VAGUE CONVERGENCE OF SPECTRAL SHIFT FUNCTIONS FOR PERIODIC RESTRICTIONS OF ONE-DIMENSIONAL SCHRÖDINGER OPERATORS

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ABSTRACT

We prove weak and vague convergence results for spectral shift functions associated with self-adjoint one-dimensional Schrödinger operators on intervals of the form $(-\ell, \ell)$ with periodic boundary conditions to the full-line spectral shift function in the infinite volume limit $\ell \to \infty$. The approach employed relies on the use of a Krein-type resolvent identity to relate the resolvent of the operator with periodic boundary conditions to the corresponding operator with Dirichlet boundary conditions in combination with various operator theoretic facts.

DEDICATION

I dedicate this to my grandmother, Jean Kiernan. She was the kindest person I've ever had the pleasure of knowing. She brightened up everyone's day. The world would be a much better place if more people were like her. I also dedicate this to my good friend, Kelly Smith. You inspired everyone you knew to live a better life.

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

The infinite volume limit of spectral shift functions of pairs of Schrödinger operators is a problem that has been studied by many authors in a variety of settings (e.g., [**2**], [**3**], [**5**], [**9**], [**10**], [**12**], [**7**], [**6**]). The basic problem is this: consider two Schrödinger operators H_ℓ and $H_\ell^{(0)}$ in the Hilbert space $L^2((-\ell,\ell)^n;d^nx)$, $n\in\mathbb{N}$ fixed, which are self-adjoint realizations of the differential expressions $-\Delta+V$ and $-\Delta$, respectively, with appropriate fixed boundary conditions on $\partial(-\ell,\ell)^n$ and $V:\mathbb{R}^n\to\mathbb{R}$ a measurable function which decays in some appropriate sense at infinity. If $\xi(\,\cdot\,;H_\ell,H_\ell^{(0)})$ denotes the spectral shift function (which is rigorously defined in Definition 3.4) for the pair $(H_\ell,H_\ell^{(0)})$, $\ell\in\mathbb{N}$, normalized to vanish identically in a neighborhood of $-\infty$, in what manner does the sequence $\{\xi(\,\cdot\,;H_\ell,H_\ell^{(0)})\}_{\ell=1}^\infty$ converge to the normalized spectral shift function $\xi(\,\cdot\,;H,H^{(0)})$ for the pair H and $H^{(0)}$, the self-adjoint realizations of $-\Delta+V$ and $-\Delta$ in $L^2(\mathbb{R}^n;dx)$?

A first natural conjecture would be that perhaps $\{\xi(\,\cdot\,;H_\ell,H_\ell^{(0)})\}_{\ell=1}^\infty$ converges to $\xi(\,\cdot\,;H,H^{(0)})$ pointwise almost everywhere in the sense that

$$\lim_{\ell\to\infty}\xi(\lambda;H_\ell,H_\ell^{(0)})=\xi(\lambda;H,H^{(0)})\text{ for a.e. }\lambda\in\mathbb{R}. \tag{1.0.1}$$

However, one cannot expect pointwise convergence of spectral shift functions in the infinite volume limit for the simple reason that, as the difference of the eigenvalue counting functions for H_ℓ and $H_\ell^{(0)}$ (cf., e.g., the remarks following [**19**, Theorem 8.7.2]), $\xi(\,\cdot\,;H_\ell,H_\ell^{(0)})$ is necessarily integer-valued, while $\xi(\,\cdot\,; H, H^{(0)})$, which coincides with the scattering phase for H and $H^{(0)}$ up to a constant multiple (cf., e.g., [**20**, eq. (1.2)]), is a non-constant continuous function of $\lambda > 0$. Therefore, one must weaken the notion of convergence.

In the case of spectral shift functions, the notion of *vague convergence* has proven to be an appropriate mode of convergence. A sequence $\{f_\ell\}_{\ell=1}^\infty \subset L^1_{\rm loc}(\mathbb{R}; dx)$ converges *vaguely* to $f \in L^1_{\rm loc}(\mathbb{R}; dx)$ if for every $g \in C_0(\mathbb{R})$, the set of all compactly supported continuous functions on $\mathbb{R}$, one has

$$\lim_{\ell\to\infty} \int_{\mathbb{R}} f_\ell(\lambda)\, g(\lambda)\, d\lambda = \int_{\mathbb{R}} f(\lambda)\, g(\lambda)\, d\lambda. \tag{1.0.2}$$

In 2009, Borovyk and Makarov ([**3**], see also [**2**]) considered the infinite volume limit problem for spectral shift functions with the half-line $(0,\infty)$ acting as the infinite volume and finite intervals of the form $(0,r)$, with $V \in L^1((0,\infty); (1+x)\, dx)$ and Dirichlet boundary conditions at the endpoints,

$$u(0) = 0 \quad \text{and} \quad u(r) = 0. \tag{1.0.3}$$

Borovyk and Makarov proved

$$\lim_{r\to\infty} \int_{\mathbb{R}} \xi(\lambda; H_r, H_r^{(0)})\, g(\lambda)\, d\lambda = \int_{\mathbb{R}} \xi(\lambda; H, H^{(0)})\, g(\lambda)\, d\lambda, \quad g \in C_0(\mathbb{R}), \tag{1.0.4}$$

as well as a remarkable result that the infinite volume spectral shift function may be recovered pointwise in terms of the following Cesáro limit:

$$\lim_{r\to\infty} \frac{1}{r} \int_0^r \xi(\lambda; H_\varrho, H_\varrho^{(0)})\, d\varrho = \xi(\lambda; H, H^{(0)}), \quad \lambda \in \mathbb{R}\backslash(\sigma_p(H) \cup \{0\}). \tag{1.0.5}$$

Shortly thereafter, [**7**] extended (1.0.4) beyond Dirichlet boundary conditions to include all separated self-adjoint boundary conditions,

$$\cos(\alpha)u(0) + \sin(\alpha)u'(0) = 0, \quad \cos(\beta)u(\ell) + \sin(\beta)u'(\ell) = 0, \tag{1.0.6}$$

where $\alpha, \beta \in [0, \pi)$ are fixed, under the slightly weaker assumption that $V \in L^1((0,\infty); dx)$. Actually, convergence is strengthened in [**7**] to

$$\lim_{r\to\infty} \int_{\mathbb{R}} \frac{\xi(\lambda; H_r, H_r^{(0)})}{1+\lambda^2} f(\lambda)\, d\lambda = \int_{\mathbb{R}} \frac{\xi(\lambda; H, H^{(0)})}{1+\lambda^2} f(\lambda)\, d\lambda \tag{1.0.7}$$

for every bounded continuous function f on $\mathbb{R}$ and any set of separated self-adjoint boundary conditions. Moreover, the arguments in [**7**] readily extend to the case where the infinite volume limit is $\mathbb{R}$, and the finite intervals take the form $(-\ell, \ell)$, with separated self-adjoint boundary conditions at the endpoints (cf. [**6**, §4**(I)**]),

$$\cos(\alpha)u(-\ell) + \sin(\alpha)u'(-\ell) = 0, \quad \cos(\beta)u(\ell) + \sin(\beta)u'(\ell) = 0. \tag{1.0.8}$$

The approach in [**7**] is based on infinite Fredholm determinants and convergence properties of resolvent operators in the infinite volume limit, and the arguments involved are of an abstract nature. This approach led to the development of abstract criteria in [**6**] for weak convergence of spectral shift functions in terms of convergence of associated sequences of Birman–Schwinger-type operators (i.e., resolvents conjugated from the left and/or right with factors of the perturbation) in the Hilbert–Schmidt or trace classes–an abstract situation which mirrors the concrete infinite volume limit problem for Schrödinger operators.

While vague convergence of spectral shift functions is settled in [**7**] for all separated self-adjoint boundary conditions (1.0.8), which includes both Dirichlet (viz., $\alpha = \beta = 0$) and Neumann (viz., $\alpha = \beta = \pi/2$) boundary conditions as special cases, the analogous problem for coupled self-adjoint boundary conditions, which include periodic boundary conditions as a special case, is not addressed. Moreover, coupled boundary conditions are not discussed in the applications in [**6**].

Periodic boundary conditions at the endpoints of $(-\ell,\ell)$,

$$u(-\ell) = u(\ell) \quad \text{and} \quad u'(-\ell) = u'(\ell), \tag{1.0.9}$$

are an important example of coupled boundary conditions, and they feature prominently in applications in connection with the modeling of periodic phenomena. Given the results of [**7**] and [**6**] for arbitrary separated self-adjoint boundary conditions, one is naturally led to ask the question:

QUESTION 1.1. *Can one extend* (1.0.4) *and* (1.0.7) *to the case where the infinite volume is* $\mathbb{R}$ *and the finite intervals take the form* $(-\ell,\ell)$, $\ell \in \mathbb{N}$, *with periodic boundary conditions at the endpoints?*

In this thesis, we answer Question 1.1 affirmatively and provide, to our knowledge, the first vague convergence results for spectral shift functions of one-dimensional Schrödinger operators with coupled self-adjoint boundary conditions.

Our approach to extending (1.0.4) and (1.0.7) to periodic boundary conditions of the form (1.0.9) and for $V \in L^1(\mathbb{R};dx)$ is to employ the machinery of Krein-type resolvent identities, in particular their precise form for regular Sturm–Liouville operators developed in [**4**], in order to verify and apply the convergence criteria from [**6**]. A Krein-type resolvent identity relates the resolvent operators of two self-adjoint extensions of a symmetric operator, and abstract identities of this type have been presented in a number of sources (cf., e.g., [**1**, §VII.84], [**15**, §14.6], and [**17**, Lemma 2.30]).

