C_n is given by*

$$|C_n| = \frac{\prod_{1\le i<j\le n}(x_j - x_i)(y_j - y_i)}{\prod_{1\le i,j\le n}(x_i + y_j)}. \tag{2.2.17}$$

The proof of Lemma 2.2 is by mathematical induction on n and uses the well-known ways in which elementary row operations affect the determinant. Due to the length and technical nature of the proof, we sketch the proof in Appendix A.

Comparing (2.2.15) and (2.2.16) confirms that the matrix in (2.2.15) is, indeed, a Cauchy matrix with $n = k$ and $\{x_j\}_{j=1}^k$ and the finite sequences $\{y_j\}_{j=1}^k$ given by

$$x_j = y_j = j, \quad 1 \le j \le k. \tag{2.2.18}$$

By Lemma 2.2, the determinant of S_k may be computed as follows:

$$\begin{aligned}|S_k| &= \frac{\prod_{1\le i<j\le k}(x_j - x_i)(y_j - y_i)}{\prod_{1\le i,j\le k}(x_i + y_j)} \\ &= \frac{\prod_{j=1}^{k}\prod_{i=1}^{j-1}(j-i)^2}{\prod_{j=1}^{n}\prod_{i=1}^{n}(i+j)}, \quad k \in \mathbb{N}.\end{aligned} \tag{2.2.19}$$

In a similar fashion, the matrix under the determinant sign in (2.1.6) is also a Cauchy matrix (this time with $x_j = j$ and $y_j = j - 1$ for $1 \le j \le k+1$), so Lemma 2.2 permits D_k to be recast in the form

$$D_k = \frac{\prod_{j=1}^{k+1}\prod_{i=1}^{j-1}(j-i)^2}{\prod_{j=1}^{n+1}\prod_{i=0}^{n}(i+j)}, \quad k \in \mathbb{N}. \tag{2.2.20}$$

Using (2.2.19) and (2.2.20) in (2.2.14), the value $P_k(0)$ may be calculated for each $k \in \mathbb{N}$ as follows:

$$P_k(0) = (-1)^k \sqrt{\frac{\prod_{j=1}^{k}\prod_{i=0}^{k-1}(i+j)}{\prod_{j=1}^{k}\prod_{i=1}^{j-1}(j-i)^2} \cdot \frac{\prod_{j=1}^{k+1}\prod_{i=0}^{k}(i+j)}{\prod_{j=1}^{k+1}\prod_{i=1}^{j-1}(j-i)^2}\,\frac{\prod_{j=1}^{k}\prod_{i=1}^{j-1}(j-i)^2}{\prod_{j=1}^{n}\prod_{i=1}^{n}(i+j)}}, \tag{2.2.21}$$

which, after taking cancellations into account, simplifies to

$$P_k(0) = (-1)^k\sqrt{2k+1}, \quad k \in \mathbb{N}. \tag{2.2.22}$$

Since $P_0(0) = 1$ by (2.2.13), the formula given in (2.2.22) extends to $k \in \mathbb{N}_0$; that is,

$$P_k(0) = (-1)^k \sqrt{2k+1}, \quad k \in \mathbb{N}_0, \tag{2.2.23}$$

which confirms that $P_k(0) \neq 0$ for all $k \in \mathbb{N}_0$. Finally, applying (2.2.23) and (2.2.3) in (2.2.12), we obtain:

$$\begin{aligned} a_k &= \frac{-\dfrac{(-1)^{k-1}k\sqrt{2k-1}}{2\sqrt{2k-1}\sqrt{2k+1}} - \dfrac{(-1)^{k+1}\sqrt{2k+3}}{2\sqrt{2k+1}\sqrt{2k+3}}}{(-1)^k\sqrt{2k+1}} \\ &= \frac{1}{\sqrt{2k+1}}\left[\frac{k}{2\sqrt{2k+1}} + \frac{k+1}{2\sqrt{2k+1}}\right] \\ &= \frac{k}{2(2k+1)} + \frac{k+1}{2(2k+1)} \\ &= \frac{\cancel{2k+1}}{2(\cancel{2k+1})} \\ &= \frac{1}{2}, \quad k \in \mathbb{N}, \end{aligned} \tag{2.2.24}$$

and for $k = 0$:

$$a_0 = -b_0 P_1(0) = -\frac{1}{2\sqrt{3}}(-1)\sqrt{3} = \frac{1}{2}. \tag{2.2.25}$$

Therefore,

$$a_k = \frac{1}{2}, \quad k \in \mathbb{N}_0. \tag{2.2.26}$$

Summarizing the findings of (2.2.3) and (2.2.26), we have proved the following theorem that explicitly identifies the Jacobi matrix $\mathscr{J}_{\mathfrak{h}}$ uniquely associated with the Hilbert sequence $\mathfrak{h}$.

