CONTINUUM MODELING OF THE DECELERATION
TRANSIENT STATE IN STOCHASTIC TRAFFIC FLOW

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A Dissertation Submitted to the Faculty of the University of Tennessee at Chattanooga in Partial Fulfillment of the Requirements of the Degree of Doctor of Philosophy in Computational Engineering

The University of Tennessee at Chattanooga
Chattanooga, Tennessee

August 2015
ABSTRACT

Due to individual driver behavior, a traffic system is subject to many stochastic factors. Deterministic partial differential equations and their extensions, traditionally used in traffic flow modeling, may not be sufficient in applications such as real-time estimation and prediction of traffic flow conditions. In previous studies, the issue of physical relevance has received little attention in efforts to introduce a stochastic component into deterministic equilibrium-based traffic models. In this work, the stochastic component is derived directly from realistic driver behavior implemented as a fully discrete fine-grained agent-based model and combined into the deterministic Aw-Rascle system of equations via ensemble averaging. The same approach can be applied to any second-order traffic model of a similar form. Solutions for the stopping and deceleration cases are obtained using the second-order Lax-Wendroff scheme and compared to the results obtained from the agent-based simulation.
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Throughout the world, urban populations are increasing for both economic and social reasons. The increase in GDP, population growth, decrease in fuel prices and cost of vehicle ownership all result in an increased number of vehicles on the roads. The resulting traffic congestion in urban areas poses many challenges. As a consequence of increasing number of traffic jams, cities are facing growing pollution and a higher frequency of accidents. A report released by Seattle-based INRIX and the Centre for Economics and Business Research estimates that, in 2013, the cost of direct and indirect losses due to traffic congestion in the USA was $124 billion, and predicts this number will rise to $186 billion by 2030 [1].

Reliable computational tools have become invaluable in prediction and resolution of problematic traffic situations. The idea behind traffic modeling is that, if the behavior of a traffic system can be correctly predicted given an initial set of data, by identifying critical areas, adjustments can be made to maximize the overall throughput of traffic along a stretch of road. Over the years, many different theoretical models of traffic flow have been developed. Generally, they fall into one of the two types: continuum (macroscopic) models, and car-following (microscopic) models. The former is the analogue of hydrodynamic models of fluid flow, while the latter is the analogue of the microscopic models in statistical mechanics [2]. Because microscopic models distinguish and trace individual driver-vehicle units, they are computationally very intensive and time consuming.
On the other hand, years of research in the field of fluid dynamics have resulted in very effective and efficient computational techniques for obtaining solutions to large continuum (macroscopic) models.

However, due to their nature and the nature of traffic systems, continuum models have limited fidelity in the field of traffic modeling. Traffic phenomena are complex and nonlinear, and depend on the interactions of a large number of vehicles. Due to the behavior of individual human drivers, vehicles do not interact simply following the laws of mechanics, but exhibit phenomena of cluster formation and forward and backward shock propagation. To date, there has not been a satisfactory general theory that can be consistently applied to real flow conditions.

While there have been efforts to introduce stochastic behavior in deterministic flow models, these attempts have mainly focused on the small stochastic fluctuations around the theoretical equilibrium speed. Little attention has been given to the physical relevance of the stochastic component, and the existing models do not consider large deviations from the equilibrium speed that occur in the transient phases of traffic flow (i.e. acceleration and deceleration). The goal of this work is to improve fidelity of continuum traffic models, with the primary focus on the stochastic behavior arising in the transient deceleration phase in a congested traffic stream. The model capturing the stochastic flow features developed in this work is integrated into the Aw-Rascle traffic model [3] via ensemble averaging, but it could be applied to any other continuum model.

Considering that the collection and processing of empirical traffic data are quite demanding in both time and resources, a microscopic fine-grained agent-based traffic model is also developed and used to provide realistic estimates of stochastic driver behavior. The model is a hybrid, consisting of a particle-hopping model and a fully discretized version of the Intelligent Driver
Model. It has been developed specifically for the purpose of this study, aiming to preserve computational efficiency similar to that of cellular automata, while providing the necessary level of detail and realistic behavior.

The content of this dissertation is as follows: Chapter 2 is an overview of the relevant macroscopic and microscopic traffic models. It contains a detailed description of the Aw-Rascle model and its predecessors, as well as the mechanisms behind car-following models, cellular automata models, and the Intelligent Driver Model. In Chapter 3, the ensemble averaging and the fine-grained agent-based model are discussed. The details of the numerical approach for solving the Aw-Rascle system of equations are given, as well as the details of the implementation of the fine-grained agent-based model. The specifics of the individual driver behavior are discussed, and the driver populations used in this work are described. Chapter 4 provides numerical results and comparison of the original Aw-Rascle model with the Aw-Rascle with the ensemble averaged source term and the agent-based models for the cases in the domain of the transient deceleration phase. The drawbacks of the original Aw-Rascle model are identified and discussed. In Chapter 5, a summary of the results and recommendations for possible further improvements of the continuum traffic models are given.
CHAPTER 2
LITERATURE REVIEW

Research on the subject of mathematical traffic flow modeling started with the work of Lighthill and Whitham in 1955 [4,5], in which they proposed a continuity equation model based on the analogy of vehicles in traffic flow and particles in fluid flow. Many models describing various aspects of traffic flow operations have been developed since. These models may be classified based on different criteria, such as nature of the independent variables (continuous, discrete, semi-discrete), level of detail (submicroscopic, microscopic, mesoscopic, macroscopic), representation of the processes (deterministic, stochastic), operationalization (analytical, simulation), and scale of application (networks, stretches, links, and intersections) [6]. For the purposes of this research, the classification will be restricted to two categories: continuum models and car-following models. Their details are discussed in the following paragraphs.

2.1 Continuum Traffic Models

Continuum traffic flow models are analogous to hydrodynamic models of fluid flow. They fall in the class of macroscopic models and describe traffic as a flow without distinguishing its constituent parts. Individual vehicles are not explicitly represented, and the traffic stream is described using continuous (in time) variables such as traffic density, velocity, and flow rate. The relationship between traffic speed or flow and density has to be specified externally, and one big
drawback of this type of models is that there is no agreement on the functional form of the speed-density relationship. Continuum models are generally systems of hyperbolic partial differential equations and are classified according to the number of equations, which is typically referred to as the order of the model [6]. They can be formulated as either deterministic or stochastic: a deterministic model assumes no random behavior, while a stochastic model incorporates random processes. Deterministic models can describe processes at different levels of temporal variation, resulting in steady-state or dynamic models. Steady-state simulations assume no temporal variations in the system, while dynamic simulations are describing the spatio-temporal variations over time, and are often used for prognostic applications. In deterministic models, unique initial and boundary conditions lead to unique solutions. In stochastic models unique initial conditions lead to a different solution for each model run, which, depending on an application, may not be a desirable property. Usually, the stochastic element in the simulation is introduced via a random number generator for each trial. This method is referred to as the Monte Carlo method. Many runs with different random numbers are needed in order to estimate the probability distribution behind the random process. An alternative approach is to represent the stochastic nature of a system by the ensemble averages of the fluctuating dependent variables in the governing equations. As far as has been established by this literature review, this approach, as presented here, has never been used in traffic flow analysis. The ensemble averaging approach is expected to provide a faster and more computationally efficient way to predict the mean traffic flow, and it removes a number of objections regarding continuum models that neglect the stochastic nature of traffic, possibly extending their region of validity. Continuum models relevant to this work are discussed in more detail in the following sections.
2.1.1 Lighthill-Whitham-Richards Traffic Model (LWR)

The mathematical model, referred to as Lighthill-Whitham-Richards (LWR) model \([4, 5, 7]\), is the simplest macroscopic traffic model, and as such, is only concerned with average behavior on a large scale. It is based on the one-dimensional continuity equation,

\[
\rho_t + (\rho v)_x = 0,
\]

(2.1)

where \(\rho(x, t)\) is the density (in number of vehicles per unit length of a road at time \(t\)), and \(v\) is the traffic stream velocity. The subscripts denote partial differentiation. Since the model consists of one equation with two dependent variables, an additional relationship between traffic speed or flow and density needs to be specified for the system to be closed. For a deterministic model, it is assumed that there is a one-to-one relationship between speed or flow and density, which only exists at the traffic equilibrium state. For example,

\[
v = V_{eq}(\rho) = v_{max}(1 - \frac{\rho}{\rho_{max}}),
\]

(2.2)

where \(v_{max}\) is some maximal velocity as a property of the road, and \(\rho_{max}\) is the maximal bumper to bumper density. One big drawback in this type of models is that there is no agreement on the functional form of the speed-density (or flow-density) relationship. Different functional relationships, such as equation 2.2, result in different traffic flow models and are referred to as the fundamental diagram because they are the integral part of these models. A more detailed discussion of the fundamental diagram is provided in Appendix A.
The system consisting of 2.1 and 2.2 is a non-linear first order partial differential equation describing a hyperbolic conservation law where mass, or in this case density of traffic, is conserved. In this sense, although the distribution of vehicles may vary with time, the overall number of vehicles in the system will only depend on the flux into and out of the domain. Because the LWR model is a closed model consisting of a single equation and one algebraic relationship, it is classified as a first-order model.

LWR model has several drawbacks. Due to the implicit assumption that the system is in equilibrium, local speed is statically coupled to the density by the speed-density relation (the fundamental diagram). This implies instantaneous adaptation to new circumstances and leads to formation of shocks (equivalent to unbounded acceleration), lack of hysteresis effect, and lack of instabilities observed in real traffic [8]. In order to address some of these deficiencies, the “dynamic” fundamental diagram was introduced into the model. This results in a system of partial differential equations, where the dynamic speed-density relation accounts for driver reaction time and anticipation of traffic conditions ahead [9]. Such models are referred to as second-order models, and the Aw-Rascle model is one of them.

2.1.2 Aw-Rascle Traffic Model (AR)

As stated previously, the assumption in the LWR model is that the velocity and the car density are related by some given function. In fluid mechanics, there is no prescribed relation between speed and density, but there is an additional equation representing the conservation of momentum. With the presumption that there is a relation between the traffic pressure and the density, and that the flow is unidirectional and isentropic (i.e. the change in flow variables is small
and gradual and the value of entropy is constant), a system of two equations with two unknowns (density and speed) can be derived. The main idea behind the second-order models is to try to mimic the momentum equation instead of prescribing the fixed relationship between speed and density. However, it is important to note that the conservation of momentum has little sense in theory of macroscopic traffic flow, since vehicles which act as sinks and sources of momentum (by braking or acceleration) are not traced individually. Nonetheless, adding the second equation, which is in traffic flow theory commonly referred to as the momentum equation, has been a relatively successful attempt to repair the basic deficiencies of the LWR model [10].

Early second-order continuum traffic models did not appropriately address the anisotropy problem (i.e. unlike in subsonic flow, drivers mainly react to vehicles in front of them, not to vehicles behind them) arising in traffic flow. The Payne-Whitham (PW) type models [11, 12] attempted to replace the so-called “pressure” term in the momentum equation with an anticipation factor describing how an average driver would react to a variation of density of cars with respect to space. The PW model is given by

\[
\begin{align*}
\rho_t + (\rho v)_x &= 0 \\
v_t + vv_x + \frac{c_0^2}{\rho} \rho_x &= \frac{V_{eq}(\rho) - v}{\tau}
\end{align*}
\]

where \(v\) and \(\rho\) are traffic speed and density, the constant \(c_0 > 0\) is the so-called traffic sound speed, \(V_{eq}(\rho)\) is the equilibrium speed, and \(\tau\) is the speed adaptation time. The definition of traffic sound speed is the speed of waves that propagate the information through the traffic flow minus the speed of traffic that carries these waves. The speed and density are dimensionless, scaled by the maximum
speed and density which are determined by the properties of the road on which the model is run. The system 2.3 has two characteristic speeds \( \lambda_{1,2}(\rho, v) = v \mp c_0 \). Since \( v \geq 0 \) and \( c_0 > 0 \), \( \lambda_1 < \lambda_2 \) and \( \lambda_2 > 0 \). This type of a model exhibits characteristic speeds greater than the speed of the traffic flow implying that a driver will react to stimuli coming from the rear and move backward.

At this point, Daganzo published a paper titled “Requiem for Second Order Fluid Approximations of Traffic Flow” [13], strongly criticizing the second-order models for their fundamental logical flaws. As a response, Aw and Rascle published their work under the title “Resurrection of Second-Order Models of Traffic Flow?” [3], and proposed a model that overcomes the issues noted by Daganzo. Independently and following a different rationale, Zhang introduced a similar model somewhat later [14]. In the literature, these models are often combined and referred to as the Aw-Rascle-Zhang model. In order to fix the issue of negative velocities, Aw and Rascle suggested an alternative equation to be coupled with the continuity equation [3]. They argue that, due to the qualitative differences between traffic flow and fluid flow, it is insufficient to simply replace the pseudo pressure term with the anticipation factor (subsequently, consistent with the common traffic flow theory practice the term “pseudo” will be dropped). They note that solving the Riemann Problem for the PW-type models will produce intermediate states with negative velocities. Since the characteristic speeds in the PW model will always result in some part of information traveling faster than the velocity of cars, they note that the main issue with the PW-type models is that the anticipation factor involves the derivative of pressure with respect to \( x \). To demonstrate the incorrectness of this assumption, they offer the following example: “Assume for instance that in front of a driver traveling with speed \( v \) the density is increasing with respect to \( x \), but decreasing with respect to \( (x - vt) \). Then the PW type of model predicts that this driver would slow down,
since the density ahead is increasing with respect to $x$! On the contrary, any reasonable driver would accelerate, since this denser traffic travels faster than him." [3]. They conclude that the correct expression must involve the convective derivative $\partial_t + v \partial_x$ of pressure $P$, where pressure is still an increasing function of the density. The pressure function considered in the original paper by Aw and Rascle is of the form

$$P = P(\rho) = \rho^\gamma, \gamma > 0$$

where $\gamma$ is a nonphysical parameter determining the shape of the pressure function. Aw and Rascle state that the only qualitatively important conditions are the behavior of this function near the vacuum and the (strict) convexity of the function $\rho P(\rho)$. Therefore, their results are presented under the following assumptions:

$$P(\rho) \sim \rho^\gamma \text{ near } \rho = 0$$

$$\gamma > 0$$

$$\forall \rho, \rho \frac{d^2P(\rho)}{d\rho^2} + 2\frac{dP(\rho)}{d\rho} > 0$$

Aw and Rascle (in Section 4 of their paper, [3]) extensively discussed the lack of stability of the solutions for vanishingly small density, sometimes referred to as vacuum problems. They claim that the model can explain instabilities in car flow observed near a vacuum, i.e. for very light traffic with few slow drivers. However Lebacque et al. [15] noted that the solutions to the Riemann problem defined by the AR system of equations is in fact not always guaranteed, the consequences being that the model does not admit solutions for certain initial conditions which can arise in real traffic situations, and that the model cannot be discretized using the Godunov scheme.
Lebacque showed that the inverse equilibrium speed-density relationship $V_{eq}^{-1}(\cdot)$ must be properly extended to avoid difficulties at high densities AND the instabilities that appear near vanishingly small densities. For details, refer to [15].