General coupled self-adjoint boundary conditions on $(-\ell,\ell)$ take the form (cf., e.g., [**4**])

$$\begin{pmatrix} u(\ell) \\ u'(\ell) \end{pmatrix} = e^{i\phi} \begin{pmatrix} R_{1,1} & R_{1,2} \\ R_{2,1} & R_{2,2} \end{pmatrix} \begin{pmatrix} u(-\ell) \\ u'(-\ell) \end{pmatrix}, \tag{1.0.10}$$

where $\phi \in [0, 2\pi)$ and the matrix $R = [R_{j,k}]_{1\leq j,k\leq 2}$ belongs to $\mathrm{SL}_2(\mathbb{R})$, that is $R \in \mathbb{R}^{2\times 2}$ and $\det(R) = 1$. If $W \in L^1((-\ell,\ell); dx)$ and $H_{\ell,R,\phi}$ denotes the self-adjoint realization of $-d^2/dx^2 + W$ with the boundary conditions in (1.0.10) and $H_{\ell,D}$ denotes the self-adjoint realization with Dirichlet boundary conditions, then by Krein's resolvent identity, the difference of the resolvents of $H_{\ell,R,\phi}$ and $H_{\ell,D}$ is finite rank with rank at most equal to two. In fact, if $R_{1,2} = 0$, which is precisely the case for periodic boundary conditions, then the difference is rank one and (cf. [**4**])

$$\begin{aligned}&\big(H_{\ell,D} - zI_{L^2((-\ell,\ell);dx)}\big)^{-1} - \big(H_{\ell,R,\phi} - zI_{L^2((-\ell,\ell);dx)}\big)^{-1} \\ &\quad = q_{\ell,R,\phi}(z)^{-1}(u_{R,\phi}(\overline{z},\,\cdot\,),\,\cdot\,)_{L^2((-\ell,\ell);dx)}u_{R,\phi}(z,\,\cdot\,), \quad z \in \rho(H_{\ell,D}) \cap \rho(H_{\ell,R,\phi}),\end{aligned} \tag{1.0.11}$$

where $u_{R,\phi}(z,\,\cdot\,)$ is an appropriate vector, and $q_{\ell,R,\phi}(\,\cdot\,)$ is a nonvanishing complex-valued function on $\rho(H_{\ell,D}) \cap \rho(H_{\ell,R,\phi})$. As a basic input, the abstract convergence criteria of [**6**] requires one to prove appropriate convergence results for Birman–Schwinger-type operators for the finite interval Schrödinger operator with coupled boundary conditions in the limit $\ell \to \infty$.

In order to do this, we apply the Krein identity in (1.0.11) to relate to the corresponding Birman–Schwinger-type operators for the finite interval Schrödinger operator with Dirichlet boundary conditions, plus a rank one term. The required convergence properties of the Dirichlet Birman–Schwinger-type operators as $\ell \to \infty$ are known from [**7**], so we are left to analyze the limiting behavior as $\ell \to \infty$ of the remaining rank one term. Precise knowledge, in particular the ℓ-dependence, of the factor $q_{\ell,R,\phi}(z)$ and the function $u_{\ell,R,\phi}(z,\,\cdot\,)$ is critical to this approach, and this analysis is carried out in detail for periodic boundary conditions in Chapter 2.

We briefly summarize the contents of each section of this thesis. Section 1.1 recalls several basic definitions pertaining to Hilbert spaces, linear operators, resolvents, and spectra, and Section 1.2 provides a basic introduction to the subject of compact operators and the trace ideals. In Section 2.1, we rigorously define the self-adjoint Schrödinger operators H, $H^{(0)}$ in $L^2(\mathbb{R}; dx)$ acting formally as $-d^2/dx^2 + V$ and $-d^2/dx^2$, respectively, and their restrictions H_ℓ, $H_\ell^{(0)}$ to $(-\ell, \ell)$ with periodic self-adjoint boundary conditions of the form (1.0.9). We also introduce the restrictions $H_{\ell,D}$, $H_{\ell,D}^{(0)}$ to $(-\ell, \ell)$ with Dirichlet boundary conditions at the endpoints. We discuss their basic properties and recall Krein's resolvent identity which relates the resolvents of H_ℓ and $H_{\ell,D}$ via a rank one term. In Section 2.2, which contains the bulk of our major analysis, we use the Krein resolvent identity to study convergence properties in the limit $\ell \to \infty$ of the Birman–Schwinger-type operators associated to H_ℓ and $H_\ell^{(0)}$. The new results in Theorems 2.9, 2.10, and 2.11 are fundamental to our approach, and they are precisely the results that ultimately yield vague convergence of spectral shift functions and the analogue of (1.0.7).

Section 3.1 recalls the basic abstract theory of the Krein spectral shift function. In Section 3.2, we apply the abstract theory of the Krein spectral shift function to the operators $H_\ell^{(0)}$ and H_ℓ. Finally, in Section 3.3, we combine the new convergence results from Section 2.2 with the abstract convergence criteria from [**6**] to obtain vague convergence of spectral shift functions in the infinite volume limit for the periodic boundary conditions in (1.0.9). To our knowledge, these are the first results of their type for coupled boundary conditions. Section 3.4 contains conclusory remarks and possible ideas for future work.

Appendix A contains a proof of the Krein resolvent identity for Schrödinger operators on a finite interval with periodic boundary conditions. For completeness, Appendix B contains a summary of the convergence criteria from [**6**], suitably tailored for the applications to Schrödinger operators in $L^2(\mathbb{R}; dx)$ and $L^2((-\ell, \ell); dx)$ studied in this thesis.

Finally, we comment on some of the basic notation used throughout this thesis. Let $\mathcal{H}$ be a separable complex Hilbert space, $(\cdot,\cdot)_{\mathcal{H}}$ the scalar product in $\mathcal{H}$ (linear in the second argument), and $I_{\mathcal{H}}$ the identity operator in $\mathcal{H}$. If T is a linear operator mapping (a subspace of) a Hilbert space into another, then $\operatorname{dom}(T)$ and $\ker(T)$ denote the domain and kernel (i.e., null space) of T. The closure of a closable operator S is denoted by $\overline{S}$. The spectrum and resolvent set of a closed linear operator in a Hilbert space will be denoted by $\sigma(\cdot)$ and $\rho(\cdot)$, respectively. The quadratic form sum of two self-adjoint operators A and W will be denoted by $A +_{\mathfrak{q}} W$.

The convergence of bounded operators in the strong operator topology (i.e., pointwise limits) will be denoted by s-lim. The Banach spaces of bounded and compact linear operators on a separable complex Hilbert space $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_\infty(\mathcal{H})$, respectively; the corresponding ℓ^p-based trace ideals will be denoted by $\mathcal{B}_p(\mathcal{H})$, their norms are abbreviated by $\|\cdot\|_{\mathcal{B}_p(\mathcal{H})}$, $p \in [1,\infty)$. Moreover, $\operatorname{tr}_{\mathcal{H}}(A)$ denotes the corresponding trace of a trace class operator $A \in \mathcal{B}_1(\mathcal{H})$.

For any closed finite interval $[a,b] \subset \mathbb{R}$, $AC([a,b])$ denotes the set of absolutely continuous functions defined on $[a,b]$. The symbol $\operatorname{sgn}(\cdot)$ denotes the signum function on $\mathbb{R}$,

$$\operatorname{sgn}(x) = \begin{cases} \dfrac{x}{|x|}, & x \in \mathbb{R}\backslash\{0\}, \\ 0, & x = 0. \end{cases} \tag{1.0.12}$$

We denote by $C(\mathbb{R})$ the space of continuous functions on $\mathbb{R}$, by $C_0(\mathbb{R})$ the continuous functions on $\mathbb{R}$ with compact support, and by $C_b(\mathbb{R})$ the bounded continuous functions on $\mathbb{R}$. $L^1_{\text{loc}}(\mathbb{R}; dx)$ denotes the set of (equivalence classes of) locally integrable (with respect to Lebesgue measure) functions on $\mathbb{R}$, and $H^1(\mathbb{R})$ (resp., $H^1(a,b)$) is the Sobolev space of order one on $\mathbb{R}$ (resp., $(a,b) \subset \mathbb{R}$) (cf., e.g., [**15**, Appendix E]). If u is a function on a set Σ, then the restriction of u to a subset $\Omega \subset \Sigma$ will be denoted by $u|_\Omega$. Finally, if $z \in \mathbb{C}$, then $\overline{z}$ denotes the *complex conjugate* of z. Throughout the text, we use the symbol a.e. to abbreviate the phrases "almost every" and "almost everywhere" (with respect to Lebesgue measure) as the need arises.

1.1. Resolvents and Spectra of Closed Operators

In this section, we recall several basic definitions pertaining to Hilbert spaces, linear operators, resolvents, and spectra. The concepts introduced are standard and may be found in a number of standard references (e.g., [**11**], [**14**], [**17**], and [**18**]). The topics and concepts introduced here will be used extensively in subsequent sections.

If $\mathcal{V}$ is a vector space over the scalar field $\mathbb{C}$, then a map $(\,\cdot\,,\,\cdot\,)_{\mathcal{V}} : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ is called an *inner product* on $\mathcal{H}$ if the following conditions are satisfied:

(i) $(v,v)_{\mathcal{V}} \geq 0$ for all $v \in \mathcal{V}$, with equality if and only if $v = 0$, the zero vector in $\mathcal{V}$,

(ii) $(u, \alpha v + \beta w)_{\mathcal{V}} = \alpha(u,v)_{\mathcal{V}} + \beta(u,w)_{\mathcal{V}}$ for all $u,v,w \in \mathcal{V}$ and all $\alpha, \beta \in \mathbb{C}$, and

(iii) $\overline{(u,v)_{\mathcal{V}}} = (v,u)_{\mathcal{V}}$ for all $u, v \in \mathcal{V}$.

When equipped with an inner product, a vector space $\mathcal{V}$ is called an *inner product space*.

An inner product $(\cdot\,,\,\cdot)_{\mathcal{V}}$ on $\mathcal{V}$ induces a norm $\|\cdot\|_{\mathcal{V}}$ on $\mathcal{V}$ defined by

$$\|v\|_{\mathcal{V}} = (v,v)_{\mathcal{V}}^{1/2}, \quad v \in \mathcal{V}, \tag{1.1.1}$$

which satisfies the axioms of a norm:

(i) $\|v\|_{\mathcal{V}} \geq 0$ for all $v \in \mathcal{V}$, with equality if and only if $v = 0$, the zero vector in $\mathcal{V}$,

(ii) $\|\alpha v\|_{\mathcal{V}} = |\alpha|\|v\|_{\mathcal{V}}$ for all $v \in \mathcal{V}$ and all $\alpha \in \mathbb{C}$, and

(iii) $\|u+v\|_{\mathcal{V}} \leq \|u\|_{\mathcal{V}} + \|v\|_{\mathcal{V}}$ for all $u, v \in \mathcal{V}$.