THEOREM 2.3. *If* $\mathfrak{h} = \left\{\dfrac{1}{k+1}\right\}_{k=0}^{\infty}$ *denotes the positive definite Hilbert sequence, then the Jacobi matrix* $\mathscr{J}_{\mathfrak{h}}$ *associated to* $\mathfrak{h}$ *is given by the right-hand side of* (2.1.3) *with*

$$a_k = \frac{1}{2}, \quad b_k = \frac{k+1}{2\sqrt{(2k+1)(2k+3)}}, \quad k \in \mathbb{N}_0. \tag{2.2.27}$$

2.3. Locating the Essential Spectrum

Before we delve into finding the essential spectrum of $\mathscr{J}_{\mathfrak{h}}$, there must be a just explanation of what exactly an essential spectrum is. Firstly, we recall the definition of a Hilbert space. In [**7**, Sections 1.1], a Hilbert space is defined in the following way.

Definition 2.4. *A (complex) Hilbert space is a vector space* $\mathfrak{H}$ *over* $\mathbb{C}$ *paired with an inner product* $\langle \cdot, \cdot \rangle$ *such that* $\mathfrak{H}$ *is complete in the metric* $d(x, y) = \|x - y\| = \langle x - y, x - y \rangle^{\frac{1}{2}}$, $x, y \in \mathfrak{H}$.

Example 2.5. *The vector space of square summable sequences of complex numbers*

$$\ell^2(\mathbb{N}_0) = \left\{ \{u_n\}_{n=0}^{\infty} \subset \mathbb{C} \;\middle|\; \sum_{n=0}^{\infty} |u_n|^2 < \infty \right\} \tag{2.3.1}$$

is a Hilbert space when paired with the inner product

$$\left\langle \{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty} \right\rangle_2 = \sum_{n=0}^{\infty} u_n \overline{v_n}, \quad \{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{N}_0). \tag{2.3.2}$$

The associated norm $\|\cdot\|_2$ *is given by*

$$\|u\|_2 = \left[\sum_{n=0}^{\infty} \|u_n\|^2 \right]^{1/2}, \quad u = \{u_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{N}_0). \tag{2.3.3}$$

A function $A : \mathfrak{H} \to \mathfrak{H}$ is called a *linear operator* if

$$A(\alpha x + \beta y) = \alpha A x + \beta A y, \quad \alpha, \beta \in \mathbb{C},\ x, y \in \mathfrak{H}. \tag{2.3.4}$$

Following MacCluer [**7**, Section 2.1], we define a *bounded linear operator* as follows.

Definition 2.6. *A linear operator* $A : \mathfrak{H} \to \mathfrak{H}$ *is a bounded linear operator in* $\mathfrak{H}$ *if* $\|Ah\| \le C\|h\|$, *for all* $h \in \mathfrak{H}$ *and for some finite constant* $C \ge 0$. *The set of all bounded linear operators in* $\mathfrak{H}$ *is denoted by* $\mathfrak{B}(\mathfrak{H})$.

Example 2.7. *The operator* $I : \mathfrak{H} \to \mathfrak{H}$ *defined by* $Ix = x$ *for all* $x \in \mathfrak{H}$ *is a bounded linear operator.* I *is called the identity operator in* $\mathfrak{H}$*. Since* $\|Ix\| = \|x\| \leq 1 \cdot \|x\|$ *for all* $x \in \mathfrak{H}$*, the operator* I *is bounded, so* $I \in \mathfrak{B}(\mathfrak{H})$*.*

Using Definitions 2.4 and 2.6, we define the *resolvent set* and *spectrum* of $A \in \mathfrak{B}(\mathfrak{H})$ as follows.