The homogeneous form of the AR model is given by:

$$
\begin{align*}
\rho_t + (\rho v)_x &= 0 \\
(v + P(\rho))_t + v(v + P(\rho))_x &= 0
\end{align*}
$$

(2.4)

In conservative form, in addition to density $\rho$, the second conserved variable is introduced: $y = \rho(v + P)$. This variable represents the relative flow, i.e. the difference between the actual flow $q = \rho v$, and the equilibrium flow $\rho V_{eq}(\rho) = Q_{eq}(\rho)$ [15]. The conservative form of the homogeneous system is given by

$$
\begin{align*}
\rho_t + (y - \rho P)_x &= 0 \\
y_t + \left(\frac{y^2}{\rho} - y P\right)_x &= 0
\end{align*}
$$

(2.5)

The two-equation conservative model in vector form is given by

$$
\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{R}
$$

where

$$
\mathbf{u} = \begin{pmatrix} \rho \\ y \end{pmatrix}, \quad \mathbf{f}(\mathbf{u}) = \begin{pmatrix} y - \rho P \\ \frac{y^2}{\rho} - y P \end{pmatrix}, \quad \mathbf{R} = 0
$$
which then can be written in the quasi-linear form as:

\[
\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} = 0
\]

where

\[
A(u) = \frac{\partial f}{\partial u} = \begin{pmatrix}
-(\gamma + 1)P & 1 \\
-\frac{y^2}{\rho^2} - \frac{\gamma P y}{\rho} & \frac{2y}{\rho} - P
\end{pmatrix}
\]

The eigenvalues for the AR model are real and distinct: \(\lambda_1 = v - \rho \frac{dP(\rho)}{d\rho}\), \(\lambda_2 = v\). Therefore, the system is strictly hyperbolic, except in vacuum when \(\rho = 0\). Since \(P(\rho)\) is an increasing function, it is ensured that \(\lambda_1 \leq \lambda_2\), and the faster wave will always move at the speed equal to the velocity of the vehicles and no faster. The model is consistent with discrete car following models [16, 17] since they assume that a reaction of each driver to the distance to the next car means (at the macroscopic level) that the correct modeling involves the convective derivative of the density, and not its derivative with respect to \(x\). Aw and Rascle [3] also recommend adding the relaxation term in the momentum equation to overcome the deficiency of the maximal speed reached by vehicles on an empty road being dependent on the initial data. Therefore, a more complete model includes the relaxation term of the form \(\frac{V_{eq}(\rho) - v}{\tau}\), where \(\tau\) is a relaxation or adjustment time and \(V_{eq}(\rho)\) is a preferred velocity corresponding to each density \(\rho\). In this case, the system becomes inhomogeneous. Since the maximum wave speed equals the speed of the traffic flow, the model possesses the anisotropic property.

For the purposes of this work, it is assumed that the non-homogeneous form of the AR model modified by Lebacque et al. provides a general form of a second-order system and will be used
as such. The modifications developed for the AR model are applicable to any second-order traffic model. Additionally, the pressure function is modified according to the work presented in [18]. In this paper, the pressure is referred to as the hesitation function \( h(\rho) \), a term more consistent with the physics of traffic flow. Various authors ([19, 20]) proposed to choose the hesitation function dependent on the equilibrium speed function, i.e. \( h(\rho) = V_{eq}(0) - V_{eq}(\rho) \). In this case, the solution relaxes towards the equilibrium speed \( V_{eq}(\rho) \). In this work, the same approach is used. To avoid any confusion, from now on, the term “pressure” will continue to be used to describe the function \( P(\rho) = V_{eq}(0) - V_{eq}(\rho) \).

### 2.2 Car-Following Models

Car-following traffic models describe the mechanisms of one vehicle following another. Since vehicles are modeled individually, these models are classified as microscopic. Also known as time-continuous, these models generally use a set of ordinary differential equations describing the dynamics of the vehicle positions \( x_\alpha \) and velocities \( v_\alpha \). It is assumed the input stimuli of the drivers are restricted to their own velocity \( v_\alpha \), the bumper-to-bumper distance \( s_\alpha = x_{\alpha-1} - x_\alpha - l_{\alpha-1} \) to the leading vehicle \( \alpha - 1 \) (where \( l_{\alpha-1} \) denotes the vehicle length), and the velocity \( v_{\alpha-1} \) of the leading vehicle. The equation of motion for each vehicle is characterized by an acceleration function dependent on these stimuli: \( \dot{v}_\alpha(t) = F (v_\alpha(t), s_\alpha(t), v_{\alpha-1}(t)) \). The driving behavior of a single driver-vehicle unit \( \alpha \) does not have to be restricted to only one, but may depend on the \( n_\alpha \) vehicles in front. Obviously, unlike continuum fluid models, the car-following models trivially satisfy anisotropy (i.e. drivers react to vehicles in front of them, not to vehicles behind them) and can easily accommodate driver and vehicle heterogeneity.
Car-following models form a bridge between individual “car following” behavior and the macroscopic view of a line of vehicles and their corresponding flow and stability properties [6]. The speed-density relationship still has to be specified externally. However, as discussed in Appendix A, the single-valued speed-density relationship is a theoretical time-independent relation that can only be defined for steady-state homogeneous traffic. Since the speed adjustment for traffic conditions in car-following models is not instantaneous, as it is assumed in the first-order continuum models, the speed-density relationship only holds for the steady-state flow consisting of identical driver-vehicle units.

In [6], three types of car-following models are described, namely safe-distance models, stimulus-response models, and psycho-spacing models. Safe-distance models describe the dynamics of each vehicle in relation to the vehicle in front of it. The approach suggested by Forbes at al. [21] is based on the very simple Pipes’ rule [22]: “A good rule for following another vehicle at a safe distance is to allow yourself at least the length of a car between you and the vehicle ahead for every ten miles an hour of speed at which you are traveling”. Later, a more realistic stimulus-response model was proposed Leutzbach [23]. This approach assumes that there is a stimulus-response relationship that describes the control process of a driver-vehicle unit. The desired distance between vehicles should depend on driver’s perception time, decision time, and braking time, where the braking time is a function of the driver’s reaction time and the maximal deceleration of the vehicle. In stimulus-response models, the follower aims to conform to the behavior of the leading vehicle. The stimulus is a result of the velocity difference between the leader and the follower, and the response is braking or acceleration. However, the model has two main drawbacks: it presumes that the following driver will react to very small changes in relative
velocity, even when the headway (i.e. the distance between the front of a vehicle and the front of the next vehicle in units of length) is large, and that, if there is no difference in velocity, there will be no response regardless of the distance between the cars. Psycho-spacing models aim to correct this by incorporating insights from perceptual psychology [24] showing that drivers are subject to limits in their perception of the stimuli to which they respond. Furthermore, it was shown by Krauss et al. [25] that, in order for a microscopic model to more accurately represent the capacity drop, slow-to-start driving rules had to be added. This addition enabled car-following models to more accurately represent the congested traffic flow [6].

Compared to the continuum approach, microscopic models are in general computationally very intensive, especially for large-scale simulations. Car-following models implementing both time and space discretization and known as the particle hopping models, began to receive more attention after the formulation proposed by Nagel and Schreckenberg [26]. These models resemble the cellular automata models, which makes them computationally efficient and ideal for parallel implementations. However, the simple rules that contribute to the efficiency also decrease the fidelity of these models.

In the following subsections, car-following models relevant to this work are discussed in more detail.

2.2.1 Intelligent Driver Model (IDM)

The Intelligent Driver Model is a time-continuous car-following model for the simulation of freeway and urban traffic. The model was developed by Treiber et al. [27]. It is a deterministic follow-the-leader model formulated as an ordinary differential equation and characterized by an
acceleration function \( \dot{v} = \frac{dv}{dt} \) that depends on the actual vehicle speed \( v(t) \), the gap \( s(t) \), and the velocity difference \( \Delta v(t) \) to the leading vehicle. By the virtue of being an agent-based model, the IDM can easily incorporate different driving characteristics for different classes of drivers. The model has also been extended to include psychological aspects of driving (i.e. driver frustration after spending time in a traffic jam), and can model driving behavior with a high degree of realism.

However, the implementation via time-continuous partial differential equations and the heavy logic involved in modeling of the realistic driver behavior result in a very computationally intensive model. As such, in its original form, the model was not appropriate for the work presented here, and has been modified as presented in Chapter 3.

### 2.2.2 Nagel-Schreckenberg Particle-Hopping Model

Theoretically, the methodology of particle hopping models lies between fluid-dynamical and car-following theories and helps clarify the connections between these two approaches [28]. Particle hopping models can be formulated as either stochastic or deterministic. On a very coarse scale, the discretized space of a microscopic model approximates continuous space, and the fluctuations due to stochastic behavior become smaller by the averaging effect. This is known as the hydrodynamic limit of the microscopic models [28]. The traffic system is represented by a lattice of cells of equal size \( \Delta x \), and the time is discretized into time steps \( \Delta t \). Generally, the behavior of the driver-vehicle units in these models is described by a set of if-then update rules of the form

\[
\begin{align*}
v_{\alpha}^{t+1} &= f \left( s_{\alpha}^{t}, v_{\alpha}^{t}, v_{\alpha-1}^{t}, \ldots \right) \\
x_{\alpha}^{t+1} &= x_{\alpha}^{t} + v_{\alpha}^{t+1}
\end{align*}
\]
The time scale is given by the reaction time of a human driver, $\Delta t = 1\, s$. Typically, the cell size is 7.5 meters (typical length of a road occupied by a vehicle at standstill). The movement of vehicles from cell to cell is discrete, and a vehicle can only assume a limited number of discrete velocity values ranging from zero to $v_{\text{max}}$, where $v_{\text{max}}$ is the maximum number of cells that can be traversed by the vehicle in one time step (generally taken to be 5, corresponding to $135\, \frac{km}{h}$). At each time step, all particles are updated either synchronously or asynchronously. If all particles are updated simultaneously (synchronous, parallel update), then such particle hopping models are similar to cellular automata (CA) (for in-depth discussion of CA see [29]).

Similarities with the CA models and the ability to, in some cases, take advantage of bit arithmetic, make particle hopping models ideal for computationally efficient parallel implementations. Unfortunately, very simple driving rules which contribute to the efficiency also decrease the fidelity of these models. The coarse discretization leads to a limited number of speeds that can be assumed by the vehicles and the unrealistic smallest acceleration of $5\, \frac{m}{s^2}$. As such, the particle hopping model in its original form was unsuitable for the purpose of this work and needed to be further extended. The extension is presented in Chapter 3.

### 2.3 Stochastic Traffic Flow

A variety of applications of traffic simulation require probabilistic models of traffic flow. These include estimation of traffic conditions along freeways and signalized arterials when measurement data are limited (whether in real-time or not) and short-term predictions of traffic flow (such as adaptive traffic signal control). A large body of literature in the field of traffic modeling focuses on short-term predictions of traffic systems. Different approaches have been taken, such as the use
of time series [30,31], Kalman filtering [32], pattern recognition [33], least-square algorithm [34], Brownian motion [35], and stochastic partial differential equations [36]. As noted in [37], the issue that received little attention in the literature pertaining to the stochastic modeling of traffic flow is that of the physical relevance of the stochastic component. If the stochastic behavior is added to a deterministic traffic model, the usual approach is to add noise terms to the model, either to the conservation equation [38] or to the dynamic speed-density relationship [39–41]. However, when the deterministic dynamics are nonlinear, it cannot be guaranteed that the result of these approaches will be consistent with the original deterministic dynamics [37]. In their work, Jabari and Liu [37] propose to base the stochastic model on the randomness of headway choices.

While there is a clear need for a physical basis for the stochastic component in the deterministic traffic flow, the method of ensemble averaging applied directly to the equations of the second-order traffic models, has not been used prior to this work. The details of this approach are discussed in the following chapter.
CHAPTER 3

METHODOLOGY

Traffic phenomena are complex and nonlinear, and depend on the interactions of a large number of vehicles. Due to the behavior of individual human drivers, vehicles do not interact simply following the laws of mechanics, but exhibit phenomena of cluster formation and forward and backward shock propagation, depending on vehicle density in a given area. Nonetheless, to date, there has not been a satisfactory general theory that can be consistently applied to real flow conditions. There are, however, common spatio-temporal empirical features of traffic flow that are qualitatively the same for different highways in different countries measured during years of traffic observations. Microscopic processes constituting the characteristics of a traffic flow take time and depend on the behaviors of individual drivers (i.e. time to adjust speed when traffic conditions change). However, it is generally assumed that if the conditions remain unchanged for a sufficient period of time, traffic flow will converge to an average state, referred to as the equilibrium state [42]. In traffic flow theory, the equilibrium state is expressed in the form of the fundamental diagram. Classical traffic flow theory, based on the fundamental diagram, distinguishes two phases of traffic flow, free-flow and congested traffic (fig. 3.1).
In free traffic flow, empirical data show a positive correlation between the flow rate and vehicle density. This relationship stops at the maximum free flow ($Q_{\text{max}}(\rho)$) corresponding to critical density ($\rho_{\text{crit}}$). This point divides the empirical data on the flow-density plane into two regions: the free-flow region on the left side, and the congested region on the right. In the congested region, the vehicle speed is lower than the lowest vehicle speed encountered in free flow. The interactions between individual drivers become more prominent, their individual behaviors have greater effect on the drivers directly behind them, and as a consequence, on the overall traffic flow. From the empirical data, it can be observed that many of the collected data points are not on the fundamental diagram. Some of these points can be explained by stochastic fluctuations around the equilibrium (e.g. different sizes of vehicles, individual drivers have different desired speeds and following distances). Others, however, are structural, and result from the dynamic properties.
of traffic flow. They reflect the transient states, that is, acceleration and deceleration phase. These changes in the traffic state are not described by the fundamental diagram, and the mean speeds observed in the transient states will generally be different from the equilibrium speed. In fact, the term “equilibrium” reflects the fact that the observed speeds will eventually, over sufficient time, converge to the equilibrium speed, given that the average traffic conditions remain the same. In other words, the average speed does not instantaneously adapt to the equilibrium speed [42]. As discussed in Section 2.3, there have been various efforts to introduce the stochastic behavior in deterministic traffic flow models, mainly in the LWR model. These attempts have mainly been focused on the stochastic fluctuations around the equilibrium speed. As noted in [37], an issue that has received little attention is that of the physical relevance of the stochastic component, and to the knowledge of the author, the structural fluctuations have never been considered separately. The goal of the work described in this chapter is to derive the stochastic component directly from the realistic driver behavior of a non-homogeneous driver population implemented as an agent-based traffic model. The primary focus is on the stochastic behavior arising in the transient deceleration phase in the congested traffic, i.e. a stream of vehicles approaching an obstacle on the road causing them to decelerate or stop. This case emphasizes the microscopic structural fluctuations, as opposed to the fluctuations around the equilibrium present in the free flow regime. The stochastic model is integrated into the Aw-Rascle traffic model, but it could be applied to any other macroscopic traffic model. In chapter 4, the impact of the stochastic component is evaluated by comparing the speed of the resulting backward-propagating stopping wave in the original AR model and the AR model with the stochastic ensemble averaged component, and comparing them to the wave speed in the agent-based model.
3.1 Aw-Rascle System of Equations - Summary

For the reasons noted in Section 3.1.3, the following form of the Aw-Rascle model with the modification by Lebacque et al. [15] will be used in this work:

\[
\begin{align*}
\rho_t + (y - \rho P)_x &= 0 \\
y_t + \left( \frac{y^2}{\rho} - y P \right)_x &= \rho \left( \frac{V_{eq}(\rho) - v}{\tau} \right)
\end{align*}
\]

The pressure function is defined as \( P(\rho) = V_{eq}(0) - V_{eq}(\rho) \), with the appropriate equilibrium function. In the vector form, the model is given by

\[
u_t + f(u)_x = R
\]

where

\[
u = \begin{pmatrix} \rho \\ y \end{pmatrix}, \quad f(u) = \begin{pmatrix} y - \rho P \\ \frac{y^2}{\rho} - y P \end{pmatrix}, \quad R = \begin{pmatrix} 0 \\ \rho \left( \frac{V_{eq}(\rho) - v}{\tau} \right) \end{pmatrix}
\]

and written in the quasi-linear form as:

\[
\frac{\partial \nu}{\partial t} + A(u) \frac{\partial \nu}{\partial x} = 0
\]

where

\[
A(u) = \frac{\partial f}{\partial \nu} = \begin{pmatrix} -\rho \frac{dp}{d\rho} + P & 1 \\ -\frac{y^2}{\rho^2} - y P' & 2y \rho - P \end{pmatrix}
\]
Since it is rare that higher order numerical schemes are used in traffic modeling, the system will be solved using the second-order Lax-Wendroff numerical scheme (Section 3.1.3). The use of the second-order upwind scheme with Roe flux vector splitting shown to be impractical because of the difficulties arising with the definition of the Roe averages with different pressure functions [43]. Namely, the calculation of the Roe average $\tilde{v}$ involves solving a quadratic equation, which is not guaranteed to have a solution for all choices of $P(\rho)$.