A sequence of vectors $\{v_n\}_{n=1}^{\infty} \subset \mathcal{V}$ *converges* to the vector $v \in \mathcal{V}$ if for every $\varepsilon > 0$, there exists an $M(\varepsilon) \in \mathbb{N}$ such that

$$\|v_n - v\|_{\mathcal{V}} < \varepsilon, \quad n > M(\varepsilon). \tag{1.1.2}$$

In this case, the vector v is called the *limit* of the sequence $\{v_n\}_{n=1}^{\infty}$. A sequence of vectors $\{v_n\}_{n=1}^{\infty} \subset \mathcal{V}$ is called a *Cauchy sequence* if for every $\varepsilon > 0$, there exists an $N(\varepsilon) \in \mathbb{N}$ such that

$$\|v_n - v_m\|_{\mathcal{V}} < \varepsilon, \quad n, m > N(\varepsilon). \tag{1.1.3}$$

The inner product space $\mathcal{V}$ is called a *Hilbert space* if every Cauchy sequence in $\mathcal{V}$ has a limit in $\mathcal{V}$.

A countable collection of vectors $\{v_n\}_{n=1}^{N} \subset \mathcal{H}$, with $N \in \mathbb{N} \cup \{\infty\}$, is called an *orthonormal set* if

$$(u_n, u_m)_{\mathcal{H}} = \begin{cases} 0, & n \neq m, \\ 1, & n = m. \end{cases} \tag{1.1.4}$$

A Hilbert space is *separable* if it has a countable orthonormal basis $\{v_n\}_{n=1}^{d}$, where $d \in \mathbb{N} \cup \{\infty\}$ is called *dimension* of the Hilbert space and we write $d = \dim(\mathcal{H})$. Henceforth, the symbol $\mathcal{H}$ will be used to denote a Hilbert space equipped with the inner product $(\cdot\,,\,\cdot)_{\mathcal{H}}$, and we will always assume that $\mathcal{H}$ is separable.

A function $A : \mathrm{dom}(A) \subseteq \mathcal{H} \to \mathcal{H}$ from a subspace $\mathrm{dom}(A) \subseteq$ to $\mathcal{H}$ is called a *linear operator* if

$$A(\alpha u + \beta v) = \alpha A u + \beta A v, \quad u, v \in \mathrm{dom}(A),\ \alpha, \beta \in \mathbb{C}. \tag{1.1.5}$$

If there exists a constant $C > 0$ such that

$$\|Au\|_{\mathcal{H}} \leq C\|u\|_{\mathcal{H}}, \quad u \in \mathrm{dom}(A), \tag{1.1.6}$$

then the linear operator A is said to be *bounded*, and one then defines the operator norm of A by

$$\|A\| = \sup_{\substack{u \in \mathrm{dom}(A), \\ \|u\|_{\mathcal{H}}=1}} \|Au\|_{\mathcal{H}}. \tag{1.1.7}$$

The set of bounded linear operators defined on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. A linear operator A belongs to $\mathcal{B}(\mathcal{H})$ if and only if A is bounded and $\mathrm{dom}(A) = \mathcal{H}$. Subsequently, the operator norm of $A \in \mathcal{B}(\mathcal{H})$ will be denoted by $\|A\|_{\mathcal{B}(\mathcal{H})}$.

A linear operator $A : \mathrm{dom}(A) \subseteq \mathcal{H} \to \mathcal{H}$ is *densely defined* if $\mathrm{dom}(A)$ is a dense subspace of $\mathcal{H}$. When A is densely defined, it has a well-defined adjoint operator $A^* : \mathrm{dom}(A^*) \subseteq \mathcal{H} \to \mathcal{H}$ defined by

$$\begin{aligned} &\mathrm{dom}(A^*) = \big\{\psi \in \mathcal{H} \,\big|\, \text{there exists } \tilde{\psi} \in \mathcal{H} \text{ such that } (\psi, A\phi)_{\mathcal{H}} = (\tilde{\psi}, \phi)_{\mathcal{H}} \text{ for all } \phi \in \mathrm{dom}(A)\big\}, \\ &\quad A^*\psi = \tilde{\psi}. \end{aligned} \tag{1.1.8}$$

By its very definition, A^* has the following property

$$(A^*\psi, \phi)_{\mathcal{H}} = (\psi, A\phi)_{\mathcal{H}}, \quad \phi \in \mathrm{dom}(A),\ \psi \in \mathrm{dom}(A^*). \tag{1.1.9}$$

A densely defined operator $A : \mathrm{dom}(A) \subseteq \mathcal{H} \to \mathcal{H}$ is called *self-adjoint* if it is equal to its adjoint, that is if $A = A^*$.

It is important to note that A^* need not be densely defined in general. In fact, $\mathrm{dom}(A^*) = \{0\}$ is possible. A densely defined linear operator $A : \mathrm{dom}(A) \subseteq \mathcal{H} \to \mathcal{H}$ is called *closable* if A^* is densely defined. In this case, A^* possesses a well-defined adjoint $(A^*)^*$, which is called the *closure* of A and is denoted by $\overline{A}$,

$$\overline{A} := (A^*)^*. \tag{1.1.10}$$

A closable operator A is called *closed* if it coincides with its closure, that is if $A = \overline{A}$.

If $A : \mathrm{dom}(A) \subseteq \mathcal{H} \to \mathcal{H}$ is densely defined and closed, then the *resolvent set* of A is defined by

$$\rho(A) := \big\{ z \in \mathbb{C} \,\big|\, (A - zI_{\mathcal{H}})^{-1} \in \mathcal{B}(\mathcal{H}) \big\}, \tag{1.1.11}$$

where $I_{\mathcal{H}} \in \mathcal{B}(\mathcal{H})$ denotes the identity operator with $I_{\mathcal{H}} u = u$ for all $u \in \mathcal{H}$. For each $z \in \rho(A)$, the bounded operator $(A - zI_{\mathcal{H}})^{-1}$ is called the *resolvent operator of A at z*. One may relate the resolvent of A at $z_1 \in \rho(A)$ to the resolvent of A at $z_2 \in \rho(A)$ via the *first resolvent identity.*

Theorem 1.2 (First Resolvent Identity, (2.84) in [**17**]). *If $A : \mathrm{dom}(A) \subseteq \mathcal{H} \to \mathcal{H}$ is closed, then*

$$\begin{aligned} (A - z_1 I_{\mathcal{H}})^{-1} - (A - z_2 I_{\mathcal{H}})^{-1} &= (z_1 - z_2)(A - z_2 I_{\mathcal{H}})^{-1}(A - z_1 I_{\mathcal{H}})^{-1} \\ &= (z_1 - z_2)(A - z_1 I_{\mathcal{H}})^{-1}(A - z_2 I_{\mathcal{H}})^{-1}, \quad z_1, z_2 \in \rho(A). \end{aligned} \tag{1.1.12}$$

The complement of $\rho(A)$ in the complex plane is called the *spectrum* of A,

$$\sigma(A) = \mathbb{C} \backslash \rho(A). \tag{1.1.13}$$

The spectrum of a self-adjoint operator must be real.

Theorem 1.3 (Theorem 2.9 and Problem 3.5 in [**17**]). *If* $A : \mathrm{dom}(A) \subseteq \mathcal{H} \to \mathcal{H}$ *is self-adjoint, then* $\sigma(A) \subseteq \mathbb{R}$. *In particular,* $\mathbb{C}\backslash\mathbb{R} \subseteq \rho(A)$. *Moreover, if* $\mathrm{dist}(\zeta, \Omega)$ *denotes the distance from a point* $\zeta \in \mathbb{C}$ *to a subset* $\Omega \subset \mathbb{C}$*, then*

$$\left\|(A - zI_{\mathcal{H}})^{-1}\right\|_{\mathcal{B}(\mathcal{H})} = \mathrm{dist}(z, \sigma(A))^{-1}, \quad z \in \rho(A). \tag{1.1.14}$$

1.2. Compact Operators and Their Properties

In this section, we compile many of the important definitions and results in the theory of compact operators that will be applied later. Throughout, $\mathcal{H}$ denotes a separable Hilbert space and $\mathcal{B}(\mathcal{H})$ is the set of bounded linear operators defined on $\mathcal{H}$.

Definition 1.4. *An operator* $K \in \mathcal{B}(\mathcal{H})$ *is called a finite rank operator if its range is a finite dimensional subspace of* $\mathcal{H}$. *The dimension of the range of* K *is called the rank of* K *and is denoted by* $\mathrm{rank}(K)$.

It is a simple matter to construct a finite rank operator in $\mathcal{H}$. If $\{u_n\}_{n=1}^N$ is an orthonormal set of vectors and $\{v_n\}_{n=1}^N$ is any set of vectors in $\mathcal{H}$ for some $N \in \mathbb{N}$, then the operator K defined by

$$Ku = \sum_{n=1}^{N} (v_n, u)_{\mathcal{H}} u_n, \quad u \in \mathcal{H}, \tag{1.2.1}$$

is a finite rank operator since the range of K clearly belongs to the subspace spanned by $\{u_n\}_{n=1}^N$. In fact, every bounded finite rank operator is of the form (1.2.1). Indeed, if $K \in \mathcal{B}(\mathcal{H})$ is finite rank and $\{u_n\}_{n=1}^N$ is an orthonormal basis for the range of K, then

$$Ku = \sum_{n=1}^{N} (u_n, Ku)_{\mathcal{H}} u_n, \quad u \in \mathcal{H}, \tag{1.2.2}$$

which is merely the Fourier expansion of Ku in terms of the orthonormal basis $\{u_n\}_{n=1}^N$. Making use of the adjoint operator in (1.2.2),

$$Ku = \sum_{n=1}^{N} (K^* u_n, u)_{\mathcal{H}} u_n, \quad u \in \mathcal{H}, \tag{1.2.3}$$

so that K is of the form (1.2.1) with $v_n = K^* u_n$, $1 \le n \le N$.