Definition 2.8. *If* $A \in \mathfrak{B}(\mathfrak{H})$*, then the resolvent set of* A *is*

$$\rho(A) = \{\lambda \in \mathbb{C} \,|\, A - \lambda I \text{ is a bijection}\}. \tag{2.3.5}$$

The complement of $\rho(A)$ *in* $\mathbb{C}$ *is denoted by* $\sigma(A)$ *and is called the spectrum of* A*:*

$$\sigma(A) = \mathbb{C} \backslash \rho(A). \tag{2.3.6}$$

A number $\lambda \in \mathbb{C}$ is an *eigenvalue* of A if there exists a vector $h \in \mathfrak{H}$ such that $Ah = \lambda h$. In this case, h is an eigenvector corresponding to λ. If $\lambda \in \mathbb{C}$ is an eigenvalue of A, then $A - \lambda I$ is not a bijection since $A - \lambda I$ is not one-to-one, so $\lambda \in \sigma(A)$. Therefore, every eigenvalue of A belongs to the spectrum of A. However, the converse to this statement is generally false, as there may exist points in $\sigma(A)$ that are not eigenvalues of A. There are two separate parts of the spectrum of A: the set of isolated eigenvalues of A with finite dimensional eigenspaces form the *discrete spectrum* of A and the complement of the discrete spectrum in $\sigma(A)$ is called the *essential spectrum* of A.

Definition 2.9. *The discrete spectrum* $\sigma_{\text{disc}}(A)$ *of* A *is the set of all eigenvalues of* A *that are isolated in* $\sigma(A)$ *and have finite dimensional eigenspaces. The essential spectrum* $\sigma_{\text{ess}}(A)$ *of* A *is*

$$\sigma_{\text{ess}}(A) = \sigma(A) \backslash \sigma_{\text{disc}}(A).$$

If $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are bounded sequences of real numbers, then the corresponding Jacobi matrix $\mathscr{J}$ given by (2.1.3) defines a bounded linear operator in $\ell^2(\mathbb{N}_0)$ in the following way: for each $u = \{u_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{N}_0)$,

$$\mathscr{J} : u = \{u_n\}_{n=0}^{\infty} \mapsto \mathscr{J} u = \left\{(\mathscr{J} u)_n\right\}_{n=0}^{\infty}, \tag{2.3.7}$$

where

$$(\mathscr{J} u)_n = \begin{cases} a_0 u_0 + b_0 u_1, & n = 0, \\ b_{n-1} u_{n-1} + a_n u_n + b_n u_{n+1}, & n \in \mathbb{N}. \end{cases} \tag{2.3.8}$$

In particular, $\mathscr{J}$ has an essential spectrum, $\sigma_{\mathrm{ess}}(\mathscr{J})$. Teschl [**6**, Section 3.2] contains the following theorem.

Theorem 2.10. *Let $\mathscr{J}$ be a Jacobi matrix with corresponding sequences $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$. Define the quantities*

$$\begin{aligned} \underline{c_-} &= \liminf_{n\to\infty} |c_-(n)|, \\ \overline{c_+} &= \limsup_{n\to\infty} |c_+(n)|, \\ c_\pm(n) &= a_n \pm (|b_n| + |b_{n-1}|), \quad n \in \mathbb{N}. \end{aligned} \tag{2.3.9}$$

If

$$\lim_{n\to\infty} (|b_{n+1}| - |b_n|) = \lim_{n\to\infty} (|a_{n+1}| - |a_n|) = 0, \tag{2.3.10}$$

then

$$\sigma_{\mathrm{ess}}(\mathscr{J}) = [\underline{c_-}, \overline{c_+}]. \tag{2.3.11}$$

Using Theorem 2.10, we explicitly calculate the essential spectrum of the Jacobi matrix $\mathscr{J}_{\mathfrak{h}}$ corresponding to the Hilbert sequence $\mathfrak{h}$.

Theorem 2.11. *If $\mathscr{J}_{\mathfrak{h}}$ is the Jacobi matrix corresponding to* (2.2.27)*, then*

$$\sigma_{\mathrm{ess}}(\mathscr{J}_{\mathfrak{h}}) = [0, 1]. \tag{2.3.12}$$

PROOF. It is easily seen that for our $a_k = \frac{1}{2}$, that

$$\lim_{k\to\infty}(|a_{k+1}| - |a_k|) = \lim_{k\to\infty}\left(\frac{1}{2} - \frac{1}{2}\right) = 0. \tag{2.3.13}$$

Now for the more cumbersome b_k:

$$\begin{aligned}\lim_{k\to\infty}(|b_k+1| - |b_k|) &= \lim_{k\to\infty}\left\{\frac{k+2}{2\sqrt{(2k+3)(2k+5)}} - \frac{k+1}{2\sqrt{(2k+1)(2k+3)}}\right\}\\ &= \lim_{k\to\infty}\frac{k+2}{2\sqrt{(2k+3)(2k+5)}} - \lim_{k\to\infty}\frac{k+1}{2\sqrt{(2k+1)(2k+3)}}\\ &= \frac{1}{4} - \frac{1}{4} = 0. \end{aligned} \tag{2.3.14}$$