### 3.1.1 Convergence of the Aw-Rascle to the Lighthill-Whitham-Richards Traffic Model

Analogous to the discussion found in [18], consider the homogeneous Aw-Rascle equation given by equation 2.5, with $y$ defined as $y = \rho(v + P(\rho))$. Given a decreasing LWR velocity function $V_{eq}(\rho)$, define $P(\rho) = V_{eq}(0) - V_{eq}(\rho)$ (clearly $P'(\rho) > 0$ and $P(0) = 0$). Then it is obvious that, if the vehicle velocity in the AR model $v(\rho) = V_{eq}(\rho)$, $y = \rho$, and both equations in 3.1 reduce to

$$\rho_t + (\rho - \rho P(\rho))_x = 0$$

Replacing $P(\rho)$ with $P(\rho) = V_{eq}(0) - V_{eq}(\rho)$ (with the scaled maximum velocity $V_{eq}(0) = 1$) results in

$$\rho_t + (\rho V_{eq}(\rho))_x = 0$$

which is the LWR model.

In the inhomogeneous case, given by equation 3.1, the speed relaxation term $\rho \left( \frac{V_{eq}(\rho) - v}{\tau} \right)$ adds temporal relaxation of local speed towards the $V_{eq}(\rho)$ driving it towards the LWR equilibrium velocity function $V_{eq}(\rho)$ unless the system is driven away from the equilibrium by another effect. If
the second equation of the system 3.1 is multiplied by \( \tau \) and the limit \( \tau \to 0 \) is applied, the equation reduces to the LWR condition \( v = V_{eq}(\rho) \). This is plausible since the limit \( \tau \to 0 \) implies that the speed is tightly coupled with density (i.e. speed adaptation is instantaneous), which is a defining feature of the class of first-order (LWR) models.

3.1.2 The Influence of the Speed Relaxation Source Term on the Solution of the Aw-Rascle System of Equations

The speed relaxation (or speed adaptation) source term, inherited from the Payne-Whitham model (eqn. 2.3), is generally added to the Aw-Rascle model in order to overcome the deficiency of the maximal speed reached by vehicles on an empty road being dependent on the initial data. In that sense, if the the local speed is below or above the equilibrium speed, the source term provides acceleration or deceleration respectively. The parameter \( \tau \) represents a characteristic speed adaptation time in which the distance to the equilibrium is \( 1/\exp \) times the original distance [44]. Depending on the traffic context (city streets, minor or major roads, highways), the speed adaptation time is of the order of few seconds on city streets and up to 20-30s on highways.

The following test cases examine the impact of different values of \( \tau \) on the solution of the inhomogeneous Aw-Rascle model (eqn. 3.1). The system is solved using the second-order Lax-Wendroff numerical method (Section 3.1.3), with the \( CFL = 0.25 \) and \( \Delta x = 0.0025 \).

The first case is that of the initial condition with the speed on the left below the equilibrium speed (fig. 3.2). In this case, the adaptation term will act as acceleration. After 500 time steps (equivalent to 115 seconds), the solution is compared to the solution initialized with the speed equal to the equilibrium speed, in which the adaptation source term is inactive.
First, note that without the adaptation term and the initial speed below the equilibrium speed, the computed solution is nonphysical (fig. 3.3). Due to the initial velocity value, the density wave ($\rho = 1.0$) is not propagating correctly towards left, but a wave of a lower density (corresponding to the initial velocity $v_R = 0.1$) is formed.
In figure 3.4, the results after 500 time steps from the non-equilibrium initial condition are compared with the solution initialized with the equilibrium speed. Two different values of \( \tau \) are used, 10 and 30 seconds. Since the initial speed is below the equilibrium speed, both density waves propagate more slowly than in the solution started from equilibrium. Additionally, since the adaptation time is exponential, in the case where \( \tau = 30 \) seconds, the speed has not yet been relaxed to the equilibrium speed.

(a) Density/speed after 10 seconds

(b) Density/speed after 30 seconds

Figure 3.4  Adaptation from low speed to the equilibrium speed, adaptation time \( \tau \) 10 and 30 seconds, simulation time: 500 time steps (115 seconds)
The second case is initialized with the speed above the equilibrium speed, and the speed adaptation term provides deceleration. As in the first case, without the adaptation term and without the equilibrium initial speed, a nonphysical solution is obtained. The initial conditions are given in fig. 3.5, and the results in fig. 3.6. In this case, the top of the density wave is propagating faster, as expected. However, since the speed profile was not exactly the same in the initial condition, the speed corresponding to the bottom region of the density wave in the case of the non-equilibrium initial condition retained the sharper gradient which resulted in the faster propagation of the bottom part of the density wave. It should also be noted that, in the first case, the speed wave profiles in the initial condition matched exactly, however, the adaptation still caused deformation which resulted in the irregular propagation speed of the top and the bottom regions of the density wave (fig. 3.4).

![Figure 3.5 Initial condition: initial density: $\rho_L = 0.28, \rho_R = 1.0$; initial velocity: $v_R = 0.5(v_R = V_{eq}(\rho_R)$ for the equilibrium case), $v_L = 0.0$](image)
3.1.3 Second-Order Lax-Wendroff Numerical Scheme

The Lax-Wendroff method for the solution of hyperbolic partial differential equations is an explicit finite-difference method, second-order accurate in both space and time. For a general non-linear equation

\[
\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u(x,t))}{\partial x} = 0
\]
the conservative form of Lax-Wendroff for a general nonlinear equation is given by

$$ u_{i}^{n+1} = u_{i}^{n} - \frac{\Delta t}{2\Delta x} \left[ f \left( u_{i+1}^{n} \right) - f \left( u_{i-1}^{n} \right) \right] + \frac{\Delta t^2}{2\Delta x^2} \left[ A_{i+\frac{1}{2}} \left( f \left( u_{i+1}^{n} \right) - f \left( u_{i}^{n} \right) \right) - A_{i-\frac{1}{2}} \left( f \left( u_{i}^{n} \right) - f \left( u_{i-1}^{n} \right) \right) \right] $$

where $A \left( u \right) = f' \left( u \right) = \frac{\partial f}{\partial u}$ is the flux Jacobian, and $A_{i\pm\frac{1}{2}}$ is the Jacobian matrix evaluated at $\frac{1}{2} \left( U_{i}^{n} + U_{i\pm\frac{1}{2}}^{n} \right)$. Subscript $i$ denotes the spatial index, and $n$ is the temporal index.

The Lax-Wendroff scheme is conditionally stable, with the Courant-Friedrichs-Lewy (CFL) stability condition given by $\frac{c \Delta t}{\Delta x} \leq 1 \iff c \leq \frac{\Delta x}{\Delta t}$, i.e. it is stable if and only if the physical velocity $c$ is not greater than the spreading velocity of the numerical method, $\frac{\Delta x}{\Delta t}$. In other words, the time step $\Delta t$ must be smaller than the time taken for the theoretical wave to travel the distance of the spatial step $\Delta x$.

Since the Lax-Wendroff method is a finite-difference scheme, both dissipation and dispersion errors are present, and since it is a second-order scheme, the dispersion error is dominant. Therefore the Lax-Wendroff method can produce spurious oscillations near sharp gradients and a phase lag. For more details see [45–47].

With the CFL number of 0.25, and the mesh discretization $\Delta x = 0.0025$, this did not cause problems in the test cases presented in Chapter 4.
3.2 Fine-Grained Agent-Based Model

3.2.1 Nagel-Schreckenberg Particle-Hopping Model and the Fine-Grained Agent-Based Model

Following the work of Nagel and Schreckenberg [26, 28], so called particle-hopping models began receiving attention. They are time and space discretized car-following models, describing the mechanism of one vehicle following another in a very coarse manner. Theoretically, the methodology of particle-hopping models lies between fluid-dynamical and car-following theories and helps clarify the connections between these two approaches [28]. On a very coarse scale, the discretized space of a microscopic model approximates continuous space, and the fluctuations due to stochastic behavior are being reduced by the averaging effect. This is known as the hydrodynamic limit of the microscopic models [28]. In a particle-hopping model, each small section of the road can either be occupied by a vehicle (particle) or be empty. At each time step, all particles are updated either synchronously or asynchronously, which leads to different model behaviors. If all particles are updated simultaneously (synchronous or parallel update), the particle-hopping model is similar to cellular automata (CA) (for an in-depth discussion of cellular automata, see [29]). However, because particle-hopping models are defined by the dynamics they are supposed to describe, they are not identical to CA [28]. Similarities with the CA models and the ability to, in some cases, take advantage of bit arithmetic, makes particle-hopping models ideal for computationally efficient parallel implementations. As such, they are widely used in traffic modeling, especially for large simulations of urban areas. The Nagel-Schreckenberg model, only describes a minimal set of driver behavior. Each particle (vehicle) occupies one 7.5 meter cell (assuming a typical car length of 5 meters + headway) and can assume an integer velocity between 0 and $v_{max}$, defined as the number
of cells the particle can “hop” during one time step. The model includes the randomization step at each update which is designed to reduce a vehicle’s velocity with some probability. This serves to incorporate three properties of human driving: fluctuations at maximum speed, overreactions at braking, and noisy acceleration. If the vehicle is accelerating, each time step the speed will be increased by 1 cell until the desired speed is reached. If we assume a time step to be one second, this results in a linear acceleration of \(7.5 \frac{m}{s^2}\). When a vehicle needs to brake due to an obstacle in front of it, it will move the available number of cells and come to a sudden stop. Additionally, the model assumes the maximum vehicle speed to be 5 cells per time step. This would result in the maximum speed of \(37.5 \frac{m}{s}\) (135 \(\frac{km}{h}\)) and a very coarse scale of possible speeds. While this approach allows for very fast run times even on large problems (entire city areas), the physical characteristics of individual vehicles are completely lost, and the fidelity of the model is drastically reduced. This may work if one is only interested in phenomenological large scale questions. However, it is not suitable for modeling heterogeneous driver populations and details of physical interactions between vehicles.

The fine-grained agent model used here is based on the approach taken to develop the fine-grid particle hopping scheme recently presented in the area of pedestrian traffic modeling [48]. The goal of the fine-grid scheme is to be able to represent smoother and more accurate movements of pedestrians of different shapes. The evaluations of the model have shown that it is able to match the empirical pedestrian speed-density data with good accuracy. This approach translates well to vehicular traffic. While the evaluations of the fine-grid particle hopping scheme for pedestrian traffic reported that the simulation speed was compromised in favor of higher accuracy, vehicular traffic problems are essentially one-dimensional, and the fine-grid scheme is still able to retain
some of the benefits of simple computation. Compared to the Nagel-Schreckenberg model, the fine-grained model allows for a much finer speed adjustment. In the work reported here, the length of a road section (or a cell) is defined to be $\Delta x = 0.5m$, which allows for more realistic acceleration and deceleration values ($0.5 m/s^2$), and a much finer resolution of available driver speeds.

### 3.2.2 Car-Following Intelligent Driver Model and Fine-Grained Agent-Based Model

Traditionally, car-following models are time-continuous and defined by ordinary differential equations describing the dynamics of the vehicles' positions and velocities. As explained previously, the set of very simple driving rules implemented in a particle-hopping model, while contributing to the efficiency, decreases the fidelity of the model. Being able to, for example, simulate deviations from the optimal driving behavior (identified by maximum speed, braking probability, time headway between drivers, comfortable acceleration and deceleration, etc.) appears to be a crucial aspect of traffic modeling, as they lead to jam formations and influence the maximum flow out of jams [49].

In order to add the physical characteristics that will differentiate between different types of drivers, a discretized version of the acceleration formula from the Intelligent Driver Model (IDM) [50] is used here within the fine-grained agent model to calculate the speed changes caused by the driver’s environment. The more sophisticated features, such as decision making, were not used in this work although could be added in the future. The overtaking functionality was implemented based on the MOBIL algorithm [51], however, the vehicles tended to separate into clusters based on the driver characteristics before reaching the equilibrium speed (i.e. the faster drivers tended to end up at the front of the traffic stream). While this is a realistic behavior, it was an additional complication since the clustering among different types of drivers prevents development of a uniform average
equilibrium speed-density wave. As such, the overtaking functionality was not appropriate in the cases used to compare the Aw-Rascle model with the agent-based model and is not used.

The IDM acceleration is a continuous function which, besides the distance to the leading vehicle $s_\alpha$ and the actual speed $v_\alpha$, also takes into account velocity differences $\Delta v_\alpha$, which play an essential stabilizing role in real traffic, especially when approaching traffic jams and avoiding rear-end collisions. The suggested IDM acceleration function [50] is given by

$$\frac{dv_\alpha}{dt} = f(s_\alpha, v_\alpha, \Delta v_\alpha) = a \left[ 1 - \left( \frac{v_\alpha}{v_0} \right)^\delta - \left( \frac{s^* (v_\alpha, \Delta v_\alpha)}{s_\alpha} \right)^2 \right]$$

In this expression, the acceleration strategy towards a desired speed $v_0$ on a free road $\dot{v}_{\text{free}}(v) = a \left[ 1 - \left( \frac{v_\alpha}{v_0} \right)^\delta \right]$ is combined with a braking strategy $\dot{v}_{\text{brake}}(s, v, \Delta v) = -a \left( \frac{v_\alpha}{s} \right)^2$, which acts as a repulsive force when a vehicle comes too close to the vehicle ahead. The parameter $a$ is the acceleration rate. If the distance to the leading vehicle is large, the interaction term $\dot{v}_{\text{brake}}$ is negligible and the IDM equation is reduced to the free-road acceleration $\dot{v}_{\text{free}}$, which is a decreasing function of the velocity with the maximum value $\dot{v}(0) = a$, and the minimum value $\dot{v}(v_0) = 0$ at the desired speed $v_0$. The acceleration exponent $\delta$ specifies how the acceleration decreases when approaching $v_0$. The limiting case $\delta \to \infty$ is equivalent to approaching $v_0$ with a constant acceleration $a$, and $\delta = 1$ corresponds to an exponential relaxation to the desired speed. In [50], it is stated that the most realistic behavior is expected between the two limiting cases, and the acceleration exponent is set to 4 in the IDM model. Therefore, in this project, $\delta = 4$ is also used.