It is important to note that the set of finite rank operators is not closed in $\mathcal{B}(\mathcal{H})$ under the operator norm. That is to say, if $\{K_n\}_{n=1}^\infty$ is a sequence of finite rank operators in $\mathcal{B}(\mathcal{H})$ which converges to $K \in \mathcal{B}(\mathcal{H})$ as $n \to \infty$:

$$\lim_{n\to\infty} \|K_n - K\|_{\mathcal{B}(\mathcal{H})} = 0, \tag{1.2.4}$$

then K need not be a finite rank operator in general. However, since $\mathcal{B}(\mathcal{H})$ is a Banach space, one may take the closure of the set of finite rank operators.

Definition 1.5. *The closure of the set of all finite rank operators in $\mathcal{B}(\mathcal{H})$ is called the set of compact operators and is denoted by $\mathcal{B}_\infty(\mathcal{H})$.*

Remark 1.6. By the very definition of $\mathcal{B}_\infty(\mathcal{H})$, it is clear that $\mathcal{B}_\infty(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$. Moreover, if $\dim(\mathcal{H}) = \infty$, one can also show that $\mathcal{B}_\infty(\mathcal{H}) \neq \mathcal{B}(\mathcal{H})$, so the set of compact operators is a proper subset of the set of bounded linear operators. Since $\mathcal{B}_\infty(\mathcal{H})$ is the closure of the set of finite rank operators by definition, it follows that if $K \in \mathcal{B}_\infty(\mathcal{H})$, then there exists a sequence of finite rank operators $\{K_n\}_{n=1}^\infty$ such that (1.2.4) holds.

The set of compact operators is closed under the operations of taking the adjoint and multiplying by a bounded linear operator.

Theorem 1.7 (Lemma 5.6 in [**17**]). *If* $K \in \mathcal{B}_\infty(\mathcal{H})$*, then* $K^* \in \mathcal{B}_\infty(\mathcal{H})$*. If, in addition,* $A \in \mathcal{B}(\mathcal{H})$*, then*

$$AK, KA \in \mathcal{B}_\infty(\mathcal{H}). \tag{1.2.5}$$

Thus, $\mathcal{B}_\infty(\mathcal{H})$ *is a two-sided ideal in* $\mathcal{B}(\mathcal{H})$*.*

Theorem 1.8 (Canonical Expansion of a Compact Operator, Theorem 6.7 in [**17**]). *If* $K \in \mathcal{B}_\infty(\mathcal{H})$*, then there exist orthonormal sets* $\{u_n\}_{n=1}^{N(K)}$ *and* $\{v_n\}_{n=1}^{N(K)}$ *(here* $N(K) \in \mathbb{N} \cup \{\infty\}$*) and positive numbers* $s_n = s_n(K)$*,* $1 \leq n \leq N$*, ordered so that* $s_n \geq s_{n+1}$ *with*

$$K = \sum_{n=1}^{N(K)} s_n(u_n, \cdot\,)_{\mathcal{H}} v_n, \quad \text{and} \quad K^* = \sum_{n=1}^{N(K)} s_n(v_n, \cdot\,)_{\mathcal{H}} u_n. \tag{1.2.6}$$

In particular, $Ku_n = s_n v_n$ *and* $K^* v_n = s_n u_n$*,* $1 \leq n \leq N(K)$*, and therefore*

$$K^*Ku_n = s_n^2 u_n \quad \text{and} \quad KK^*v_n = s_n^2 v_n, \quad 1 \leq n \leq N(K), \tag{1.2.7}$$

so that KK^* *and* K^*K *have the same nonzero eigenvalues, namely* $\{s_n^2\}_{n=1}^{N(K)}$*.*

Definition 1.9. *The positive numbers* $\{s_n(K)\}_{n=1}^{N(K)}$ *in the canonical decomposition of* $K \in \mathcal{B}_\infty(\mathcal{H})$ *(cf.* (1.2.6)*) are called the singular values of* K*. There are either finitely many singular values (this happens precisely when* K *is finite rank) or they converge to zero.*

The next theorem contains some basic results on the singular values of a compact operator. In particular, the largest singular value equals the operator norm.

Theorem 1.10 (Lemma 6.8 in [**17**]). *If* $K \in \mathcal{B}_\infty(\mathcal{H})$ *with singular values* $\{s_n(K)\}_{n=1}^{N(K)}$*, then*

$$\|K\|_{\mathcal{B}(\mathcal{H})} = s_1(K) \tag{1.2.8}$$

and

$$s_n(AK) \le \|A\|_{\mathcal{B}(\mathcal{H})} s_n(K), \quad \textit{and} \quad s_n(KA) \le \|A\|_{\mathcal{B}(\mathcal{H})} s_n(K), \quad 1 \le n \le N(K). \tag{1.2.9}$$

In general, the nature of the spectrum of a bounded linear operator may be quite complicated, but it turns out that the structure of the spectrum of a compact operator is particularly simple. The following classical result is known as the *Riesz–Schauder Theorem* and shows that the spectrum of a compact operator is analogous to that of an operator in a finite-dimensional space.

THEOREM 1.11 (Riesz–Schauder Theorem). *If $K \in \mathcal{B}_\infty(\mathcal{H})$, then $\sigma(K)$ is a countable subset of $\mathbb{C}$ with no accumulation point different from 0. Each nonzero element $\lambda \in \sigma(K)$ is an eigenvalue of K with finite algebraic multiplicity, and $\overline{\lambda}$ is an eigenvalue of K^* with the same algebraic multiplicity. If $\operatorname{rank}(K) = \infty$, then 0 is an accumulation point of $\sigma(K)$. If $\dim(\mathcal{H}) = \infty$, then $0 \in \sigma(K)$.*

A closed operator T will possess a spectrum with a simple structure even when T is not compact if T has a compact resolvent. Note that in the following result, T need not be a bounded operator.

THEOREM 1.12 (Theorem III.6.29 in [**11**]). *If T (not necessarily bounded) is a closed operator in $\mathcal{H}$ with $\rho(T) \neq \emptyset$ and $(T - z_0 I_{\mathcal{H}})^{-1} \in \mathcal{B}_\infty(\mathcal{H})$ for some $z_0 \in \rho(T)$, then $(T - z I_{\mathcal{H}})^{-1} \in \mathcal{B}_\infty(\mathcal{H})$ for all $z \in \rho(T)$ and the spectrum of T consists entirely of isolated eigenvalues with finite algebraic multiplicities.*

Next, we introduce important classes of compact operators called the *Schatten–von Neumann classes*. These classes will play a fundamental role later on in our specific applications.

DEFINITION 1.13. *Let $p \in [1, \infty)$ be fixed. An operator $K \in \mathcal{B}_\infty(\mathcal{H})$ is said to belong to the class $\mathcal{B}_p(\mathcal{H})$ if $\sum_{n=1}^{N(K)} s_n(K)^p$ is a convergent numerical series. If $K \in \mathcal{B}_p(\mathcal{H})$, then one defines*

$$\|K\|_{\mathcal{B}_p(\mathcal{H})} = \left[\sum_{n=1}^{N(K)} s_n(K)^p \right]^{1/p}. \tag{1.2.10}$$

The class $\mathcal{B}_1(\mathcal{H})$ is called the trace class, and $\mathcal{B}_2(\mathcal{H})$ is called the Hilbert–Schmidt class.

Many basic properties of the Schatten–von Neumann classes are summarized in the next theorem. Note that property (*iv*) below is an immediate consequence of (1.2.8).

THEOREM 1.14 (Lemma 6.12 and Corollary 6.13 in [**17**]). *If $p \in [1, \infty)$ is fixed, then the following items (i)–(v) hold:*

(i) $\mathcal{B}_p(\mathcal{H})$ is a subspace of $\mathcal{B}(\mathcal{H})$.

(ii) $\| \cdot \|_{\mathcal{B}_p(\mathcal{H})}$ defines a norm on $\mathcal{B}_p(\mathcal{H})$.

(iii) $(\mathcal{B}_p(\mathcal{H}), \| \cdot \|_{\mathcal{B}_p(\mathcal{H})})$ is a Banach space.

(iv) If $K \in \mathcal{B}_p(\mathcal{H})$, then $\|K\|_{\mathcal{B}(\mathcal{H})} \le \|K\|_{\mathcal{B}_p(\mathcal{H})}$.

(v) If $A \in \mathcal{B}(\mathcal{H})$ and $K \in \mathcal{B}_p(\mathcal{H})$, then $AK, KA \in \mathcal{B}_p(\mathcal{H})$.

In addition to the basic properties listed above, the Schatten–von Neumann classes have the following nesting property which follows immediately from their definition.

THEOREM 1.15. *If $p, q \in [1, \infty)$ with $p \ge q$ and $K \in \mathcal{B}_q(\mathcal{H})$, then $K \in \mathcal{B}_p(\mathcal{H})$. Therefore, $\mathcal{B}_1(\mathcal{H}) \subset \mathcal{B}_p(\mathcal{H})$ for any $p \in [1, \infty)$ and, in particular, $\mathcal{B}_1(\mathcal{H}) \subset \mathcal{B}_2(\mathcal{H})$.*

THEOREM 1.16 (Lemma 6.14 in [**17**]). *An operator* $K \in \mathcal{B}_\infty(\mathcal{H})$ *belongs to the trace class* $\mathcal{B}_1(\mathcal{H})$ *if and only if it can be expressed as the product of two operators from the Hilbert–Schmidt class* $\mathcal{B}_2(\mathcal{H})$, $K = K_1K_2$, *with* $K_1, K_2 \in \mathcal{B}_2(\mathcal{H})$. *In this case,*

$$\|K\|_{\mathcal{B}_1(\mathcal{H})} \leq \|K_1\|_{\mathcal{B}_2(\mathcal{H})}\|K_2\|_{\mathcal{B}_2(\mathcal{H})}. \tag{1.2.11}$$

The inequality in (1.2.11) is a special case of a more general Hölder-type inequality for the trace ideals which is the content of the following result.