Now to calculate the $\underline{c_-}$,

$$\begin{aligned}\underline{c_-} &= \liminf_{n\to\infty}|c_-(n)|\\ &= \lim_{n\to\infty}|a_n - (|b_{n+1}| + |b_n|)|\\ &= \frac{1}{2} - \left(\frac{1}{4} + \frac{1}{4}\right) = 0. \end{aligned} \tag{2.3.15}$$

Now to calculate the $\overline{c_+}$,

$$\begin{aligned}\overline{c_+} &= \liminf_{n\to\infty}|c_+(n)|\\ &= \lim_{n\to\infty}|a_n + (|b_{n+1}| + |b_n|)|\\ &= \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1. \end{aligned} \tag{2.3.16}$$

Therefore, applying (2.10), specifically (2.3.11), we obtain $\sigma_{\text{ess}}(\mathscr{J}_{\mathfrak{h}}) = [0, 1]$. □

2.4. Conclusion

In conclusion, in this thesis we have explicitly determined the Jacobi matrix corresponding to the Hilbert sequence according to Akhiezer's construction. We have also found the essential spectrum of the Jacobi matrix. However, as explained in Definition 2.9, the spectrum of $\mathscr{J}_{\mathfrak{h}}$ consists of two disjoint subsets:

$$\sigma(\mathscr{J}_{\mathfrak{h}}) = \sigma_{\text{disc}}(\mathscr{J}_{\mathfrak{h}}) \cup \sigma_{\text{ess}}(\mathscr{J}_{\mathfrak{h}}). \tag{2.4.1}$$

We have not attempted to determine the discrete spectrum $\sigma_{\text{disc}}(\mathscr{J}_{\mathfrak{h}})$, which is the set of all isolated eigenvalues of $\mathscr{J}_{\mathfrak{h}}$ that have finite dimensional eigenspaces. By Stone's result [**5**, Theorem 7.13], any eigenvalue of $\mathscr{J}_{\mathfrak{h}}$ has a one-dimensional eigenspace, so $\mathscr{J}_{\mathfrak{h}}$ precisely consists of the isolated eigenvalues of $\mathscr{J}_{\mathfrak{h}}$. We suggest the following problem for future study.

PROBLEM 2.12. *Determine the isolated eigenvalues of* $\mathscr{J}_{\mathfrak{h}}$ *or show that none exist; that is, explicitly determine* $\sigma_{\text{disc}}(\mathscr{J}_{\mathfrak{h}})$.

APPENDIX A

Calculation of Cauchy Determinant

Let $\{x_j\}_{j=1}^n$ and $\{y_j\}_{j=1}^n$ be sets of positive numbers. The Hilbert sequence can be substituted with the following generic matrix by letting $x_i = i$ and $y_j = j - 1$, with $1 \leq i, j \leq k + 1$. Also, let the Cauchy determinant of order n be,

$$\begin{vmatrix} \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \cdots & \frac{1}{x_1+y_n} \\ \frac{1}{x_2+y_1} & \frac{1}{x_2+y_2} & \cdots & \frac{1}{x_2+y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_n+y_1} & \frac{1}{x_n+y_2} & \cdots & \frac{1}{x_n+y_n} \end{vmatrix} = D_n \tag{A.1}$$

[**8**] indicates that the value of D_n determinant is,

$$\frac{\prod_{1\leq i<j\leq n}(x_j - x_i)(y_j - y_i)}{\prod_{1\leq i,j\leq n}(x_i + y_j)} \tag{A.2}$$

Now we will prove by induction that the Hilbert sequence is Cauchy,

Proof. Base case n = 1: $\{x_1\}$, $\{y_1\}$. Then,

$$D_1 = |\frac{1}{x_1 + y_1}| = \frac{1}{x_1 + y_1}$$
$$\frac{\prod_{1\leq i<j\leq 1}(x_j - x_i)(y_i - y_j)}{\prod_{1\leq i,j\leq 1}(x_i + y_j)} = \frac{1}{x_1 + y_1}. \tag{A.3}$$

Since the numerator is an empty product. Inductive Step: Let $n \in \mathbb{N}$ and $\{x_j\}_{j=1}^n$, $\{y_j\}_{j=1}^n$ Assume

$$D_{n-1} = \frac{\prod_{1\leq i<j\leq n-1}(x_j - x_i)(y_i - y_j)}{\prod_{1\leq i,j\leq n-1}(x_i + y_j)}, \text{ where } \{x_j\}_{j=1}^{n-1}, \{y_j\}_{j=1}^{n-1} \tag{A.4}$$