For a denser traffic, the deceleration term is relevant and depends on the ratio between the effective
desired minimum gap $s^*$ and the actual gap $s_a$. The effective desired minimum gap is defined by

$$s^*(v, \Delta v) = \left( s_0 + vT + \frac{v\Delta v}{2\sqrt{ab}} \right)$$

The minimum distance $s_0$ is only significant for low velocities in congested traffic. In stationary traffic, the main contribution is the term $vT$ which corresponds to following the leading vehicle with a constant desired time gap $T$. The last term is only active in non-stationary traffic when $\Delta v \neq 0$ and implements a braking strategy that in nearly all situations limits braking decelerations to the comfortable deceleration $b$. The IDM will brake stronger than $b$ if the gap becomes too small. This particular strategy makes the IDM collision free. For more details and extensions, see [50] and [52]. While not used in this work, instead of the formula, a coarse version of the discretized acceleration and deceleration values could be stored as a look-up table to avoid the repeated calculations and improve computational efficiency.

3.2.2.1 Effects of Discretization on the Intelligent Driver Model

The Intelligent Driver Model evaluated fairly well against real traffic cases [50]. Since the equation used for speed in the IDM is an ordinary differential equation, it has to be solved numerically. Generally this is done by using an explicit numerical scheme assuming constant acceleration within each update time interval $\Delta t$. Typically $\Delta t$ is chosen between 0.1 and 0.2 seconds.

For the purpose of comparing the effect that the time and space discretization used in the fine-grained agent-based model have on the equations of the IDM, the IDM equations are solved
numerically and the solution is compared against the values obtained from the agent-based model.

The expression for acceleration given by 3.2

$$\frac{dv_\alpha}{dt} = f(s_\alpha, v_\alpha, \Delta v_\alpha)$$  \hspace{1cm} (3.3)

coupled with the general equation of motion

$$\frac{dx_\alpha}{dt} = v_\alpha$$

represents a (locally) coupled system of ordinary differential equations (ODEs) for the positions $x_\alpha$ and velocities $v_\alpha$ of all vehicles $\alpha$. As the considered acceleration function $f$ is nonlinear, the set of the ODEs has to be solved by means of numerical integration. As noted in [50], in the context of car-following models, it is natural to use an explicit scheme and assume constant accelerations within each update time interval $\Delta t$. The explicit update rule

$$v_\alpha (t + \Delta t) = v_\alpha (t) + \dot{v}_\alpha (t) \Delta t$$

$$x_\alpha (t + \Delta t) = x_\alpha (t) + v_\alpha (t) \Delta t + \frac{1}{2} \dot{v}_\alpha (t) (\Delta t)^2$$

is used, consistent with the work presented in [50]. $\dot{v}_\alpha$ is an abbreviation for the acceleration function $f(s_\alpha, v_\alpha, \Delta v_\alpha)$. For $\Delta t \to 0$, the scheme locally converges to the exact solution of 3.3 with consistency order 1 for the velocities, equivalent to the first order Euler scheme used here (fig. 35).
3.7), and consistency of order 2 for the positions ("modified Euler update") [50] with respect to the $L^2$-norm. As the typical update time interval for the IDM is 0.1s and 0.2s, $\Delta t = 0.1$ is used here.

![Convergence of the IDM acceleration function solved using the explicit first-order Euler method. $L^2$-norm with respect to the solution obtained with $\Delta t = 1.0E - 10$](image)

The effects of the discretization on IDM are presented in detail in Appendix B.

In conclusion, the error incurred by discretization is small and physically negligible, and due to the incorporated elements from the IDM, the fine-grained agent-based model, developed specifically for the purpose of this work, is able to better approximate the realistic driver behavior than the simple particle-hopping model, while still preserving some of the computational efficiency of the CA models. Further improvements in the performance of the model can be made by coarser discretization of the acceleration values and avoidance of repeated calculations, but at the expense of an increase in error compared to the IDM solutions.
3.3 Multi-Class Driving Behavior and Driver Population

As stated previously, deviations from the optimal driving behavior appear to be the crucial aspect of traffic modeling [49]. Specifically, the deviation in driver behavior lead to jam formations and variable acceleration rates influence the maximum flow out of jams, which in turn produce phenomena observed in the empirical traffic data. Jensen et al. [53] proposed a general framework to classify driver performance in six different groups: timid, cautious, conservative, neutral, assertive, and aggressive. Extreme classifications, timid and aggressive, constitute dangerous driving behavior, while assertive and cautious behaviors may be classified as unsafe. For example, characteristics associated with the assertive behavior, like tailgating, speeding above the traffic flow, or rapidly changing lanes, are all unsafe. Similarly, the characteristics of cautious driving, such as traveling below the speed of traffic to maintain the minimum posted speed limit, over-scanning before making turns or lane changes, and not anticipating traffic patterns while maintaining vehicle speed, all may be classified as unsafe. In their work, Jensen at al. assume the normal distribution for general driver behavior, and the percentages of each class of drivers are based on the number of standard deviations about the mean (table 3.1).

By the virtue of being a microscopic model, the fine-grained agent-based model allows easy configuration of individual driver characteristics. In order to address the heterogeneity of the real traffic, and following the work of Jensen et al. [53], the driver characteristics in this work are implemented using the parameters given in [50]. The classification is simplified to three categories, namely “timid”, “normal” and “aggressive” (table 3.1), and the parameters associated with each group, consistent with [50] are given in table 3.2.
Table 3.1  Driver safety classification based on behind-the-wheel operating behaviors with assumed Gaussian distribution population percentages [53]

<table>
<thead>
<tr>
<th>Population Percentage</th>
<th>5%</th>
<th>20%</th>
<th>50%</th>
<th>20%</th>
<th>5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Safety Level</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Characteristics</td>
<td></td>
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<tr>
<td>- too slow</td>
<td></td>
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<tr>
<td>- unconfident</td>
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<td>- disturbs</td>
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<tr>
<td>- traffic flow</td>
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<tr>
<td>- unpredictable</td>
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<td></td>
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<tr>
<td>- always obeys speed limits regardless of traffic flow</td>
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<tr>
<td>- not instinctive</td>
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<tr>
<td>- over scans</td>
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<tr>
<td>- follows traffic flow</td>
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<tr>
<td>- predictable</td>
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<td>- confident</td>
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<td>- proper scanning technique</td>
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<td>- routinely speeds</td>
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<tr>
<td>- tailgates</td>
<td></td>
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<tr>
<td>- under scans</td>
<td></td>
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<tr>
<td>- too fast</td>
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<tr>
<td>- overconfident</td>
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<td>- unpredictable</td>
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</tbody>
</table>

Driver Classification in FGAB model

<table>
<thead>
<tr>
<th>Driver Classification</th>
<th>Timid</th>
<th>Cautious</th>
<th>Conservative</th>
<th>Neutral</th>
<th>Assertive</th>
<th>Aggressive</th>
</tr>
</thead>
</table>

Table 3.2  IDM parameters used to distinguish different classes of drivers [50]

<table>
<thead>
<tr>
<th>IDM Parameter</th>
<th>Timid</th>
<th>Normal</th>
<th>Aggressive</th>
</tr>
</thead>
<tbody>
<tr>
<td>Desired speed $v_0$ in km/h</td>
<td>100</td>
<td>120</td>
<td>140</td>
</tr>
<tr>
<td>Desired time gap $T$ in s</td>
<td>1.8</td>
<td>1.5</td>
<td>1.0</td>
</tr>
<tr>
<td>Jam distance $s_0$ in m</td>
<td>4.0</td>
<td>2.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Maximum acceleration $a$ in m/s$^2$</td>
<td>1.0</td>
<td>1.4</td>
<td>2.0</td>
</tr>
<tr>
<td>Desired deceleration $b$ in m/s$^2$</td>
<td>1.0</td>
<td>2.0</td>
<td>3.0</td>
</tr>
</tbody>
</table>

The parameter $s_0$ describes the minimum bumper-to-bumper distance at standstill. Typical gaps in a queue of vehicles standing at traffic lights are in the range between $1m$ and $5m$. While a normal driver typically keeps a minimum gap of $2m$, a cautious driver prefers larger gaps and an aggressive driver likes tailgating. Vehicle length is not a model parameter. It is assumed that a typical length of a passenger car is $5m$. The desired acceleration $a$ describes the acceleration behavior of the driver. Since the acceleration behavior is based on a physical movement, the value of $a$ has to be physically reasonable. The acceleration exponent $\delta = 4$ is kept constant for all driver cases. The comfortable braking deceleration $b$ determines how the driver will approach slower
leaders or stationary objects such as traffic lights or obstacles on the road. The IDM tries to limit the braking deceleration to $b$, although in some cases it will exceed $b$ in order to avoid collisions.

The heterogeneous synthetic driver population used in the simulations cases here is generated from the total driver population (500 vehicles in each case) and assigning the characteristics based on a random selection from a normal distribution (fig. 3.8), $\mu = 0.0, \sigma = 3.0$.

![Figure 3.8 Assumed Gaussian distribution for the synthetic driver population in heterogeneous case](image)

In the test cases where homogeneous driver population is assumed, the synthetic population consists only of “normal” drivers. In the next section, the fundamental diagrams for the homogeneous and the heterogeneous populations are presented.
3.3.1 Fundamental Diagrams for the Homogenous and Heterogeneous Driver Populations

As discussed in Chapter 2, the idealized form of the flow, speed, density relationship in traffic theory is referred to as the fundamental diagram. While there is no true agreement on its functional form, macroscopic traffic modeling presumes the existence of a reproducible fundamental diagram [4, 5, 13, 54] with an understanding that the fundamental diagram is only a theoretical relation between density and flow in stationary homogeneous traffic, i.e. the steady state equilibrium of identical driver-vehicle units [8]. In this work, the functional relationship used for the fundamental diagram is based on [55] and is of the form:

\[ V_{eq}(\rho) = \frac{\pi}{2} \arctan \left( \frac{f_1(\rho - f_2)}{\rho - 1} \right) + \arctan(f_1f_2) \]

The parameters \( f_1 \) and \( f_2 \) specify the exact shape of the curve and are derived from the experimental data sets using Eureqa software (the software uses symbolic regression combined with a genetic algorithm to determine mathematical equations that describe sets of data in its simplest form). The data presented here are produced by the fine-grained agent-based model. For each run, 500 vehicles are generated, and in the case of the heterogeneous population, assigned their characteristics as described in section 3.3. The range of possible speeds (0 to \( 38 \frac{m}{s} \) for the heterogeneous, and 0 to \( 34 \frac{m}{s} \) for the homogeneous population) is discretized in integer values. The first vehicle (the leader) entering the simulation domain is assigned the integer speed for which the equilibrium density is wanted, and is not allowed to accelerate or decelerate. The following vehicles are then placed behind the first vehicles at relatively small random distances with the initial speed half of the first vehicle’s speed. This is done in order to control the initial density. Since all
the vehicles, except for the leader, are allowed to freely adjust their speed, large initial separation due to the acceleration phase would prevent the formation of the equilibrium wave. At 700 seconds, the traffic stream at a certain length (domain of influence) behind the leader becomes stable, speed synchronization is achieved, and the state of the traffic stream is recorded (fig. 3.9). The simulation is repeated 2000 times, and the results are ensemble averaged. At this point, the equilibrium density is measured and associated with the leader’s speed. It needs to be noted that, for higher speeds, the domain of influence (i.e. the length at which the vehicles are affected by the speed of the leader) is very short (approximately 4-5 cars are affected at the speeds close to the maximum speed). The domain of influence becomes longer as the speed decreases. For speed $0\frac{m}{s}$, it becomes infinite.

Figure 3.9 Fine-grained agent-based model, homogeneous driver population, leader’s speed: $12\frac{m}{s}$. Speed and density are normalized by the maximum speed ($34\frac{m}{s}$) and maximum density ($0.7143, 5m$ car length + $2m$ jam spacing)
Since two synthetic populations are used in this work (homogeneous and heterogeneous), two different fundamental diagrams are used (figures below).

Figure 3.10  Equilibrium flow and speed for the homogeneous driver population, $f_1 = 5.4$, $f_2 = 0.15$

Figure 3.11  Equilibrium flow and speed for the heterogeneous driver population, $f_1 = 3.5345$, $f_2 = 0.27896$
The driving parameters for the timid and aggressive drivers are more or less symmetrical with the mean values close (but not exactly the same) to the normal driver values, and these drivers compose roughly 50% of the total population. However, from the graphs above, it can be seen that the maximum equilibrium flow increases for the heterogeneous population. This illustrates the effects of the small differences in the microscopic parameters on the overall macroscopic system.

### 3.4 Ensemble Averaged Aw-Rascle Equations

#### 3.4.1 Ensemble Averaging

Due to parameters of drivers’ behavior, vehicular flow is inherently stochastic. The observed variations in speed can be treated as noise. Noise is a stochastic process consisting of variations in functions of time and space, and therefore is only statistically characterized. In a noisy system, one cannot argue a single event at a certain time or position. Only averaged quantities of a single system over a certain space or time interval or the averaged quantity of many identical systems at a certain time instance or spatial position can be discussed. The former is referred to as space or time average, and the latter as ensemble average (fig. 3.12).

The concept of an ensemble average is based on the existence of an independent statistical event. The common simple example to illustrate the ideas behind averaging is the experiment with coin flipping. If value of 1 is assigned to a head and value of 0 to a tail, then the arithmetic average of the numbers generated by coin flipping is defined as

\[ X_N = \frac{1}{N} \sum_{n=1}^{N} x_n \]

where the \( n^{th} \) flip is denoted as \( x_n \), and \( N \) is the total number of flips. If all the coins are the same, flipping a single (not the same) coin \( N \) times, or \( N \) coins a single time would be equivalent. However, flipping the same coin \( N \) times would not, because the events would not be independent. Now consider the case where 100
coins are flipped. If the coins are divided in groups of 10, then each of these groups is an ensemble having a probability value \( x_n \) associated with it, and this value will differ from an ensemble to an ensemble. As the size of the ensemble increases, the closer the value gets to the expected value \( X_N = 0.5 \), because, as the \( N \) becomes larger, the fluctuations in the outcome decrease.

![Figure 3.12 Time average vs. ensemble average](image)

The ensemble average of \( x \), denoted as \( \langle x \rangle \), is defined as

\[
X = \langle x \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} x_n
\]  
(3.4)

Generally, the \( x_n \) can be realizations of any random variable \( x \), and \( \langle x \rangle \) defined in eqn. 3.4 is its ensemble average. The quantity \( \langle x \rangle \) is also sometimes referred to as the expected value of the random variable \( x \), or simply as its mean. Returning to the case of traffic flow, the vehicle velocity at a given point in space \((x, t)\) in a stochastic flow can be considered to be a random variable \( v_t(x, t) \).
If there was a large number of identical experiments \((n)\), so that the \(v_i^{(n)}(x, t)\) in each of them was identically distributed, the ensemble average of \(v_i^{(n)}(x, t)\) is given by

\[
\langle v_i(x, t) \rangle = V_i(x, t) \equiv \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} v_i^{(n)}(x, t)
\]

While in the case of ergodic dynamical systems, which broadly speaking, have the same behavior averaged over time as averaged over the space of all the system’s states, the ensemble average is the same as the time average. However, in a non-stationary flow this is not the case, and the ensemble averaging has to be applied to the case of the stochastic traffic flow. Formally, it is possible to consider taking the limit \(N \to \infty\), but in practice, in the context of either experiments or numerical simulations, \(N\) is necessarily finite. So the question becomes “How large must \(N\) be before \(\langle v \rangle\) no longer changes with increasing \(N\)?” Practical experience in fluid dynamics has shown that the convergence occurs very slowly [56].