LEMMA 1.17 (Theorem 2.8 in [**16**]). *Let* $p, q, r \in [1, \infty)$ *with* $p^{-1} + q^{-1} = r^{-1}$. *If* $A \in \mathcal{B}_p(\mathcal{H})$ *and* $B \in \mathcal{B}_q(\mathcal{H})$, *then* $AB \in \mathcal{B}_r(\mathcal{H})$ *and*

$$\|AB\|_{\mathcal{B}_r(\mathcal{H})} \leq \|A\|_{\mathcal{B}_p(\mathcal{H})}\|B\|_{\mathcal{B}_q(\mathcal{H})}. \tag{1.2.12}$$

The class $\mathcal{B}_1(\mathcal{H})$ is called the trace class because every operator $K \in \mathcal{B}_1(\mathcal{H})$ has a well-defined trace.

LEMMA 1.18 (Lemma 6.15 in [**17**]). *If* $K \in \mathcal{B}_1(\mathcal{H})$, *then for any orthonormal basis* $\{u_n\}_{n=1}^{\dim(\mathcal{H})}$ *of* $\mathcal{H}$,

$$\sum_{n=1}^{\dim(\mathcal{H})} (u_n, Ku_n)_{\mathcal{H}} \tag{1.2.13}$$

is convergent; the sum is independent of the orthonormal basis chosen and satisfies

$$\left| \sum_{n=1}^{\dim(\mathcal{H})} (u_n, Ku_n)_{\mathcal{H}} \right| \leq \|K\|_{\mathcal{B}_1(\mathcal{H})}. \tag{1.2.14}$$

DEFINITION 1.19. *If* $K \in \mathcal{B}_1(\mathcal{H})$, *then the trace of* K *is defined to be*

$$\operatorname{tr}_{\mathcal{H}}(K) = \sum_{n=1}^{\dim(\mathcal{H})} (u_n, Ku_n)_{\mathcal{H}}, \tag{1.2.15}$$

where $\{u_n\}_{n=1}^{\dim(\mathcal{H})}$ *is any orthonormal basis of* $\mathcal{H}$.

REMARK 1.20. In light of Lemma 1.18, the trace satisfies

$$|\operatorname{tr}_{\mathcal{H}}(K)| \leq \|K\|_{\mathcal{B}_1(\mathcal{H})}, \quad K \in \mathcal{B}_1(\mathcal{H}). \tag{1.2.16}$$

LEMMA 1.21 (Properties of the Trace, Lemma 6.16 in [**17**]). *Let $K, K_1, K_2 \in \mathcal{B}_1(\mathcal{H})$.*

(i) The trace is linear:

$$\operatorname{tr}_{\mathcal{H}}(\alpha K_1 + \beta K_2) = \alpha \operatorname{tr}_{\mathcal{H}}(K_1) + \beta \operatorname{tr}_{\mathcal{H}}(K_2), \quad \alpha, \beta \in \mathbb{C}. \tag{1.2.17}$$

(ii) $\operatorname{tr}_{\mathcal{H}}(K^*) = \overline{\operatorname{tr}_{\mathcal{H}}(K)}$.

(iii) If $(u, K_1 u)_{\mathcal{H}} \leq (u, K_2 u)_{\mathcal{H}}$ for all $u \in \mathcal{H}$, then $\operatorname{tr}_{\mathcal{H}}(K_1) \leq \operatorname{tr}_{\mathcal{H}}(K_2)$.

(iv) The trace is cyclic: $\operatorname{tr}_{\mathcal{H}}(AK) = \operatorname{tr}_{\mathcal{H}}(KA)$ for all $A \in \mathcal{B}(\mathcal{H})$.

In the following proposition, we show how to compute the $\mathcal{B}_p(\mathcal{H})$ norm of a rank one operator. We will make extensive use of this result later on.

PROPOSITION 1.22. *Let $\mathcal{H}$ denote a Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{H}}$. If $\phi, \psi \in \mathcal{H}$, then $A = (\psi, \cdot)_{\mathcal{H}}\phi$, $\operatorname{dom}(A) = \mathcal{H}$, defines a trace class operator, that is $A \in \mathcal{B}_1(\mathcal{H})$, and*

$$\|A\|_{\mathcal{B}_p(\mathcal{H})} = \|\psi\|_{\mathcal{H}} \|\phi\|_{\mathcal{H}}, \quad p \in [1, \infty). \tag{1.2.18}$$

PROOF. If either of ψ or ϕ is the zero vector, the result is trivial. Therefore, let us assume that $\psi, \phi \in \mathcal{H} \setminus \{0_{\mathcal{H}}\}$. Since A is rank one, it is trace class, that is $A \in \mathcal{B}_1(\mathcal{H})$. In light of the identities

$$\begin{aligned}(f, Ag)_{\mathcal{H}} &= \big(f, (\psi, g)_{\mathcal{H}}\phi\big)_{\mathcal{H}} = (\psi, g)_{\mathcal{H}}(f, \phi)_{\mathcal{H}} \\ &= \big(\overline{(f, \phi)_{\mathcal{H}}}\psi, g\big)_{\mathcal{H}} = \big((\phi, f)_{\mathcal{H}}\psi, g\big)_{\mathcal{H}}, \quad f, g \in \mathcal{H},\end{aligned} \tag{1.2.19}$$

the adjoint of A is $A^* = (\phi, \cdot)_{\mathcal{H}}\psi$. Therefore,

$$A^*Af = A^*(\psi, f)_{\mathcal{H}}\phi = \|\phi\|_{\mathcal{H}}^2 (\psi, f)_{\mathcal{H}}\psi, \quad f \in \mathcal{H}, \tag{1.2.20}$$

that is, $A^*A = \|\phi\|_{\mathcal{H}}^2(\psi, \cdot)\psi$. It follows that the lone nonzero eigenvalue of A^*A is $\|\phi\|_{\mathcal{H}}^2\|\psi\|_{\mathcal{H}}^2$, and A has one singular value, namely $s_1 = \|\phi\|_{\mathcal{H}}\|\psi\|_{\mathcal{H}}$. By (1.2.10), we have $\|A\|_{\mathcal{B}_p(\mathcal{H})} = s_1$, and the result follows. □

REMARK 1.23. Since $\|K\|_{\mathcal{B}(\mathcal{H})} = s_1(K)$ for any $K \in \mathcal{B}_\infty(\mathcal{H})$, for the operator A in Proposition 1.22, one also infers

$$\|A\|_{\mathcal{B}(\mathcal{H})} = \|\psi\|_{\mathcal{H}}\|\phi\|_{\mathcal{H}}. \tag{1.2.21}$$

The next two results pertain to convergence of products of operators in the trace ideals, and they will also play an important role in our applications.

LEMMA 1.24. *Let $p, q, r \in [1, \infty)$ with $p^{-1} + q^{-1} = r^{-1}$. If $\{A_\ell\}_{\ell=1}^\infty \subset \mathcal{B}_p(\mathcal{H})$, $\{B_\ell\}_{\ell=1}^\infty \subset \mathcal{B}_q(\mathcal{H})$, $A \in \mathcal{B}_p(\mathcal{H})$, and $B \in \mathcal{B}_q(\mathcal{H})$ with*

$$\lim_{\ell\to\infty} \|A_\ell - A\|_{\mathcal{B}_p(\mathcal{H})} = 0 \quad \text{and} \quad \lim_{\ell\to\infty} \|B_\ell - B\|_{\mathcal{B}_q(\mathcal{H})} = 0, \tag{1.2.22}$$

then

$$\lim_{\ell\to\infty} \|A_\ell B_\ell - AB\|_{\mathcal{B}_r(\mathcal{H})} = 0. \tag{1.2.23}$$

PROOF. The containment $[A_\ell B_\ell - AB] \in \mathcal{B}_r(\mathcal{H})$ follows from Lemma 1.17, and one infers

$$\begin{aligned} \|A_\ell B_\ell - AB\|_{\mathcal{B}_r(\mathcal{H})} &\leq \|A_\ell(B_\ell - B)\|_{\mathcal{B}_r(\mathcal{H})} + \|(A_\ell - A)B\|_{\mathcal{B}_r(\mathcal{H})} \\ &\leq \|A_\ell\|_{\mathcal{B}_p(\mathcal{H})}\|B_\ell - B\|_{\mathcal{B}_q(\mathcal{H})} + \|A_\ell - A\|_{\mathcal{B}_p(\mathcal{H})}\|B\|_{\mathcal{B}_q(\mathcal{H})}. \end{aligned} \tag{1.2.24}$$

Since $\|A_\ell\|_{\mathcal{B}_p(\mathcal{H})} \leq C$, $\ell \in \mathbb{N}$, for some ℓ-independent constant $C > 0$, the claim in (1.2.23) follows from (1.2.22) and (1.2.24) by an application of the Squeeze Theorem. □

In order to state the second result, we first recall the notion of strong convergence of a sequence of bounded operators.

Definition 1.25. *A sequence* $\{B_n\}_{n=1}^{\infty} \subset \mathcal{B}(\mathcal{H})$ *is said to converge strongly to* $B \in \mathcal{B}(\mathcal{H})$ *if*

$$\lim_{n\to\infty} \|B_n v - Bv\|_{\mathcal{H}} = 0, \quad v \in \mathcal{H}. \tag{1.2.25}$$

In this case, one writes

$$\operatorname*{s-lim}_{n\to\infty} B_n = B, \tag{1.2.26}$$

and B *is called the strong limit of* $\{B_n\}_{n=1}^{\infty}$.

With this definition, we are prepared to state the second result.