Let D_n denote the determinant in (A.1). We perform row operations on the matrix under the determinant in (A.1). Subtract the first column from the kth column for $2 \le k \le n$. Then the $(j,1)$ entry of the resulting matrix is

$$\frac{1}{x_j + y_k} - \frac{1}{x_j + y_1} = \frac{y_1 - y_k}{x_j + y_1} \frac{1}{x_j + y_k} \tag{A.5}$$

if $k \neq 1$. If $k = 1$, then the $(j,k) = (j,1)$ entry is

$$\frac{1}{x_j + y_1}, \quad 1 \le j \le n.$$

Since adding a multiple of one column to another does not chance the value of the determinant, we have

$$\begin{vmatrix} \frac{1}{x_1+y_1} & \frac{y_1-y_2}{x_1+y_1}\frac{1}{x_1+y_2} \cdots & \frac{y_1-y_k}{x_1+y_1}\frac{1}{x_1+y_k} & \cdots & \frac{y_1-y_n}{x_1+y_1}\frac{1}{x_1+y_n} \\ \frac{1}{x_2+y_1} & \frac{y_1-y_2}{x_2+y_1}\frac{1}{x_2+y_2} \cdots & \frac{y_1-y_k}{x_2+y_1}\frac{1}{x_2+y_k} & \cdots & \frac{y_1-y_n}{x_2+y_1}\frac{1}{x_2+y_n} \\ \vdots & & \vdots & \ddots & \vdots \\ \frac{1}{x_n+y_1} & \frac{y_1-y_2}{x_n+y_1}\frac{1}{x_n+y_2} \cdots & \frac{y_1-y_k}{x_n+y_1}\frac{1}{x_n+y_k} & \cdots & \frac{y_1-y_n}{x_n+y_1}\frac{1}{x_n+y_n} \end{vmatrix} \tag{A.6}$$

Factor $\frac{1}{x_j+y_1}$ from the jth row for each $1 \le j \le n$, and factor $y_1 - y_k$ from the kth column for each $2 \le k \le n$ to obtain:

$$D_n = [\prod_{1\le j\le n} \frac{1}{x_j + y_1}][\prod_{2\le j\le n} (y_1 - y_k)] \begin{vmatrix} 1 & \frac{1}{x_1+y_1} \cdots & \frac{1}{x_1+y_k} & \cdots & \frac{1}{x_1+y_n} \\ 1 & \frac{1}{x_2+y_1} \cdots & \frac{1}{x_2+y_k} & \cdots & \frac{1}{x_2+y_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1}{x_n+y_1} \cdots & \frac{1}{x_n+y_k} & \cdots & \frac{1}{x_n+y_n} \end{vmatrix} \tag{A.7}$$

Subtract row 1 from row j for $2 \le j \le n$ in the matrix under the determinant in (A.7). The jth tow of the new matrix is:

$$\begin{aligned} j = 1 &: (1 \quad \frac{1}{x_1+y_2} \cdots \frac{1}{x_1+y_k} \cdots \frac{1}{x_1+y_n}) \\ 2 \le j \le n &: (0 \quad \frac{1}{x_j+y_2} - \frac{1}{x_1+y_2} \cdots \frac{1}{x_j+y_k} \frac{1}{x_1+y_k} \cdots \frac{1}{x_j+y_n} - \frac{1}{x_1+y_n}) \end{aligned} \tag{A.8}$$

Then (A.8) becomes,

$$D_n = [\prod_{1 \le j \le n} \frac{1}{x_j+y_1}][\prod_{1 \le j \le n} (y_1 - y_k)] \times \begin{vmatrix} 1 & \frac{1}{x_1+y_1} \cdots & \frac{1}{x_1+y_k} & \cdots & \frac{1}{x_1+y_n} \\ 0 & \frac{x_1-x_2}{(x_2+y_2)(x_1+y_2)} \cdots & \frac{x_1-x_2}{(x_2+y_k)(x_1+y_k)} & \cdots & \frac{x_1-x_2}{(x_2+y_n)(x_1+y_n)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{x_1-x_n}{(x_n+y_2)(x_1+y_2)} \cdots & \frac{x_1-x_n}{(x_n+y_k)(x_1+y_k)} & \cdots & \frac{x_1-x_n}{(x_n+y_n)(x_1+y_n)} \end{vmatrix} \tag{A.9}$$