In the context of traffic modeling, deriving ensemble averages from a real traffic data would be nearly impossible. This is where the idea of using the microscopic traffic simulation becomes important. This work is not focused on the calibration and validation of the microscopic traffic models, and it is assumed that there is a microscopic model that approximates the realistic driver behavior fairly well. The fine-grained agent-based model, discussed in section 3.2, is used as such an approximation, even though it is understood that it may not be perfect. It does, however, take into account realistic variability present in the real driver behavior and derived from empirical observations [50]. Additionally, since the large number of experiments is necessary for the derivation of the ensemble averages, it is desirable that such a microscopic model be as
computationally efficient as possible, which in the case of the fine-grained agent-based model is achieved by both time and space discretization. Turning back to the question of the number of runs necessary to obtain the convergence of the ensemble averages, it was experimentally determined that, after 2000 independent simulations, in the case of a traffic stream coming to a stop, the convergence is achieved. It needs to be noted that with an increase of variability in driver behavior, or introduction of more classes of drivers, this number will necessarily become larger.

A fluctuating variable can be represented as the sum of its mean value and the fluctuation. For example, if the fluctuating variable is velocity \( v \), then

\[
v(x, t) = \overline{v(x, t)} + v'(x, t)
\]

where the \( \overline{v(x, t)} \) denotes the mean, and \( v'(x, t) \) the fluctuating part (or the perturbation). It can be easily seen that the average of the fluctuation is zero, i.e., \( \langle v' \rangle = 0 \).

However, the ensemble average of the square of the fluctuation is not zero, and is in fact the variance of the variable, denoted by \( \langle (v')^2 \rangle \) or \( \text{var}[v] \). Mathematically, the variance is defined as

\[
\text{var}[v] = \langle (v')^2 \rangle = \langle [v - \overline{v}]^2 \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} [v_n - \overline{v}]^2
\]

Therefore, the variance, like the ensemble average, can never really be measured as it would require an infinite number of members of the ensemble. From the definition of ensemble average, the variance can be written as

\[
\text{var}[v] = \langle v^2 \rangle - \overline{v}^2
\]
or, in other words, the variance is the second moment minus the square of the first moment (or the mean), and in this naming convention, the ensemble mean is the first moment. The variance is also related to the standard deviation of the random variable, in this case, \( v \):

\[
\sigma_v^2 = \text{var}[v]
\]

The term “root mean square” or RMS is often used as a synonym for standard deviation. Statistically, in cases where a discrete random variable \( X \) takes random values from a finite set of data \( x_1, x_2, \ldots, x_N \), with each value having the same probability, the standard deviation is defined as

\[
\sigma(x) = \sqrt{\frac{1}{N} \sum_{n=1}^{N} (x_n - \mu)^2}
\]

and \( \mu \) is the mean.

The RMS value of a set of data \( x_1, x_2, \ldots, x_N \), is defined as

\[
x_{\text{RMS}} = \sqrt{\frac{1}{N} \sum_{n=1}^{N} x_n^2}
\]  \hspace{1cm} (3.5)

If the set of data points \( x_1, x_2, \ldots, x_N \) in eqn. 3.5 is replaced by the fluctuation in a random variable \( X \) (i.e. in general terms \( [x_n - \bar{x}] \)) it is clear that the RMS value corresponds to the definition of standard deviation.
The covariance between two random variables $X$ and $Y$ can also be defined in these terms as:

$$
\sigma(x, y) = \frac{\sum_{n=1}^{N} (x_n - \bar{x})(y_n - \bar{y})}{\sqrt{\sum_{n=1}^{N} (x_n - \bar{x})^2} \sum_{n=1}^{N} (y_n - \bar{y})^2} = \frac{\langle (x_n - \bar{x})(y_n - \bar{y}) \rangle}{x_{RMS}y_{RMS}}
$$

Additionally, to facilitate further discussion and application of ensemble averaging to the Aw-Rascle system of equations, it is necessary to define the set of conventional rules of averaging known as the Reynolds conditions [57]:

- The average of the sum is the sum of the averages
  $$\bar{f + g} = \bar{f} + \bar{g}$$

- Constants do not affect and are not affected by averaging
  $$\bar{af} = af, \text{where } a = const.$$

- $\bar{a} = a, \text{where } a = const.$
  (these properties imply that the averaging operator is linear)

- The average of the time or space derivative of a quantity is equal to the corresponding derivative of the average
  $$\bar{\left( \frac{\partial f}{\partial s} \right)} = \frac{\partial \bar{f}}{\partial s}, \text{where } s \text{ can be either a space coordinate or time;}$$

- The average of the product of an average and a function is equal to the product of the averages
  $$\bar{\bar{fg}} = \bar{f} \bar{g}$$

These requirements are just that, the requirements that are imposed on the averaging operator, and they cannot be derived. They exist for the reason of practical simplicity, and the averaging operators
that do not satisfy the Reynolds conditions are too complex to be of much use. From the equations above, additional properties can be derived:

\[
\overline{f'} = 0
\]

\[
\overline{f'g} = \bar{f} \bar{g}
\]

\[
\overline{fg'} = 0
\]

\[
\frac{\partial f'}{\partial s} = 0, \text{ where } s \text{ can be either a space coordinate or time}
\]

### 3.4.2 Ensemble Averaged Aw-Rascle Equation

Consider the conservative form of the non-homogeneous Aw-Rascle equation with the speed relaxation source term:

\[
\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} \left( y - \rho P \right) = 0
\]

\[
\frac{\partial}{\partial t} y + \frac{\partial}{\partial x} \left( \frac{y^2}{\rho} - yP \right) = \frac{\rho \left( V_{eq}(\rho) - v \right)}{\tau}
\]

(3.6)

where \( y = \rho(v + P) \), \( P = 1 - V_{eq}(\rho) \), and \( \tau \) is the speed relaxation time.
Separating each variable in its mean and fluctuating parts, i.e. \( x = \bar{x} + x' \), the system 3.6 becomes:

\[
\frac{\partial}{\partial t} (\bar{\rho} + \rho') + \frac{\partial}{\partial x} \left( (\bar{y} + y') - (\bar{\rho} + \rho')(\bar{P} + P') \right) = 0
\]

\[
\frac{\partial}{\partial t} (\bar{y} + y') + \frac{\partial}{\partial x} \left( \frac{(\bar{y} + y')^2}{(\bar{\rho} + \rho') - (\bar{y} + y')(\bar{P} + P')} \right) = \frac{(\bar{\rho} + \rho')(V_{eq}(\rho) - (\bar{u} + u'))}{\tau} \tag{3.7}
\]

Applying the ensemble averaging operator and the rules of ensemble averaging, and ignoring the moments higher than the second moment, the ensemble averaged form of the equation with the fluctuations is obtained:

\[
\frac{\partial}{\partial t} \bar{\rho} + \frac{\partial}{\partial x} \left( \bar{y} - \rho \bar{P} \right) - \frac{\partial}{\partial x} \rho'P' = 0
\]

\[
\frac{\partial}{\partial t} \bar{y} + \frac{\partial}{\partial x} \left( \bar{y} \frac{\bar{y}'^2}{\bar{\rho}} - \bar{y} \bar{P}' \right) + \frac{\partial}{\partial x} \left( \frac{\bar{y}'^2}{\bar{\rho}} - \frac{2 \bar{y} \rho' y'}{\rho^2} - \frac{y' P'}{\rho} \right) = \frac{\bar{\rho} (V_{eq}(\rho) - u) - \rho' P' - \rho' u'}{\tau} \tag{3.8}
\]

The bar notation for the conserved variables will be subsequently dropped. In the vector form, the model is given by

\[
u_t + f(u)_x = R
\]

where

\[
u = \begin{pmatrix} \rho \\ y \end{pmatrix}, \quad f(u) = \begin{pmatrix} y - \rho P \\ \frac{y^2}{\rho} - y P \end{pmatrix}, \quad R = \begin{pmatrix} \frac{\partial}{\partial x} \rho' P' \\ \frac{\rho (V_{eq}(\rho) - u) - \rho' P' - \rho' u'}{\tau} - \frac{\partial}{\partial x} \left( \frac{y'^2}{\rho} - \frac{2y \rho' y'}{\rho^2} - y' P' \right) \end{pmatrix}
\]
and written in the quasi-linear form as:

\[
\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x} = 0
\]

where

\[
A(\mathbf{u}) = \frac{\partial f}{\partial \mathbf{u}} = \begin{pmatrix}
-(\rho \frac{\partial P}{\partial \rho} + P) & 1 \\
\frac{y^2}{\rho^2} - y \frac{\partial P}{\partial \rho} & \frac{2y}{\rho} - P
\end{pmatrix}
\]

The second-order Lax-Wendroff scheme, as presented in Section 3.1.3, is used to numerically solve the system 3.8. The speed adaptation source term is evaluated at the current time step, and the source terms involving the ensemble averaged terms are calculated using the central differences, i.e.

\[
f'(x) = \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} + O(\Delta x^2)
\]

The question that arises now is how to derive the fluctuation terms from the fine-grained agent-based model.

### 3.4.3 Ensemble Averaged Values in the Fine-Grained Agent-Based Model

In this section, the application of the theoretical elements described in section 3.4.1 is discussed. From the ensemble averaged Aw-Rascle model (eqn. 3.8), the variances and covariance terms involving the AR variables are needed to complete the model. This requires ensemble averaging of data produced by the fine-grained agent-based model. Each simulation case carried out with the fine-grained agent-based model is considered an independent event. The synthetic agent population is generated and initialized as described in section 3.3. The length of the domain
is set to 10km, and a stationary obstacle is placed at 8024.5m from the beginning of the domain. Except for the leading vehicle, all vehicles are allowed to freely adapt their speed (accelerate and decelerate). The leading vehicle sets the speed for the incoming traffic stream. After an amount of time, a stable (equilibrium) speed/density region of traffic develops behind the leading vehicle, and eventually encounters the obstacle that forces it to slow down and stop. The goal now is to capture the velocity of the vehicles in the region in which they are slowing down and stopping and relate it to density. In order to do this, a measuring point is placed in the agent-based domain 6km from the beginning of the domain. It is important to place the measuring point far enough from the beginning of the domain so that the incoming traffic has enough time to develop the equilibrium wave, and close enough to the obstacle so that the backward-propagating stopping wave can be captured. Each time a vehicle passes the point, its speed is recorded and associated with the local density in front of it. The agent-based domain is discretized in cells 500 meters in length over which the density is measured, as this is consistent with the placement of detector devices that are in practice used to collect traffic data. While the literature suggests that, on an empty straight road, drivers might scan distances in front of them that are slightly longer than this, 500 meters was used here. The local density is calculated as the ratio of the length currently occupied by the vehicles and the total length of the cell (500 meters). Both speed and density are normalized by the maximum speed and density the model allows and recorded together with the time step at which they are measured. The same experiment is carried out 2000 times, and the ensemble averaging of the recorded data is performed over each recorded time step. The RMS and covariance values are calculated according to the equations presented in the previous section, then normalized by the
corresponding ensemble averaged means. This implies, when the terms are combined with the AR system, that they need to be denormalized by the local (equation) variable values.

There are several technicalities with respect to the AR variables $y$ and $P$ that need to be considered. These values are not physical in the sense that they cannot be derived directly from observations (only velocity and density can be directly recorded). The variable $y = \rho(v + P)$ represents the relative flow, i.e. the difference between the actual flow ($q = \rho v$) and the equilibrium flow $\rho V_{eq}(\rho) = Q_{eq}(\rho)$. The quantity $P(\rho)$ represents the “traffic pressure”, by analogy with gas dynamics, and $P(\rho) = 1 - V_{eq}(\rho)$ for the choice of the pressure function used in this work. With not much choice, these variables are calculated from the local velocity and density values for each data point recorded. Since the model assumes the existence of the equilibrium velocity, the appropriate equilibrium velocity function is used in the calculations involving pressure (as presented in section 3.3.1). The resulting ensemble averages are plotted with respect to the ensemble averaged density, and the appropriate functional approximation is used for each. The results are presented in chapter 4.
CHAPTER 4
RESULTS

This chapter provides numerical results and comparison of the original Aw-Rascle model with the Aw-Rascle with the ensemble averaged terms. Both equations are compared to the fine-grained agent-based model for the cases in the domain of the transient deceleration phase in the congested traffic flow. The numerical solutions to the Aw-Rascle system of equations are obtained using the second-order Lax-Wendroff scheme, described in section 3.1.3. The original Aw-Rascle model is given by

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (y - \rho P) = 0
\]
\[
\frac{\partial y}{\partial t} + \frac{\partial}{\partial x} \left( \frac{y^2}{\rho} - yP \right) = \rho \left( \frac{V_{eq}(\rho) - v}{\tau} \right)
\]

and the ensemble averaged version by

\[
\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x} \left( \bar{y} - \bar{\rho} \bar{P} \right) - \frac{\partial}{\partial x} \rho'\bar{P}' = 0
\]
\[
\frac{\partial \bar{y}}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\bar{y}^2}{\bar{\rho}} - \bar{y}\bar{P} \right) + \frac{\partial}{\partial x} \left( \frac{y' \bar{P}' - 2\bar{y}'y'}{\rho^2} - \frac{y'\rho'}{\rho^2} \right) = \frac{\bar{\rho} \left( V_{eq}(\rho) - v \right) - \bar{\rho}'\bar{P}' - \bar{\rho}'v'}{\tau}
\]

Variable \( y \), defined as \( y = \rho(v + P) \), represents the relative flow, i.e. the difference between the actual flow \( q = \rho v \), and the equilibrium flow \( \rho V_{eq}(\rho) = Q_{eq}(\rho) \). The pressure function
is defined as \( P(\rho) = V_{eq}(0) - V_{eq}(\rho) \), with the appropriate equilibrium function for the driver population in question given by

\[
V_{eq}(\rho) = v_{max} \left( \frac{\pi}{2} \arctan \left( f_1 \frac{\rho - f_2}{\rho + f_2} \right) + \frac{\pi}{2} \arctan(f_1f_2) \right)
\]

The parameters \( f_1 \) and \( f_2 \) specify the exact shape of the equilibrium speed curve and are derived from the data sets obtained from the fine-grained agent-based model and fitted to the appropriate functional expressions using Eureqa software (the software uses symbolic regression combined with a genetic algorithm to determine mathematical equations that describe sets of data in its simplest form) (for details, refer to section 3.3.1). The details concerning the specifics of the driver populations implemented within the fine-grained agent-based model are given in section 3.3. For convenience, the relevant fundamental diagrams are repeated here, with the congested traffic flow defined as the flow to the right of the \( \rho_{crit} \).

Figure 4.1  Equilibrium flow-density and speed-density for the homogeneous driver population, \( f_1 = 5.4, f_2 = 0.15 \)
For the methodology of measuring and collecting data from the fine-grained agent-based model, refer to section 3.4.3.

### 4.1 Homogeneous Driver Population

The main source of stochastic behavior in the fine-grained agent-based model used here is the random order of drivers on the road and their interactions. Various stochastic features certainly can be added directly to the driver specifications (such as random variation in the desired headway, or desired acceleration), but they would increase the complexity of the model and require a larger ensemble size. If the agent population is homogeneous (consisting of only one type of drivers), after reaching equilibrium, the stochastic behavior is effectively eliminated from the traffic stream. Since the Aw-Rascle model, in its original form, assumes a homogeneous driver population, this case can serve for the base comparison of the Aw-Rascle solution with the fine-grained agent-based model. All the ensemble averaged terms in this case are zero, and the model is reduced to its
original form given by equation 3.1. The fundamental diagram for the homogeneous population of “normal” drivers is given in fig. 4.1.