Theorem 1.26 (Grümm's Theorem, [**8**]). *Let* $p \in [1, \infty)$, $A \in \mathcal{B}_p(\mathcal{H})$, *and* $\{A_\ell\}_{\ell=1}^{\infty} \subset \mathcal{B}_p(\mathcal{H})$ *with* $\lim_{\ell\to\infty} \|A_\ell - A\|_{\mathcal{B}_p(\mathcal{H})} = 0$. *If* $B \in \mathcal{B}(\mathcal{H})$, $\{B_\ell\}_{\ell=1}^{\infty} \subset \mathcal{B}(\mathcal{H})$ *with* $\sup_{\ell\in\mathbb{N}} \|B_\ell\|_{\mathcal{B}(\mathcal{H})} < \infty$ *and* $\operatorname{s-lim}_{\ell\to\infty} B_\ell = B$, *then*

$$\lim_{\ell\to\infty} \|A_\ell B_\ell - AB\|_{\mathcal{B}_p(\mathcal{H})} = \lim_{\ell\to\infty} \|B_\ell A_\ell - BA\|_{\mathcal{B}_p(\mathcal{H})} = 0. \tag{1.2.27}$$

1.3. A Canonical Direct Sum Decomposition of $L^2(\mathbb{R}; dx)$

We begin this section by recalling the definition of the L^p-spaces.

Definition 1.27. *If* $\Omega \subset \mathbb{R}$ *is Lebesgue measurable and* $p \in [1, \infty)$, *then* $L^p((a, b); dx)$ *is the set of (equivalence classes of) all functions* $f : \Omega \to \mathbb{C}$ *for which*

$$\int_\Omega |f(x)|^p \, dx < \infty. \tag{1.3.1}$$

If $f \in L^p(\Omega; dx)$, *then one defines*

$$\|f\|_{L^p(\Omega; dx)} = \left[\int_\Omega |f(x)|^p \, dx \right]^{1/p}. \tag{1.3.2}$$

For each $p \in [1,\infty)$, the functional $\|\cdot\|_{L^p(\Omega;dx)}$ defines a norm on $L^p(\Omega;dx)$, and $\big(\|\cdot\|_{L^p(\Omega;dx)}, L^p(\Omega;dx)\big)$ is a Banach space.

It is a well-known fact that $L^2(\Omega;dx)$ is a Hilbert space under the canonical inner product

$$(f,g)_{L^2(\Omega;dx)} = \int_\Omega \overline{f(x)}g(x)\,dx, \quad f,g \in L^2(\Omega;dx). \tag{1.3.3}$$

In particular, for each $L \in (0,\infty)$, $L^2((-L,L);dx)$ and $L^2(\mathbb{R}\backslash(-L,L);dx)$ also constitute Hilbert spaces.

If $f \in L^2((-L,L);dx)$ and $g \in L^2(\mathbb{R}\backslash(-L,L);dx)$, then we define a function $(f \oplus_L g) \in L^2(\mathbb{R};dx)$ by declaring

$$(f \oplus_L g)(x) = \begin{cases} f(x), & \text{a.e. } x \in (-L,L), \\ g(x), & \text{a.e. } x \in \mathbb{R}\backslash(-L,L). \end{cases} \tag{1.3.4}$$

From the properties of the Lebesgue integral, it is clear that $(f \oplus_L g) \in L^2(\mathbb{R};dx)$ and that

$$\|f \oplus_L g\|^2_{L^2(\mathbb{R};dx)} = \|f\|^2_{L^2((-L,L);dx)} + \|g\|^2_{L^2(\mathbb{R}\backslash(-L,L);dx)}, \tag{1.3.5}$$

for all $f \in L^2((-L,L);dx)$ and all $g \in L^2(\mathbb{R}\backslash(-L,L);dx)$. In addition, every function $u \in L^2(\mathbb{R};dx)$ may be expressed in the form (1.3.4):

$$u = f \oplus_L g \quad \text{with} \quad f = u|_{(-L,L)} \quad \text{and} \quad g = u|_{\mathbb{R}\backslash(-L,L)}. \tag{1.3.6}$$

Thus, for each $L > 0$, $L^2(\mathbb{R};dx)$ may be expressed as a *direct sum* of $L^2((-L,L);dx)$ and $L^2(\mathbb{R}\backslash(-L,L);dx)$:

$$L^2(\mathbb{R};dx) = L^2((-L,L);dx) \oplus_L L^2(\mathbb{R}\backslash(-L,L);dx). \tag{1.3.7}$$

By the additivity property of the Lebesgue integral, the inner product of two functions in $L^2(\mathbb{R};dx)$ may be expressed as a sum of individual inner products of their corresponding components: if $u, v \in L^2(\mathbb{R}; dx)$ are given by

$$u = f_1 \oplus_L g_1 \quad \text{and} \quad v = f_2 \oplus_L g_2, \tag{1.3.8}$$

for some $f_j \in L^2((-L, L); dx)$ and $g_j \in L^2(\mathbb{R}\backslash(-L, L); dx)$, $j \in \{1, 2\}$, then evidently

$$(u, v)_{L^2(\mathbb{R};dx)} = (f_1, f_2)_{L^2((-L,L);dx)} + (g_1, g_2)_{L^2(\mathbb{R}\backslash(-L,L);dx)}. \tag{1.3.9}$$

In the sequel, we shall make use of the fact that a function $u \in L^2(\mathbb{R}; dx)$ may be decomposed according to (1.3.6). Since the decomposition obviously depends on the value of $L \in (0, \infty)$, and we intend to study various limiting phenomena as $L \to \infty$, we insist on the notation "$\oplus_L$" to bring out the explicit L-dependence of the decomposition in (1.3.6).

CHAPTER 2

ONE-DIMENSIONAL SCHRÖDINGER OPERATORS AND CONVERGENCE PROPERTIES OF RESOLVENTS

2.1. Basic Properties of One-dimensional Schrödinger Operators

In this section, we introduce notation and rigorously define the families of one-dimensional Schrödinger operators to be studied in the sequel. We begin by recalling the notion of absolute continuity for functions $f : [a, b] \to \mathbb{C}$.

Definition 2.1. *Let $a, b \in \mathbb{R}$ with $a < b$. A function $f : [a, b] \to \mathbb{C}$ is absolutely continuous on $[a, b]$ if and only if there exists a function $h \in L^1((a, b); dx)$ such*

$$f(x) = f(a) + \int_a^x h(t)\, dt, \quad x \in [a, b]. \tag{2.1.1}$$

The set of all absolutely continuous functions on $[a, b]$ is denoted by $AC([a, b])$.

If $f \in AC([a, b])$ with (2.1.1), then f is differentiable almost everywhere on (a, b) and $f'(x) = h(x)$ for a.e. $x \in (a, b)$. Absolute continuity forms the basis for defining the Sobolev space $H^1(\mathbb{R})$.

Definition 2.2. *A function $f : \mathbb{R} \to \mathbb{C}$ belongs to the class $H^1(\mathbb{R})$ if and only if $f|_{[a,b]} \in AC([a, b])$ for all intervals $[a, b] \subset \mathbb{R}$, $f \in L^2(\mathbb{R}; dx)$, and $f' \in L^2(\mathbb{R}; dx)$.*

Given these preparatory definitions, we now introduce the Schrödinger operators to be studied. Suppose

$$V \in L^1(\mathbb{R}; dx) \text{ is real-valued a.e.,} \tag{2.1.2}$$

with

$$M := \int_{-\infty}^{\infty} |V(x)|\, dx < \infty, \tag{2.1.3}$$

and define the differential expression τ by

$$\tau = -\frac{d^2}{dx^2} + V(x). \tag{2.1.4}$$

Consider the self-adjoint operator H in $L^2(\mathbb{R}; dx)$ defined by

$$\begin{aligned} &(Hf)(x) = (\tau f)(x) \text{ for a.e. } x \in \mathbb{R}, \\ &f \in \operatorname{dom}(H) = \big\{g \in L^2(\mathbb{R}; dx) \,\big|\, g, g' \in AC([-R,R]) \text{ for all } R > 0,\, \tau g \in L^2(\mathbb{R}; dx)\big\}. \end{aligned} \tag{2.1.5}$$

Alternatively, H is the unique semibounded (from below) self-adjoint operator associated via the *KLMN Theorem* (cf., e.g., [**17**, Theorem 6.24 & Corollary 9.36]) with the closed, semibounded (from below) symmetric sesquilinear form Q_H given by

$$Q_H(f,g) = \int_{-\infty}^{\infty} \Big[\overline{f'(x)} g'(x) + \overline{f(x)} V(x) g(x)\Big]\, dx, \quad f, g \in \operatorname{dom}(Q_H) = H^1(\mathbb{R}). \tag{2.1.6}$$

At this point, we introduce a factorization of V to be used in the sequel. Specifically, let

$$V(x) = u(x)v(x), \quad v(x) = |V(x)|^{1/2}, \quad u(x) = v(x)\operatorname{sgn}(V(x)) \quad \text{for a.e. } x \in \mathbb{R}, \tag{2.1.7}$$

and then for each $\ell \in \mathbb{N}$, one defines

$$\begin{aligned} V_\ell(x) = V(x)|_{(-\ell,\ell)}, \quad v_\ell(x) = v(x)|_{(-\ell,\ell)}, \quad u_\ell(x) = u(x)|_{(-\ell,\ell)}, \\ \text{for a.e. } x \in (-\ell,\ell). \end{aligned} \tag{2.1.8}$$

In light of the assumption in (2.1.2), one notes that

$$u, v \in L^2(\mathbb{R}; dx), \quad u_\ell, v_\ell \in L^2((-\ell,\ell); dx), \quad \ell \in \mathbb{N}. \tag{2.1.9}$$

For each $\ell \in \mathbb{N}$, define the differential expression

$$\tau_\ell = -\frac{d^2}{dx^2} + V_\ell(x), \tag{2.1.10}$$

and let $H_{\ell,\max}$ denote the maximally defined Sturm–Liouville operator of Schrödinger type associated with τ_ℓ in $L^2((-\ell,\ell);dx)$:

$$\begin{aligned}
&(H_{\ell,\max} f)(x) = (\tau_\ell f)(x) \text{ for a.e. } x \in (-\ell,\ell), \\
&f \in \operatorname{dom}(H_{\ell,\max}) = \{g \in L^2((-\ell,\ell);dx) \,|\, g, g' \in AC([-\ell,\ell]), \\
&\qquad\qquad \tau_\ell g \in L^2((-\ell,\ell);dx)\}, \quad \ell \in \mathbb{N}.
\end{aligned} \tag{2.1.11}$$