Factor $x_1 - x_j$ from the jth row of the matrix in (A.9) for $2 \le j \le n$. Factor $\frac{1}{x_1+y_k}$ from the kth column of the matrix in (A.9) for $2 \le k \le n$. Then

$$D_n = [\prod_{1 \le j \le n} \frac{1}{x_j+y_1}][\prod_{2 \le k \le n} (y_1 - y_k)][\prod_{2 \le j \le n} (x_1 - x_j)][\prod_{2 \le k \le n} \frac{1}{x_1+y_k}] \tag{A.10}$$

$$\times \begin{vmatrix} 1 & \frac{1}{x_1+y_2} \cdots & \frac{1}{x_1+y_k} & \cdots & \frac{1}{x_1+y_n} \\ 0 & \frac{1}{x_2+y_2} \cdots & \frac{1}{x_2+y_k} & \cdots & \frac{1}{x_2+y_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1}{x_n+y_2} \cdots & \frac{1}{x_n+y_k} & \cdots & \frac{1}{x_n+y_n} \end{vmatrix}.$$

Expand the determinant on the right hand side in (A.10) about the first column.

$$D_n = [\prod_{1\leq j\leq n} \frac{1}{x_j + y_1}][\prod_{2\leq k\leq n} (y_1 - y_k)][\prod_{2\leq j\leq n} (x_1 - x_j)][\prod_{2\leq k\leq n} \frac{1}{x_1 + y_k}] \tag{A.11}$$

$$\times \begin{vmatrix} \frac{1}{x_1+y_2} \cdots & \frac{1}{x_1+y_k} & \cdots & \frac{1}{x_1+y_n} \\ \frac{1}{x_2+y_2} \cdots & \frac{1}{x_2+y_k} & \cdots & \frac{1}{x_2+y_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_n+y_2} \cdots & \frac{1}{x_n+y_k} & \cdots & \frac{1}{x_n+y_n} \end{vmatrix},$$

which is an $(n-1)x(n-1)$ Cauchy matrix with sets $\{x_j\}_{j=1}^{n-1}$ and $\{y_j\}_{j=1}^{n-1}$ where

$$x_j = x_{j+1}, \quad y_j = y_{j+1}, \quad 1 \leq j \leq n-1$$

This determinant is calculated using the induction hypothesis, and is equal to the determinant in (A.11)

$$= \frac{\prod_{1\leq j<k\leq n-1}(x_{k+1} - x_{j+1})(y_{k+1} - y_{j+1})}{\prod_{1\leq j,k\leq n-1}(x_{j+1} + y_{k+1})}$$

$$= \frac{\prod_{2\leq j<k\leq n}(x_k - x_j)(y_k - y_j)}{\prod_{2\leq j,k\leq n}(x_j + y_k)}. \tag{A.12}$$

Hence by (A.10, A.11, A.12)

$$D_n = [\prod_{1\leq j\leq n} \frac{1}{x_j + y_1}][\prod_{2\leq k\leq n} (y_1 - y_k)][\prod_{2\leq j\leq n} (x_1 - x_j)][\prod_{2\leq k\leq n} \frac{1}{(x_1 + y_k)}] \tag{A.13}$$

$$\times \frac{\prod_{2\leq j<k\leq n}(x_k - x_j)(y_k - y_j)}{\prod_{2\leq j,k\leq n}(x_j + y_k)}.$$

We are able to combine a few of these products to produce a simpler result:

$$[\prod_{1\leq j\leq n} x_j + y_1][\prod_{2\leq k\leq n} (x_1 + y_k)][\prod_{2\leq j,k\leq n} (x_j + y_k)] = [\prod_{1\leq j,k\leq n} (x_j + y_k)] \tag{A.14}$$

Another useful reduction:

$$[\prod_{2\leq k\leq n} y_1 - y_k][\prod_{2\leq j\leq n}(x_1 - x_j)][\prod_{2\leq j<k\leq n}(x_k - x_j)(y_k - y_j)] = [\prod_{1\leq j<k\leq n}(x_k - x_j)(y_k - y_j)] \quad \text{(A.15)}$$

Finally, combing (A.13, A.15, and A.16) we yield

$$D_n = \frac{\prod_{1\leq i<j\leq n}(x_j - x_i)(y_j - y_i)}{\prod_{1\leq i,j\leq n}(x_i + y_j)} \quad \text{(A.16)}$$

□

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