4.1.1 Fine-Grained Agent-Based Model; Case: Stopping

First, the agent-based solution will be presented. In order to compare the two models, the speed of the backward propagating wave will be compared. The agent-based simulation is initialized with 500 vehicles with the “normal” driver properties. For the purpose of presentation, the domain is discretized into 1000 meter cells. Since the stochastic behavior is absent and each simulation has exactly the same outcome, ensemble averaging over a number of simulations will not reduce local variations in speed and density due to driver characteristics (i.e. desired headway and jam spacing), and the space averaging over 1000 meters serves to smooth out the solution. Note that this is different from the discretization that the fine-grained agent-based model uses, which is $\Delta x = 0.5$ meters. No important features are lost. Total length of the agent-based domain is 10km. The obstacle is placed at 8024.5m from the beginning of the domain to allow the vehicles to achieve equilibrium before they are forced to stop. The maximum speed of the leading vehicle is set to $12 \frac{m}{s}$. The vehicles behind it are allowed to accelerate to their maximum speed if that is possible. Speed and density are normalized by the maximum speed ($34 \frac{m}{s}$) and the maximum density ($\frac{10}{14}$, corresponding to 10 0.5m cells occupied by a car, and 2m jam spacing). Fig. 4.3 shows the agent-based solution after 500 time steps (500 seconds). The vehicles behind the leader reached the steady-state equilibrium corresponding to the speed of $12 \frac{m}{s}$.
Figure 4.3 Homogeneous fine-grained agent-based model at 500 seconds

Figure 4.4 shows the agent-based solution at 681 seconds. At this point, the leading car reached the obstacle at 8024.5 meters and stopped. The front of the leading car at this point is at 8022.5 meters (accounting for the 2 meters of jam spacing). At the beginning of the simulation, when the leading car was first inserted in the domain, its front was positioned at 5 meters. Therefore, the car traveled from 5 meters to 8022.5 meters in 681 seconds at the average speed of $\frac{8017.5 \text{ m}}{681 \text{s}} = 11.7731 \text{ m/s}$. Note here that the acceleration phase of the leading car (from $0 \text{ m/s}$ to $12 \text{ m/s}$) is included in the travel time. If ~10 seconds of speed adaptation time is subtracted (the time taken by the leading car to accelerate to $12 \text{ m/s}$, the average speed becomes $\frac{8017.5 \text{ m}}{671 \text{s}} = 11.9485 \text{ m/s}$, which agrees with the equilibrium speed.

An important feature to notice is the difference in the speed between the agent-based solution at 500 seconds and that at 681 seconds. It can be observed that, as the leading vehicle slowed down
and stopped, it influenced all the following vehicles, and the speed of the following wave decreased below the equilibrium speed.

Figure 4.4  Homogeneous fine-grained agent-based model at 681 seconds. The leading car reached the obstacle and stopped.

Fig. 4.5 shows the agent-based solution at 703 seconds. If the solution were allowed to develop further, it would be affected by the vehicles that were not a part of the initial equilibrium wave. It can be observed that the wave of an increased density due to the traffic stream coming to a stop has started to propagate backward. At this point in time, a vehicle came to a stop at 7916.5 meters (front), its back end being at 7911.5 meters. The stopping wave propagated from the back of the leading car at 8017.5 m to the back of the last stopped car at 7911.5 m over the time of 703s - 681s = 22 seconds. Therefore, the speed of the backward propagating wave is

\[
\frac{(7911.5m - 8017.5m)}{22s} = -4.81 \frac{m}{s}.
\]
Figure 4.5  Homogeneous fine-grained agent-based model at 703 seconds. Density wave is propagating backward

From the presented results, it can be seen that, in the transient deceleration phase, the speed of the incoming traffic stream in the agent-based model is not constant. The vehicles are not in equilibrium, but rather “chasing” the equilibrium state as the density in front of them constantly changes. This behavior will be analyzed in more detail for the inhomogeneous driver population in the cases to follow.

### 4.1.2 Aw-Rascle Model; Case: Stopping

In this section, the solution obtained from the Aw-Rascle model (eqn. 3.1) will be presented. The speed adaptation time $\tau$ was set to 10 seconds. The initial condition (fig. 4.6), corresponding to the agent-based case in the previous section, is given by
\[ \rho_{Left} = 0.28, \ v_{Left} = V_{eq}(\rho_{Left}) \]

\[ \rho_{Right} = 1.0, \ v_{Right} = V_{eq}(\rho_{Right}) = 0.0 \]

Figure 4.6 Aw-Rascle model, initial condition

In order to compare the solutions obtained from the two models, the scaling between the two needs to be discussed.

As explained in the previous section, the obstacle in the agent-based simulation is placed at 8024.5 meters. The stopping density wave in the equation initial condition is at 0.66 (top) and 0.6 (bottom). The top corresponds to the position of the obstacle, and given the spatial discretization
of the domain, $\Delta x = 0.0025$, it is positioned 264 cells from the start of the domain. Therefore, the spatial scaling is $dx = \frac{8024.5m}{264} = 30.4m$. The maximum speed at which the information can propagate in the agent-based model is the maximum speed at which a vehicle can travel, i.e. $34\frac{m}{s}$. The CFL number used is 0.25. From the CFL condition ($CFL = \lambda \frac{dt}{dx}$), it follows $0.25 = 34\frac{m}{s} \frac{dt}{30.4m}$, and each time step $dt = 0.22s$.

The equation solution after 22 seconds is given in fig. 4.7. Only the propagation speed of the stopping wave is of interest here. The top of the density wave is at 0.6575, and the bottom at 0.5675. The propagation speed of the top is given by $\frac{0.6575-0.66}{0.0025} = -1 \text{cell} \times 30.4m = -30.4m$, $\frac{-30.4m}{22s} = -1.38 \frac{m}{s}$, and of the bottom by $\frac{0.5675-0.6}{0.0025} = -13 \text{cells} \times 30.4m = -395.2m \frac{-395.2m}{22s} = -17.96 \frac{m}{s}$.

Figure 4.7 Aw-Rascle model after 22 seconds (100 time steps)
Therefore, the speed at which the wave is propagating is much higher than the speed in the agent-based solution, and the equation model predicts that the incoming traffic stream will remain at equilibrium (whereas in the agent-based model solution, it was observed that the speed will decrease soon as the leading vehicle begins to slow down prior to the stop). This issue will be revisited in the next section.

4.2 Inhomogeneous Driver Population

The considered synthetic driver population in the cases to follow is inhomogeneous and as described in section 3.3. The ensemble averaged Aw-Rascle model will be evaluated on two cases in the domain of transient deceleration phase, namely the stopping case (i.e. vehicles decelerating from the traffic stream speed to zero), and the bottleneck case (i.e. vehicles decelerating to the speed of slower traffic moving in front of them). First, the ensemble averaged terms and their functional approximations will be presented, then the solutions of the Aw-Rascle model and the Aw-Rascle model with ensemble averaged source terms will be compared with the agent-based model.

4.2.1 Correlation Terms

Values for correlation terms in the transient stopping phase of the congested traffic flow are obtained directly from the fine-grained agent-based model. Each case consisted of 500 vehicles randomly selected from the normally distributed population of three types of drivers (for details, refer to section 3.3). The vehicles were allowed to reach the equilibrium density for the prescribed velocity before they encountered the obstacle and were forced to stop. The simulation of this scenario is repeated 2000 times, and the results are ensemble averaged, as described in section 3.4.3.
The following graphs show ensemble averaged statistics for three different incoming velocities of a heterogeneous driver population. These velocities correspond to the equilibrium values in the free-flow, early congested, and congested traffic-flow phases. For the equilibrium flow phases for the heterogeneous driver population, refer to figure 4.2.

Figure 4.8 $y_{RMS}$ with respect to ensemble density
Figure 4.9 \( \bar{\rho'} y' \) with respect to ensemble density

Figure 4.10 \( \bar{\rho'} P' \) with respect to ensemble density
Figure 4.11 \( y'P' \) with respect to ensemble density

Figure 4.12 \( \rho'v' \) with respect to ensemble density

The reason for the seemingly large deviation from zero in fig. 4.12 is that small fluctuation values are normalized by an even smaller mean.
From these data sets, it is obvious that there is a relationship between the ensemble averaged fluctuation values and the initial speed with which vehicles enter the transient stopping phase. After all, the mechanism of speed adjustment is the same, and only the starting densities and velocities are different. However, at free-flow speeds, the interactions between individual drivers are limited, even when slowing down and stopping. To understand this, consider that during the equilibrium phase corresponding to the speed of $34 \frac{m}{s}$, the distance between individual vehicles in the agent-based model is on average around 200 meters. Even during the transient stopping phase, this distance is large enough to minimize effects of the intermediate interactions between vehicles, and allow each individual driver to smoothly come to a stop. As the density increases, and the equilibrium flow enters the congested phase, the pattern in the vehicle interactions becomes more obvious. Consider fig. 4.13. The sudden drop in the mean speed is due to the initial encounter between faster incoming traffic and the backward propagating stopping shock wave. This is the point at which, due to the increasing density, the vehicle is unable to maintain its equilibrium velocity and the transient phase begins. The same instability is apparent in the ensemble averaged statistics presented in the previous graphs in this section.
Figure 4.13  Mean (ensemble) speed change during stopping transient phase for the incoming equilibrium speed of $8\frac{m}{s}$, $12\frac{m}{s}$ and $22\frac{m}{s}$

While there obviously is a scaling relationship between the ensemble variables and the equilibrium speed of the traffic entering the transient phase, the relationship is non-linear with respect to mean density. The simple linear scaling was tested but does not produce sufficiently good results. This subject will not be considered at the present.

4.2.1.1 Functional Approximation for the Ensemble Averaged Terms Used in the Test Cases

Functional approximations of the ensemble averaged terms used within the ensemble averaged Aw-Rascle model were derived with Eureqa software. The results are presented in the graphs to follow. Coefficient of determination $R^2$ is included with each graph to indicate goodness of fit for each function (with maximum possible value 1.0 indicating a perfect fit).
Figure 4.14  $y_{RMS}$ with respect to ensemble density; approaching speed $12 \frac{m}{s} (R^2 = 0.954388)$

Figure 4.15  $\overline{\rho' y'}$ with respect to ensemble density; approaching speed $12 \frac{m}{s} (R^2 = 0.972355)$
Figure 4.16 $\overline{\rho'P'}$ with respect to ensemble density; approaching speed $12\,\text{m/s}$ ($R^2 = 0.9964591$)

Figure 4.17 $\overline{y'P'}$ with respect to ensemble density; approaching speed $12\,\text{m/s}$ ($R^2 = 0.99349169$)
Figure 4.18 \( \bar{\rho'v'} \) with respect to ensemble density; approaching speed \( 12 \frac{m}{s} \) \( (R^2 = 0.210876) \)

The low \( R^2 \) for the \( \bar{\rho'v'} \) does not present a problem because the overall low mean value of this term does not have a significant impact on the solution.

### 4.2.2 Case 1: Stopping

The case considered in this section is that of the congested traffic stream approaching an obstacle on the road at the equilibrium speed of \( 12 \frac{m}{s} \). The solutions of the Aw-Rascle model and the Aw-Rascle model with ensemble averaged source terms are compared with the agent-based model.

#### 4.2.2.1 Fine-Grained Agent-Based Solution

Each agent-based simulation is initialized with 500 vehicles randomly selected from the normally distributed agent population, as presented in section 3.3. For the purpose of presentation,
the domain is discretized into 30m cells, and speed and density are space averaged over each cell. Then the ensemble averaging is applied over 2000 simulations. Total length of the domain is 10km. The obstacle is placed at 8024.5m from the beginning of the domain to allow the vehicles to achieve equilibrium before they are forced to stop. The maximum speed of the leading vehicle is set to $12\frac{m}{s}$. The vehicles behind it are allowed to accelerate to their maximum speed if that is possible. Fig. 4.19 shows the agent-based simulation after 600 seconds. At this point, the agents have not yet reached the obstacle, but they have achieved the equilibrium speed and density behind the leading vehicle. Speed and density are scaled in the same way as for the fundamental diagram used in the equation model (fig. 4.2).

![Figure 4.19](image)

Figure 4.19 Agent-based simulation, at 600 seconds; speed and density before reaching the obstacle
Figure 4.20  Agent-based simulation, at 677 seconds; the leading vehicle has stopped

Figure 4.21  Agent-based simulation, at 700 seconds; backward propagating wave has formed

Note that, even though the speed is constant, the density wave has an irregular shape due to the constant headway adjustments of the drivers immediately behind the slow leading car. The first vehicle in the agent-based simulation comes to a stop after $676.744 \approx 677$ time steps (677
seconds). Prior to stopping, the leading vehicle will gradually decelerate (the transient phase). This immediately affects all of the closely following vehicles, and the speed of the incoming traffic is already decreased below the equilibrium speed (fig. 4.20 and fig. 4.21). At 700 seconds, the backward propagating wave is obvious (fig. 4.21).

Figure 4.22  Agent-based simulation; backward wave propagation in density and speed at 700s and 750s
In order to determine the speed of the backward propagating wave, the solution at 700 seconds is compared to that at 750 seconds (fig. 4.22).

At 700 seconds, the right side of both speed and density waves is at \( \sim 0.796 \), and at 750 seconds at \( \sim 0.772 \), resulting in the difference of \( \sim 0.024 \). To determine the speed of propagation on the left side, only the intersection of the density wave with the equilibrium line can be analyzed (fig. 4.23). At 700 seconds, the left side of the wave is at \( \sim 0.604 \), and at 750 at \( \sim 0.529 \), which gives the difference of \( \sim 0.075 \). Each cell is 0.003 units long, or 30m. Therefore:

\[
v_{Right} = \frac{-0.024}{0.003} = \frac{-8}{3} \text{cells} \times 30m = -240m, \quad \frac{-240m}{50s} = -4.8 \frac{m}{s}
\]

and

\[
v_{Left} = \frac{-0.075}{0.003} = \frac{-25}{3} \text{cells} \times 30m = -750m, \quad \frac{-750m}{50s} = -15.0 \frac{m}{s}
\]

However, because the speed of the incoming traffic is constantly decreasing, the wave speed is not constant per time step, and will depend on the time interval in which it is measured.

![Figure 4.23 Right side of the density wave at 700s and 750s](image-url)
4.2.2.2 Numerical Results of the Aw-Rascle Model and the Aw-Rascle Model with Ensemble Averaged Source Term

In this section, the solutions for the simple backward propagating stopping wave obtained with the original AR model and the AR model with ensemble averaged terms are compared. The initial condition (fig. 4.24) for this case is given by:

\[
\rho_{Left} = 0.424496, \quad v_{Left} = V_{eq}(\rho_{Left}) = 0.358177
\]

\[
\rho_{Right} = 1.0, \quad v_{Right} = 0.0
\]

The second order Lax-Wendroff numerical scheme was used with the mesh size \( \Delta x = 0.0025 \) and \( CFL = 0.25 \). Numerical results are presented next.

Figure 4.24 Initial condition: \( \rho_{Left} = 0.424496, v_{Left} = 0.358177, \rho_{Right} = 1.0, v_{Right} = 0.0 \)
The simulation is allowed to run for 1200 time steps (fig. 4.25). The speed adaptation parameter $\tau$ is set to 20 seconds.