Next, we introduce the self-adjoint restriction of $H_{\ell,\max}$ with periodic boundary conditions:

$$\begin{aligned}
&(H_\ell f)(x) = (\tau_\ell f)(x) \text{ for a.e. } x \in (-\ell,\ell), \\
&f \in \operatorname{dom}(H_\ell) = \big\{g \in L^2((-\ell,\ell);dx) \,\big|\, g, g' \in AC([-\ell,\ell]),\ g(-\ell) = g(\ell), \\
&\qquad\qquad g'(-\ell) = g'(\ell),\ \tau_\ell g \in L^2((-\ell,\ell);dx)\big\}, \\
&\qquad\qquad \ell \in \mathbb{N}.
\end{aligned} \tag{2.1.12}$$

In the special case $V \equiv 0$, we use the superscript "(0)" to denote the corresponding differential expressions and operators. Explicitly,

$$\tau^{(0)} = -\frac{d^2}{dx^2}, \tag{2.1.13}$$

so that

$$\begin{aligned}
&\big(H^{(0)} f\big)(x) = \big(\tau^{(0)} f\big)(x) \text{ for a.e. } x \in \mathbb{R}, \\
&f \in \operatorname{dom}\big(H^{(0)}\big) = \big\{g \in L^2(\mathbb{R};dx) \,\big|\, g, g' \in AC([-R,R]) \text{ for all } R > 0, \\
&\qquad\qquad \tau^{(0)} g \in L^2(\mathbb{R};dx)\big\},
\end{aligned} \tag{2.1.14}$$

and the free maximal Sturm–Liouville operator of Schrödinger type in $L^2((-\ell,\ell);dx)$ is given by

$$\begin{aligned}
&(H_{\ell,\max}^{(0)}f)(x) = (\tau_\ell^{(0)}f)(x) \text{ for a.e. } x\in(-\ell,\ell), \qquad (2.1.15)\\
&f\in \operatorname{dom}(H_{\ell,\max}^{(0)}) = \{g\in L^2((-\ell,\ell);dx)\,|\, g,g'\in AC([-\ell,\ell]),\\
&\qquad\qquad \tau_\ell^{(0)}g\in L^2((-\ell,\ell);dx)\}, \quad \ell\in\mathbb{N}.
\end{aligned}$$

The self-adjoint restriction of $H_{\ell,\max}^{(0)}$ with periodic boundary conditions then takes the form:

$$\begin{aligned}
&\big(H_\ell^{(0)}f\big)(x) = \big(\tau_\ell^{(0)}f\big)(x) \text{ for a.e. } x\in(-\ell,\ell), \qquad (2.1.16)\\
&f\in \operatorname{dom}\big(H_\ell^{(0)}\big) = \big\{g\in L^2((-\ell,\ell);dx)\,\big|\, g,g'\in AC([-\ell,\ell]),\, g(-\ell)=g(\ell),\\
&\qquad\qquad g'(-\ell)=g'(\ell),\, \tau_\ell^{(0)}g\in L^2((-\ell,\ell);dx)\big\},\\
&\qquad\qquad \ell\in\mathbb{N}.
\end{aligned}$$

The sesquilinear form associated to H_ℓ (cf., e.g., [**4**, (5.3)] and [**15**, §10.2]) will be denoted by Q_{H_ℓ} and it is given by

$$\begin{aligned}
&Q_{H_\ell}(f,g) = \int_{-\ell}^{\ell}\Big[\overline{f'(x)}g'(x) + \overline{f(x)}V_\ell(x)g(x)\Big]\,dx, \qquad (2.1.17)\\
&f,g\in\operatorname{dom}(Q_{H_\ell}) = \big\{h\in L^2((-\ell,\ell);dx)\,\big|\, h\in AC([-\ell,\ell]),\\
&\qquad\qquad h(-\ell)=h(\ell),\, h'\in L^2((-\ell,\ell);dx)\big\}, \quad \ell\in\mathbb{N},
\end{aligned}$$

and $Q_{H_\ell}^{(0)}$ denotes the same expression in the special case $V\equiv 0$. A close look at the quadratic form, combined with standard estimates, reveals that H_ℓ is bounded from below uniformly in $\ell\in\mathbb{N}$.

THEOREM 2.3. *If* $M=\int_{-\infty}^{\infty}|V(x)|\,dx$, *then*

$$Q_{H_\ell}(f,f) \geq -M(M+1)\|f\|^2_{L^2((-\ell,\ell);dx)}, \quad f\in\operatorname{dom}(Q_{H_\ell}),\, \ell\in\mathbb{N}. \qquad (2.1.18)$$

As a result,

$$H_\ell \geq -M(M+1), \quad \ell \in \mathbb{N}, \tag{2.1.19}$$

and therefore

$$\sigma(H_\ell) \subset [-M(M+1), \infty), \quad \ell \in \mathbb{N}. \tag{2.1.20}$$

Proof. It suffices to prove

$$(f, H_\ell f)_{L^2((-\ell,\ell);dx)} \geq -M(M+1)\|f\|^2_{L^2((-\ell,\ell);dx)}, \quad f \in \operatorname{dom}(H_\ell),\ \ell \in \mathbb{N}. \tag{2.1.21}$$

Let $\ell \in \mathbb{N}$ and $f \in \operatorname{dom}(H_\ell)$ be fixed. If $M = 0$, then clearly $H_\ell \geq 0 = -0(0+1)$. Thus, we may assume without loss that $M > 0$. Integrating by parts, one obtains:

$$\begin{aligned}
&(f, H_\ell f)_{L^2((-\ell,\ell);dx)} \\
&\quad = \int_{-\ell}^{\ell} \overline{f(x)}\big(-f''(x) + V(x)f(x)\big)\,dx \\
&\quad = -\int_{-\ell}^{\ell} \overline{f(x)}f''(x)\,dx + \int_{-\ell}^{\ell} V(x)|f(x)|^2\,dx \\
&\quad = -\bigg[\overline{f(x)}f'(x)\bigg|_{-\ell}^{\ell} - \int_{-\ell}^{\ell} |f'(x)|^2\,dx\bigg] + \int_{-\ell}^{\ell} V(x)|f(x)|^2\,dx \\
&\quad = \int_{-\ell}^{\ell} |f'(x)|^2\,dx + \int_{-\ell}^{\ell} V(x)|f(x)|^2\,dx \\
&\quad = \|f'\|^2_{L^2((-\ell,\ell);dx)} + \int_{-\ell}^{\ell} V(x)|f(x)|^2\,dx \\
&\quad \geq \|f'\|^2_{L^2((-\ell,\ell);dx)} - \int_{-\ell}^{\ell} |V(x)||f(x)|^2\,dx.
\end{aligned} \tag{2.1.22}$$

By [**17**, Lemma 9.32] with the choice $\varepsilon = M^{-1}$,

$$\begin{aligned}
&\int_{-\ell}^{\ell} |V(x)||f(x)|^2\,dx \\
&\quad = \sum_{n=-\ell}^{\ell-1} \int_n^{n+1} |V(x)||f(x)|^2\,dx \\
&\quad \leq \sum_{n=-\ell}^{\ell-1} \bigg(\bigg[\int_n^{n+1} |V(x)|\,dx\bigg] \sup_{t\in[n,n+1]} |f(t)|^2\bigg)
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n=-\ell}^{\ell-1} \Bigg(\bigg[\int_n^{n+1} |V(x)|\,dx \bigg] \bigg[M^{-1} \int_n^{n+1} |f'(x)|^2\,dx + (1+M) \int_n^{n+1} |f(x)|^2\,dx \bigg] \Bigg), \\
&\leq \sum_{n=-\ell}^{\ell-1} \Bigg(M \bigg[M^{-1} \int_n^{n+1} |f'(x)|^2\,dx + (1+M) \int_n^{n+1} |f(x)|^2\,dx \bigg] \Bigg), \\
&= \sum_{n=-\ell}^{\ell-1} \Bigg(\int_n^{n+1} |f'(x)|^2\,dx + M(1+M) \int_n^{n+1} |f(x)|^2\,dx \Bigg), \\
&= \|f'\|^2_{L^2((-\ell,\ell);dx)} + M(1+M)\|f\|^2_{L^2((-\ell,\ell);dx)}.
\end{aligned} \tag{2.1.23}
$$

Now,

$$
\begin{aligned}
&(f, H_\ell f)_{L^2((-\ell,\ell);dx)} \\
&\quad \geq \|f'\|^2_{L^2((-\ell,\ell);dx)} - \Big(\|f'\|^2_{L^2((-\ell,\ell);dx)} + M(1+M)\|f\|^2_{L^2((-\ell,\ell);dx)} \Big) \\
&\quad = -M(M+1)\|f\|^2_{L^2((-\ell,\ell);dx)}.
\end{aligned} \tag{2.1.24}
$$

□

For future reference and ease of notation, we use λ_∞ to denote the uniform lower bound:

$$
\lambda_\infty := -M(M+1). \tag{2.1.25}
$$

We also introduce the operators $H_{\ell,D}$ and $H_{\ell,D}^{(0)}$ with Dirichlet boundary conditions at the endpoints of $[-\ell,\ell]$, as they will serve as important reference operators:

$$
\begin{aligned}
&(H_{\ell,D} f)(x) = (\tau_\ell f)(x) \ \text{ for a.e. } x \in (-\ell,\ell), \\
&f \in \operatorname{dom}(H_{\ell,D}) = \big\{ g \in L^2((-\ell,\ell);dx) \,\big|\, g, g' \in AC([-\ell,\ell]),\ g(-\ell) = g(\ell) = 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \tau_\ell g \in L^2((-\ell,\ell);dx) \big\},
\end{aligned} \tag{2.1.26}
$$

and

$$
(H_{\ell,D}^{(0)} f)(x) = (\tau_\ell^{(0)} f)(x) \ \text{ for a.e. } x \in (-\ell,\ell), \tag{2.1.27}
$$

$$f \in \operatorname{dom}(H_{\ell,D}^{(0)}) = \big\{g \in L^2((-\ell,\ell);dx)\,\big|\, g, g' \in AC([-\ell,\ell]),\, g(-\ell) = g(\ell) = 0,$$
$$\tau_\ell^{(0)} g \in L^2((-\ell,\ell);dx)\big\}.$$