Initial position of the backward propagating wave is given by $x_{Left} = 0.6$ and $x_{Right} = 0.6425$. In the agent-based model, the obstacle is placed 8024.5m from the beginning of the domain, which corresponds to the $x_{Right} = 0.6425$. Given the mesh size $\Delta x = 0.0025$, $x_{Right}$
is placed 257 cells into the domain, $dx = \frac{8024.5m}{257} = 31.2m$ That is, each cell in the continuum model corresponds to 31.2 meters in the agent-based model. Given that the maximum speed of the information propagation corresponds to the maximum speed of a vehicle in the agent-based model (38 m/s for the aggressive driver), it follows from the definition of the CFL number: 

$$CFL = \frac{\lambda \Delta t}{\Delta x},$$

$dt = 0.21s$, i.e. each time step in the continuum model corresponds to 0.21 seconds in the agent-based model. At the end of the simulation (1200 time steps, 252 seconds), the coordinates for the backward propagating density wave, and the corresponding speed, are as follows:

| Wave propagation speed comparison for Aw-Rascle and Aw-Rascle with ensemble averaging |
|---------------------------------|-----------------|
| **Aw-Rascle model without the ensemble averaged source term** | **Aw-Rascle model with the ensemble averaged source term** |
| $x_L$ | $x_R$ | speed $L$ | speed $R$ |
| Start: 0.6 | Start: 0.6425 | $-166dx/1200dt$ (-18.07m/s) | $-47dx/1200dt$ (-5.82m/s) |
| End: 0.235 | End: 0.525 | | |
| Start: 0.6 | Start: 0.6425 | $-151dx/1200dt$ (-18.69m/s) | $-46dx/1200dt$ (-5.7m/s) |
| End: 0.2225 | End: 0.5275 | | |

4.2.2.3 Evaluation

In this section, the original Aw-Rascle model and the Aw-Rascle model with the ensemble averaged source term are compared to the results of the fine-grained agent-based simulation for the stopping case.
As it can be seen in table 4.2, there is no significant improvement in the propagation speed of the wave. This is not surprising considering the speed of the incoming traffic in the agent-based model decreases from the equilibrium speed even before the wave starts propagating backward. Considering the point in time when the first vehicle stopped (at 677 seconds), and referring back to fig. 4.20, one might think that setting the initial condition to the speed of the traffic stream at this point might improve the result. However, the PW speed-adaptation source term will quickly bring the speed back up to the equilibrium value, and since the speed is constant in this region, the ensemble averaged source term has very little effect. Therefore, the wave will always propagate faster in the equation model than in the agent-based model.

Since the incoming speed cannot be adjusted due to the nature of the Aw-Rascle model, another option is to match the initial density closely to the agent-based model. The density wave is fully developed at 700 seconds, so this profile will be used as the initial condition for the equation (fig. 4.26).
Figure 4.26  Agent-based solution at 700s, and the matching initial condition for the AR model

The right side of the density wave is placed at $x_{Right} = 0.6375$ ($\rho_{Right} = 1.0$). The initial density profile matches that of the agent-based model at 700 time steps, with the left side at $x_{Left} = 0.4175$ ($\rho_{Left} = 0.424496$). The result after 50 seconds is compared with the agent-based model at 750s (fig. 4.27).

From fig. 4.27, it can be seen that there is an improvement in the shape of the density wave for the Aw-Rascle model with the ensemble averaged source term in the region where the speed would be the least affected by the Aw-Rascle speed adaptation mechanism. As expected, the wave propagation speed is still faster than in the agent-based solution.
Referring back to table 4.2, as expected, changing the density profile in the initial condition does not improve the wave propagation speed of either of the AR models. However, the AR model with the ensemble averaged source term does improve the profile of the density solution to more closely match that of the agent-based solution, at least in the region where the incoming traffic stream velocity closely matches that in the agent model. It is worth noting that in [43], it is stated
that the original AR model, when compared to the real traffic data, “has the major drawback of modeling very high density waves incorrectly. For these waves speeds are predicted to travel with high positive velocity, i.e. with the flow of traffic. This contradicts the real data”. Although the pressure function and the fundamental diagram used in [43] is different from the one used in this work, the issues with the AR speed adaptation discussed in this section most likely contribute to the problem.

4.2.3 Case 2: Bottleneck (Merging with a Slower Traffic Stream)

In this section, the case considered is that of the congested traffic stream moving at the equilibrium speed of $12 \frac{m}{s}$ and merging with a slower traffic stream, moving at $6 \frac{m}{s}$. This is equivalent to the case of a bottleneck, a localized disruption of traffic due to some physical condition of a road (e.g. reduced number of lanes). The same ensemble averaged statistics as in Case 1 are used here, because the mechanism of deceleration is the same and, as explained in section 4.2.1, only depends on the speed of the incoming traffic stream. All other parameters are kept the same. In the agent-based model, instead of having a stationary obstacle at 8024.5m from the beginning of the domain, a vehicle is placed at this position. This vehicle is simulating the back of the slower traffic stream. It starts moving at the speed of $6 \frac{m}{s}$ when the leading vehicle of the approaching ($12 \frac{m}{s}$) stream is 100m behind it, ensuring that the speed difference will not immediately affect the equilibrium, and that the transition to the lower speed will be smooth.
4.2.3.1 Fine-Grained Agent-Based Solution

Figure 4.28 shows the agent-based model at 600 seconds, before the $12 \frac{m}{s}$ equilibrium traffic stream reached the slow moving vehicle. At this point, the stream is more than 100 meters away, and the vehicle is still stopped. As in the previous case, there is a local irregularity in the density immediately behind the leading car due to the headway adjustment of the vehicles closely following it.

Figure 4.28 Agent-based simulation, at 600 seconds; speed and density before reaching the slow moving vehicle

At 700 seconds, the slower vehicle ($6 \frac{m}{s}$) is moving, and the new equilibrium region at $6 \frac{m}{s}$ is starting to develop (fig. 4.29).
Finally, at 800 seconds, the new equilibrium region is fully developed, and the transition from $12 \frac{m}{s}$ to $6 \frac{m}{s}$ proceeds as the faster vehicles catch up with the slower moving vehicles in front of them (fig. 4.30).
4.2.3.2 Numerical Results and Evaluation of the Aw-Rascle Model and the Aw-Rascle Model with Ensemble Averaged Source Term

All the scaling parameters for this case are the same as in Case 1. To evaluate the AR models, the initial conditions are set to approximate the density profile of the agent-based solution at 800 seconds.

Figure 4.31 Agent-based solution at 800s, and the matching initial condition for the AR model

Figure 4.31 shows the initial condition for the equation plotted against the agent-based model at 800 second. The irregularity in the density wave is ignored as it is a feature of the agent-based model, has no impact on the solution, and most importantly, because the Aw-Rascle equations cannot model a stable local variation in density without a corresponding variation in speed.
The solution is advanced for 50 seconds, and the result compared with the agent-based solution at 850 seconds (fig. 4.32).

Figure 4.32 Aw-Rascle solution with the initial condition corresponding to the agent-based solution at 850s compared to the agent-based model after 50 seconds

In this case, as in the previous case of traffic coming to a stop, the speed of the incoming traffic in the agent-based model decreased below the equilibrium speed due to the interactions of the vehicles upstream. It can be seen that the speed of the backward propagating wave is again
greater in the equation model than in the agent-based model (in this case, the traffic is also moving forward at the constant speed of $6 \frac{m}{s}$).

Figure 4.33  Agent-based solution at 800s, and the matching initial condition (both speed and density) for the AR model

Due to the smooth and relatively small transition from the region of lower density to that of a higher density, the Aw-Rascle model with the ensemble averaged source term shows very little difference compared to the original Aw-Rascle model. In fig. 4.32, it can be seen that the speed for the Aw-Rascle model with the ensemble averaged source term is slightly below the equilibrium speed due to the correlation terms present in the speed-adaptation PW source term. However, these terms are very small and not intended to correct the solution outside of the transient region.
In order to better illustrate the effect of the speed-adaptation source term, the same case is initialized with both density and speed wave profile matching that of the agent-based model at 800 seconds. The solution is again advanced for 50 seconds and compared with the agent-based model at 850 seconds. The adaptation time $\tau$ is set to 20 seconds. The effects of the speed adaptation are visible even after 10 seconds (fig. 4.34).
Figure 4.35  Aw-Rascle with ensemble averaged source term, solution after 50 seconds; effect of the speed-adaptation source term

4.3 Discussion

It needs to be mentioned that in [55], the authors establish a connection between a microscopic follow-the-leader model and the continuum Aw-Rascle model and show that, in the case of a homogeneous driver population, the macroscopic model can be viewed as the limit of the time discretization of the microscopic model as the number of vehicles increases. The second case
presented in their work is a simulation of a bottleneck, and the equation model shows very good agreement with the microscopic follow-the-leader model. However, the microscopic model used in their study is based on the Aw-Rascle model, and the parameters are adjusted to match those of the equation. To illustrate the issue arising from this, consider the following: the fine-grained agent-based model used in this work is a space-discretized version of the Intelligent Driver Model (IDM). The speed of the individual vehicles in this model depends on the speed of the leading vehicle and the headway. If the model was adjusted in such a way that the vehicles allowed for a shorter headway before beginning to brake, the length at which the speed of the incoming traffic stream is affected by braking would be shorter, and the differences in the propagation speed of the backward moving wave would be smaller.

On the other hand, the Aw-Rascle equation model has a limited number of parameters that can be adjusted in order to approximate the behavior of any other follow-the-leader model, agent-based model, or real traffic, and these parameters are not strictly tied to the physical features of the traffic system (e.g. traffic “pressure” or the hesitation factor cannot be measured directly). The driving element behind the system of equations is the fundamental diagram, which, even if extended to account for stochastic fluctuations, still cannot model large deviations from the equilibrium speed-density relationship present in the transient phases of traffic flow. Additionally, the speed-adaptation source term, inherited from the earlier Payne-Whitham (PW) model, and introduced in the Aw-Rascle model as a starting mechanism in the cases in which the initial speed is zero, is adjusting the speed towards equilibrium even when this is not appropriate (as in the cases of stopping or slowing down). The speed-adaptation time is also a heuristic parameter.
Therefore, even though the ensemble-averaged source term does account for the physical stochastic behaviors and relates directly to the individual driver behaviors and the driver population, it is only active in the transient flow region and will not affect the upstream conditions. Having an additional selective speed-adjustment source term based on the domain of influence of the downstream traffic conditions could improve the performance of the AR model with the ensemble averaged source term, but at the cost of an increased complexity.
Fluctuations around the steady-state equilibrium in traffic flow can be attributed to stochastic fluctuations due to vehicle sizes and individual driver behaviors and structural fluctuations resulting from the dynamic properties of traffic flow. The latter reflect the transient states, that is, acceleration and deceleration phases. Modeling these stochastic components of the traffic flow is crucial in short-term traffic predictions. While there have been efforts to introduce the stochastic behavior in deterministic flow models, these attempts have mainly focused on the small stochastic fluctuations around the theoretical equilibrium speed. Little attention has been given to the physical relevance of the stochastic component, and the existing models do not consider large deviations from the equilibrium speed that occur in the transient phases of traffic flow.

The goal of this work is to improve fidelity of continuum traffic models, with the primary focus on the stochastic behavior arising in the transient deceleration phase in a congested traffic stream. The model capturing the stochastic flow features developed here is integrated into the Aw-Rascle traffic model via ensemble averaging. The same approach can be applied to any other macroscopic traffic model of similar form, but it should be noted that, besides the similar Zhang model, other currently available macroscopic models do not appropriately address the anisotropy problem (section 2.1.2).
A microscopic fine-grained agent-based traffic model has also been developed specifically for the purpose of this study, aiming to provide realistic estimates of the stochastic driver behavior while preserving computational efficiency.

Due to the nature of the Aw-Rascle model, the improvement is limited. In particular, the speed-adaptation source term, which is a necessary part of the model designed to drive the solution towards the equilibrium state, is inappropriate for the transient stopping phase and will always cause the model to predict a larger propagation speed of the stopping wave than what is observed in a realistic traffic system. Implementing a nonlocal anticipation source term in combination with the ensemble averaged source term would likely result in a greater improvement. The question that should be further explored is whether the adaptation towards the mean speed instead of the equilibrium speed is more appropriate for the transient deceleration phase.

It was observed that the ensemble averaged statistics depend on the speed of the traffic stream prior to stopping. While simple scaling works to some extent, this relationship seems to be more complex. The statistics will also depend on the particular driver population and the characteristics of each driver type. A further exploration of these relationships would improve the generality of the approach presented here.

The fine-grained agent-based model compares well with the Intelligent Driver Model. Further discretization of the acceleration and deceleration functions, as described in section 3.2.2, would lead to a greater computational efficiency, but at the expense of fidelity. While the synthetic driver population used here is fairly simple, a more complex population could be created by introducing additional driver and vehicle classes and adding explicit stochastic behavior within
each class. This would necessarily increase the ensemble size, but since each system is independent, the collection and processing of statistics could easily be implemented in parallel.
APPENDIX A

FUNDAMENTAL DIAGRAM
The relationship between traffic flow and density contains a lot of information about the macroscopic (average) behavior of driver-vehicle units. In the idealized form, this relationship is called the fundamental diagram (FD). While macroscopic traffic modeling presumes the existence of a reproducible FD [4, 5, 13, 54], it is understood that the FD is only the theoretical relation between density and flow in stationary homogeneous traffic, i.e. the steady state equilibrium of identical driver-vehicle units [8].

The most encountered form of a fundamental diagram, originating back to the work of Lighthill and Whitham, is a plot of flow versus density. An example of such diagram, taken from [58], is shown in Figure A.1. Note that in this type of FD two branches separated by the critical density $k_c$ can be distinguished: the free-flow branch on the left, along which the flow increases more or less linearly, and the congested regime, in which the flow degrades with increasing density, until the jam density $k_j$ is reached and traffic comes to a standstill, resulting in zero flow. At the critical density $k_c$, the flow reaches the maximum, called the capacity flow.

Using standard statistical techniques for data fitting, a variety of fundamental diagrams have been developed over the years with various degrees of success. The earliest model was the Greenshields model [59], a fundamental diagram in a single-equation continuous form (single-regime model). It was developed by fitting a linear curve to seven empirical observations, which is not enough to generate a good representation of a speed-density relationship. In order to improve this overly simplified relation, other models such as Greenberg [60] and Underwood model [61] were proposed. Besides those derived from empirical data, some fundamental diagrams are derived from car-following models, for example Van Aerde’s [62] and Newell’s model [16].
Figure A.1  A fundamental diagram relating the density \( k \) to the flow \( q \). The capacity flow \( q_{cap} \) is reached at the critical density \( k_c \). The space-mean speed \( v_s \) for any point on the curve, is defined as the slope of the line through that point and the origin. Taking the slope of the tangent to points on the curve, gives the characteristic wave speed \( w \) for LWR type traffic model [58].

Regardless of the derivation, single-equation fundamental diagrams fail to fit the empirical data consistently, either in a free-flow or in a congested traffic regime. In order to resolve this problem, different multi-regime models have been developed, such as Edie [63], the modified Greenberg [64], the three-regime model [64,65], and multi-regime models by cluster analysis [66]. Figure A.2 illustrates performance of multi-regime models against empirical data [41]. The idea behind these models is to use different curves to fit different phases of traffic flow. The issue, however, arises in identifying the breaking points between different traffic regimes.

Furthermore, neither single- nor multi-regime deterministic fundamental diagrams include the randomness that has been observed empirically [64,67,68]. In deterministic models, the speed-density relation is pairwise – given a density there exists one fixed speed computed from a deterministic formula. From empirical data, it is clear that multiple traffic speeds can correspond to one traffic density. Essentially, this randomness comes from two main sources [59,69], namely
from the data collection system and data processing, and from the inherent traffic dynamics. While it is known that individual driver-vehicle unit’s behaviors are not uniform, there is a general lack of knowledge regarding the details of this source of randomness.

Figure A.2 Multi-regime models and empirical data [41]

In 1951, Berry and Belmont [70] analyzed the distribution of vehicle speeds and travel times from different facilities using empirical data. Possibly the first stochastic model regarding flow-density relationship was presented in the paper by Soyster in 1973 [71]. In this work, the arrival of vehicles to the bottom and the top of the hill is modeled as a Poisson process, and a finite number of traffic states are incorporated into a Markov chain with a transition matrix. In a Poisson process, the time between consecutive events is assumed to follow an exponential distribution, and each of the arrival times is assumed to be independent of other arrival times. The stochastic behavior in this model is not based on the realistic traffic behavior. More recently, Wang
et al. [41] developed a stochastic speed-density model based on the Karhunen-Loève expansion in order to avoid computational challenges which dominate Monte Carlo simulation [71,72]. While this work showed that a deterministic continuum model could be extended to capture some traffic phenomena which are typically not modeled using the macroscopic approach (such as spontaneous congestion), it assumes an exponential correlation between different densities obtained from the raw data, does not attempt to explain the sources of stochastic behavior, and therefore does not address the objection that the understanding of the fundamental diagram model is incomplete.
APPENDIX B

EFFECTS OF DISCRETIZATION ON THE INTELLIGENT DRIVER MODEL
The open road acceleration case is used with the acceleration exponent $\delta = 4.0$. The vehicles accelerate from $0 \text{ m/s}$ to the maximum speed of $34 \text{ m/s}$ during $\sim 43$ seconds which is in agreement with the value provided in [50]. In the agent-based simulation, the acceleration value is calculated on each time step as a real number. An appropriate scaling and rounding is then applied depending on $\Delta t$ and $\Delta x$ chosen for the agent-based simulation. For example, if the calculated acceleration value is $1.4 \text{ m/s}^2$, and $\Delta x = 0.5 \text{ m}$, $\Delta t = 1 \text{s}$, the speed in that time step will increase by $1 \text{ m/s}$ (2 cells in one time step). The remaining $0.4 \text{ m/s}^2$ will be saved and added to the acceleration in the next time step.

The following graphs show the speed evaluation from $0 \text{ m/s}$ to $34 \text{ m/s}$ using the IDM model with explicit numerical integration (Forward Euler scheme with $\Delta t = 0.1 \text{s}$), and the scheme used in the agent-based model with different choices of time and space discretization. For the purpose of comparison, the relative error between the two is shown.

![Graph showing acceleration values](image)

**Figure B.1** Acceleration values on an open road ($0 \text{ m/s}$ to $34 \text{ m/s}$) calculated from the IDM equation 3.3 with $\delta = 4.0$. Agent-based solution $\Delta x = 0.5 \text{ m}$, $\Delta t = 1 \text{s}$, IDM numerical integration model $\Delta t = 0.1 \text{s}$
Figure B.2  Speed on an open road (acceleration from $0 \frac{m}{s}$ to $34 \frac{m}{s}$) calculated from the IDM equation 3.3 with $\delta = 4.0$. Agent-based solution $\Delta x = 0.5m$, $\Delta t = 1s$, IDM numerical integration model $\Delta t = 0.1s$

The following graph shows the relative error in the calculated speed values during the acceleration phase on an open road resulting from the spatial and temporal discretization in the agent-based model. The time step in the agent simulation is kept constant, $\Delta t = 1.0s$ as $\Delta x$ is decreased in the former, and the size of a cell is kept constant, $\Delta x = 0.5m$ as $\Delta t$ is decreased in the latter.
The results of simultaneous refinement of both cell size and time step in the agent-based model are shown below.
From the graphs above, it can be seen that overall, the relative error for continuous acceleration decreases with the smaller cell size, but increases with the smaller time step. This is due to the rounding applied. Consider again the acceleration rate of $\frac{1}{4} \, \text{m} \, \text{s}^{-2}$. If $\Delta x = 0.5 \, \text{m}$, $\Delta t = 1 \, \text{s}$, the speed in that time step will increase by $1 \, \text{m} / \text{s}$ (2 cells in one time step). The remaining $0.4 \, \text{m} / \text{s}^2$ will be saved and added to the acceleration in the next time step. If $\Delta x = 0.25 \, \text{m}$, $\Delta t = 1 \, \text{s}$, the speed in that time step will increase by $1.25 \, \text{m} / \text{s}$, and the remainder (error) is lower, $0.25 \, \text{m} / \text{s}^2$. If, however, $\Delta x = 0.5 \, \text{m}$, but $\Delta t = 0.5 \, \text{s}$, the calculated acceleration is applied for a shorter time, and the error is accumulated twice as fast. The fine-grained agent-based model uses $\Delta x = 0.5 \, \text{m}$, $\Delta t = 1 \, \text{s}$, as it seems to be a reasonable choice from the physical point of view. Figure B.6 shows the process of acceleration from $0 \, \text{m} / \text{s}$ to free-flow, and deceleration to $0 \, \text{m} / \text{s}$, followed by the graphs of the corresponding speed and the error for the chosen discretization ($\Delta x = 0.5 \, \text{m}$, $\Delta t = 1 \, \text{s}$).
Figure B.6 Acceleration, speed, and the relative error from the fine-grained agent-based model ($\Delta x = 0.5m$, $\Delta t = 1s$) compared with the IDM solution ($\Delta t = 0.1s$)
The graph indicates oscillations around the acceleration/deceleration value during the deceleration phase. This behavior is also due to the rounding process. However, the speed data shows a similar level of error during both acceleration and deceleration phases, and therefore this behavior does not have an impact on the final outcome. Also note the small oscillations around the maximum speed during the free-flow phase. These too are a consequence of the rounding process, but since in reality drivers do not ever maintain a constant speed, these small fluctuations add to the realism of the model. It is important to understand the physical values associated with the error in speed produced by the agent-based model. The highest relative errors appear during the acceleration and deceleration phases when the actual speed is very low. In order to illustrate how small the actual physical difference in speeds calculated by the IDM ODE and the agent-based model, the corresponding absolute error is given in fig. B.7. The largest difference in the calculated speed is \(1.27 \text{ m/s} \), which in terms of vehicle motion is negligible.

![Graph showing the absolute error corresponding to fig. B.6 (\(\Delta x = 0.5m, \Delta t = 1s\)) compared with the IDM solution (\(\Delta t = 0.1s\))]
APPENDIX C

NUMERICAL VERIFICATION OF THE SECOND-ORDER LAX-WENROFF SCHEME
The convergence rate of the second-order Lax-Wendroff scheme was verified using mesh refinement on the homogeneous Aw-Rascle equation (eqn. 2.5). The fundamental diagram for this case is given in fig. 4.2, and the pressure term defined as $P(\rho) = V_{eq}(\rho_{max}) - V_{eq}(\rho)$, where $\rho_{max} = 1$. The initial condition (fig. C.1) is given by

$$\rho(x) = a \exp \left( -\frac{(x - b)^2}{2c^2} \right)$$

$$v(x) = V_{eq}(\rho(x))$$

$a = 0.1$, $b = 0.6$, $c = 0.15$

Figure C.1 Initial speed and density

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The baseline solution at $t = 0.5$ (fig. C.2) was calculated on the mesh $\Delta x = 0.00001953125$, $\Delta t = 0.0000048828125$

![](image)

Figure C.2 Solution $\Delta x = 0.00001953125$, $\Delta t = 0.0000048828125$, $CFL = 0.25$

The CFL was kept constant (0.25) as $\Delta x, \Delta t$ was refined. The convergence rate is given in fig. C.3.
Figure C.3  Convergence for the second-order Lax-Wendroff scheme, $CFL = 0.25$
APPENDIX D

COMPARISON OF THE NUMERICAL SOLUTIONS OF THE HOMOGENEOUS AR MODEL
AND THE LWR MODEL
As stated in section 3.1.1, the homogeneous AR model converges to the LWR model when the initial velocity is given by \( v(\rho) = V_{eq}(\rho) \). Here, the comparison of the two models is presented using five distinct initial conditions (as in [73]). Additionally, the solutions to the AR model obtained with the second-order Lax-Wendroff scheme are compared to the solutions obtained with the Roe’s upwind scheme with the Mimod flux limiter. In all cases \( \Delta x = 0.0025, \Delta t = 0.000625 \).

The fundamental diagram for both models is given by

\[
V_{eq}(\rho) = 1 - \rho^\gamma, \quad \gamma = 1.0
\]

and the AR pressure function is given by (as in [3])

\[
P(\rho) = \rho^\gamma
\]

Under these conditions, the solutions of the LWR and the AW models will exhibit the same features.

The homogeneous AR model, as before, is given by

\[
\begin{align*}
\rho_t + (y - \rho P)_x &= 0 \\
y_t + \left( \frac{y^2}{\rho} - y P \right)_x &= 0
\end{align*}
\]

The Roe decomposition is performed as in [43], with Roe averages \( \tilde{P} \) and \( \tilde{v} \) given by

\[
\tilde{P} = \frac{\Delta (\rho P)}{(\gamma + 1) \Delta \rho}
\]
\[ \tilde{v} = \frac{2c}{b \mp \sqrt{b^2 - 4ac}} \]

where

\[ a = \Delta \rho \]
\[ b = (\gamma + 2)(\bar{P} \Delta \rho - 2\Delta y) \]
\[ c = \Delta \left( \frac{y^2}{\rho} - Py \right) + (\gamma + 1)(\bar{P}^2 \Delta \rho - \bar{P} \Delta y) \]

For the details of the implementation, refer to [43]. The Lax-Wendroff scheme is used as described in section 3.1.3.

The LWR model, given by

\[ \rho_t + (\rho v)_x = 0, \]
\[ v = V_{eq}(\rho) = 1 - \rho^\gamma \]

is solved using the two-stage upwind scheme, equivalent to the second-order Lax-Wendroff scheme [43], and the Minmod flux limiter is applied:

\[ \text{if } v_{j-\frac{1}{2}} > 0 \]
\[ \rho_j^{n+1} \rightarrow \rho_j^{n+1} + \frac{1}{2} \phi (r_{j-1}^+ v_{j-\frac{1}{2}} + (1 - v_{j-\frac{1}{2}}) \Delta \rho_{j-\frac{1}{2}}) \]
\[ \rho_{j-1}^{n+1} \rightarrow \rho_{j-1}^{n+1} + \frac{1}{2} \phi (r_{j-1}^+ v_{j-\frac{1}{2}} + (1 - v_{j-\frac{1}{2}}) \Delta \rho_{j-\frac{1}{2}}) \]
else if \( \nu_{j-\frac{1}{2}} < 0 \)

\[
\rho_j^{n+1} \rightarrow \rho_j^{n+1} + \frac{1}{2} \phi(r_j^-) \nu_{j-\frac{1}{2}} \left( 1 + \nu_{j-\frac{1}{2}} \right) \Delta \rho_{j-\frac{1}{2}} \\
\rho_{j-1}^{n+1} \rightarrow \rho_{j-1}^{n+1} + \frac{1}{2} \phi(r_{j-1}^-) \nu_{j-\frac{1}{2}} \left( 1 + \nu_{j-\frac{1}{2}} \right) \Delta \rho_{j-\frac{1}{2}}
\]

where

\[
r^+_j = \frac{\left( 1 - \nu_{j-\frac{1}{2}} \right) \Delta f_{j-\frac{1}{2}}}{\left( 1 - \nu_{j+\frac{1}{2}} \right) \Delta f_{j+\frac{1}{2}}}, \quad r^-_j = \frac{\left( 1 + \nu_{j+\frac{1}{2}} \right) \Delta f_{j+\frac{1}{2}}}{\left( 1 + \nu_{j-\frac{1}{2}} \right) \Delta f_{j-\frac{1}{2}}},
\]

\( \nu \) is the Courant number, indicating the direction of the wave propagation

\[
\nu_{j+\frac{1}{2}} = \frac{\Delta t}{\Delta x} \left[ \frac{f' \left( \rho_{j+1}^n \right) - f' \left( \rho_j^n \right)}{\rho_{j+1}^n - \rho_j^n} \right]
\]

and the Roe’s Minmod limiter is given by

\[
\phi(r) = \max \left( 0, \min \left( r, 1 \right) \right)
\]

**Problem 1**

The initial condition is given by (fig. D.1)

\[
\rho(x, 0) = \begin{cases} 
0.5, & x < 1.5 \\
0.8, & x > 1.55
\end{cases}
\]
\begin{equation}
v(x, 0) = \begin{cases} 
0.5, & x < 1.5 \\
0.2, & x > 1.55 
\end{cases}
\end{equation}

The exact solution of the LWR model is given by a shock wave moving to the left at the speed \( s = -0.3 \), computed from the Rankine-Hugoniot condition as

\[ s = v_{\text{max}} \left( 1 - \left( \frac{\rho_L + \rho_R}{\rho_{\text{max}}} \right) \right), \]

where \( v_{\text{max}} = \rho_{\text{max}} = 1 \).

Numerical solution at \( t = 2 \) is given in fig. D.2. As expected, the second-order Lax-Wendroff scheme without a flux limiter exhibits dispersion near sharp discontinuities. Otherwise, the solution is in agreement with [73].

![Initial speed and density for Problem 1](image)

**Figure D.1**  Initial speed and density for Problem 1
Problem 2

The initial condition is given by (fig. D.3)

\[
\rho(x, 0) = \begin{cases} 
0.8, & x < 1.5 \\
0.6, & x > 1.55 
\end{cases}
\]

\[
v(x, 0) = \begin{cases} 
0.2, & x < 1.5 \\
0.4, & x > 1.55 
\end{cases}
\]

In both LWR and AR models, the solution is a left-moving rarefaction wave in agreement with [73].
Problem 3

The initial condition is given by (fig. D.5)

\[ \rho(x, 0) = \begin{cases} 
0.4, & x < 1.5 \\
0.1, & x > 1.55 
\end{cases} \]
The exact solution of the LWR model is a right-moving rarefaction wave. The solution to the AR model in this case, with $\gamma = 1$, also simplifies to a rarefaction wave [73].
Problem 4

The initial condition is given by (fig. D.7)

\[
\rho(x, 0) = \begin{cases} 
0.5, & x < 1.5 \\
0, & x > 1.55 
\end{cases}
\]

\[
v(x, 0) = \begin{cases} 
0.5, & x < 1.5 \\
1, & x > 1.55 
\end{cases}
\]

The solution in this case for both LWR and AR models is a single rarefaction wave, as in [73].

Figure D.7 Initial speed and density for Problem 4
Problem 5

The initial condition is given by (fig. D.9)

\[
\rho(x, 0) = \begin{cases} 
0, & x < 1.5 \\ 
0.5, & x > 1.55 
\end{cases}
\]

\[
v(x, 0) = \begin{cases} 
1, & x < 1.5 \\ 
0.5, & x > 1.55 
\end{cases}
\]

In this case, the solution for the LWR model is a right-moving shock wave, and for the AR model it is a single contact discontinuity. For the case of \( \gamma = 1 \), the shock wave of the LWR coincides with the contact discontinuity, as they are moving at the same speeds: \( s = 0.5 = v_R \) [73].
Figure D.9  Initial speed and density for Problem 5

Figure D.10  Speed and density for Problem 5 at $t = 2$
REFERENCES


VITA

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