One recalls that the sesquilinear form $Q_{H_{\ell,D}}$ of the Dirichlet operator $H_{\ell,D}$ is given by (cf., e.g., [**4**])

$$Q_{H_{\ell,D}}(f,g) = \int_{-\ell}^{\ell} \Big[\overline{f'(x)}g'(x) + \overline{f(x)}V_\ell(x)g(x)\Big]\,dx, \tag{2.1.28}$$
$$f, g \in \operatorname{dom}(Q_{H_{\ell,D}}) = \big\{h \in L^2((-\ell,\ell);dx)\,\big|\, h \in AC([-\ell,\ell]),$$
$$h(-\ell) = h(\ell) = 0,\, h' \in L^2((-\ell,\ell);dx)\big\}, \quad \ell \in \mathbb{N}.$$

Actually, a simple modification to the proof of Theorem 2.3, systematically replacing H_ℓ by $H_{\ell,D}$, reveals that the Dirichlet operators $H_{\ell,D}$, $\ell \in \mathbb{N}$, are also uniformly bounded from below by $\lambda_\infty = -M(M+1)$.

THEOREM 2.4. *If* $M = \int_{-\infty}^{\infty} |V(x)|\,dx$, *then*

$$Q_{H_{\ell,D}}(f,f) \geq -M(M+1)\|f\|^2_{L^2((-\ell,\ell);dx)}, \quad f \in \operatorname{dom}(Q_{H_{\ell,D}}), \quad \ell \in \mathbb{N}. \tag{2.1.29}$$

As a result, $H_{\ell,D} \geq -M(M+1)$ *for each* $\ell \in \mathbb{N}$ *and therefore,*

$$\sigma(H_{\ell,D}) \subset [-M(M+1), \infty), \quad \ell \in \mathbb{N}. \tag{2.1.30}$$

Having defined the self-adjoint operators H, $H^{(0)}$, H_ℓ, $H_\ell^{(0)}$, $H_{\ell,D}$, and $H_{\ell,D}^{(0)}$, $\ell \in \mathbb{N}$, we introduce their resolvent operators as follows:

$$R(z) = \big(H - zI_{L^2(\mathbb{R};dx)}\big)^{-1}, \quad z \in \rho(H),$$
$$R^{(0)}(z) = \big(H^{(0)} - zI_{L^2(\mathbb{R};dx)}\big)^{-1}, \quad z \in \mathbb{C}\backslash[0,\infty), \tag{2.1.31}$$

and

$$R_\ell(z) = \big(H_\ell - zI_{L^2((-\ell,\ell);dx)}\big)^{-1}, \quad z \in \rho(H_\ell),$$

$$R_\ell^{(0)}(z) = \big(H_\ell^{(0)} - zI_{L^2((-\ell,\ell);dx)}\big)^{-1}, \quad z \in \rho(H_\ell^{(0)}),$$

$$R_{\ell,D}(z) = \big(H_{\ell,D} - zI_{L^2((-\ell,\ell);dx)}\big)^{-1}, \quad z \in \rho(H_{\ell,D}),$$

$$R_{\ell,D}^{(0)}(z) = \big(H_{\ell,D}^{(0)} - zI_{L^2((-\ell,\ell);dx)}\big)^{-1}, \quad z \in \rho(H_{\ell,D}^{(0)}),\ \ell \in \mathbb{N}. \tag{2.1.32}$$

Using Krein-type resolvent formulas, one may relate the resolvent of H_ℓ to the resolvent of $H_{\ell,D}$ for each fixed $\ell \in \mathbb{N}$. The precise form of these resolvent formulas was worked out in [**4**] for all self-adjoint restrictions of $H_{\ell,\max}$ (i.e., for all separated and non-separated boundary conditions). For completeness, we present the result in the special case of periodic boundary conditions since the resulting formula will play a crucial role later on.

In order to state Krein's formula, one introduces for each $z \in \rho(H_{\ell,D})$ a distinguished basis for $\ker(H_{\ell,\max} - zI_{L^2((-\ell,\ell);dx)})$, denoted by $\{\psi_{j,\ell}(z,\cdot)\}_{j=1,2}$, by specifying the boundary values

$$\begin{aligned} &\psi_{1,\ell}(z,-\ell) = 0, \quad \psi_{1,\ell}(z,\ell) = 1, \\ &\psi_{2,\ell}(z,-\ell) = 1, \quad \psi_{2,\ell}(z,\ell) = 0, \end{aligned} \tag{2.1.33}$$

in addition to the requirement

$$H_{\ell,\max}\psi_{j,\ell}(z,x) = z\psi_{j,\ell}(z,x), \quad x \in [-\ell,\ell],\ j \in \{1,2\}, \tag{2.1.34}$$

which simply means that $\psi_{j,\ell}(z,\cdot) \in \operatorname{dom}(H_{\ell,\max})$ satisfies the ordinary differential equation

$$-\psi_{j,\ell}''(z,x) + V_\ell(x)\psi_{j,\ell}(z,x) = z\psi_{j,\ell}(z,x), \quad x \in (-\ell,\ell),\ j \in \{1,2\},\ z \in \rho(H_{\ell,D}), \ell \in \mathbb{N}. \tag{2.1.35}$$

In the special case $V \equiv 0$, we will follow our previously adopted convention and denote this basis for $\ker\big(H_{\ell,\max}^{(0)} - zI_{L^2((-\ell,\ell);dx)}\big)$ by $\{\psi_{j,\ell}^{(0)}(z,\cdot)\}_{j=1,2}$. Actually, $\{\psi_{j,\ell}^{(0)}(z,\cdot)\}_{j=1,2}$ may be computed explicitly by solving the ordinary differential equation $-\psi_{j,\ell}''(z,x) = z\psi_{j,\ell}(z,x)$, and one finds, for each $\ell \in \mathbb{N}$,

$$\psi_{1,\ell}^{(0)}(z,x) = \frac{1}{2}\bigg[\frac{\cos(z^{1/2}x)}{\cos(z^{1/2}\ell)} + \frac{\sin(z^{1/2}x)}{\sin(z^{1/2}\ell)}\bigg], \tag{2.1.36}$$
$$\psi_{2,\ell}^{(0)}(z,x) = \frac{1}{2}\bigg[\frac{\cos(z^{1/2}x)}{\cos(z^{1/2}\ell)} - \frac{\sin(z^{1/2}x)}{\sin(z^{1/2}\ell)}\bigg], \quad x \in [-\ell,\ell],\ \operatorname{Im}(z^{1/2}) \geq 0,\ z \in \rho(H_{\ell,D}).$$

Returning to the case of general V, and with this basis $\{\psi_{j,\ell}(z,\,\cdot\,)\}_{j=1,2}$ in hand, we now state Krein's resolvent formula which connects $R_\ell(z)$ to $R_{\ell,D}(z)$ via a rank-one term.

Lemma 2.5 (Krein's Resolvent Formula). *If $\ell \in \mathbb{N}$ and $\{\psi_{j,\ell}(z,\,\cdot\,)\}_{j=1,2}$ denotes the basis of* $\ker(H_{\ell,\max} - zI_{L^2((-\ell,\ell);dx)})$ *which satisfies* (2.1.33) *for $z \in \rho(H_{\ell,D})$, then*

$$R_\ell(z) = R_{\ell,D}(z) + P_\ell(z), \quad z \in \rho(H_\ell) \cap \rho(H_{\ell,D}),\ \ell \in \mathbb{N}, \tag{2.1.37}$$

where the rank one operator $P_\ell(z)$ is defined by

$$P_\ell(z) = -q_\ell(z)^{-1}(\psi_\ell(\overline{z},\,\cdot\,),\,\cdot\,)_{L^2((-\ell,\ell);dx)}\psi_\ell(z,\,\cdot\,), \quad z \in \rho(H_\ell) \cap \rho(H_{\ell,D}),\ \ell \in \mathbb{N}, \tag{2.1.38}$$

with

$$q_\ell(z) = \psi_\ell'(z,-\ell) - \psi_\ell'(z,\ell), \quad z \in \rho(H_\ell) \cap \rho(H_{\ell,D}),\ \ell \in \mathbb{N}, \tag{2.1.39}$$

and

$$\psi_\ell(z,x) = \psi_{2,\ell}(z,x) + \psi_{1,\ell}(z,x), \quad x \in [-\ell,\ell],\ z \in \rho(H_{\ell,D}),\ \ell \in \mathbb{N}. \tag{2.1.40}$$

A proof of Lemma 2.5 is provided in Appendix A.

Example 2.6. *In the special case when $V(x) = 0$ for a.e. $x \in \mathbb{R}$, the terms in* (2.1.38), (2.1.39), *and* (2.1.40) *may be computed explicitly, and one obtains*

$$R_\ell^{(0)}(z) = R_{\ell,D}^{(0)}(z) + P_\ell^{(0)}(z), \quad z \in \rho(H_\ell^{(0)}) \cap \rho(H_{\ell,D}^{(0)}),\ \ell \in \mathbb{N}, \tag{2.1.41}$$

where

$$P_\ell^{(0)}(z) = -\frac{\cot(z^{1/2}\ell)}{2z^{1/2}}\big(\psi_\ell^{(0)}(\overline{z},\,\cdot\,),\,\cdot\,\big)_{L^2((-\ell,\ell);dx)}\psi_\ell^{(0)}(z,\,\cdot\,),$$
$$z\in\rho(H_\ell^{(0)})\cap\rho(H_{\ell,D}^{(0)}),\ \mathrm{Im}(z^{1/2})\geq 0,\ \ell\in\mathbb{N}, \tag{2.1.42}$$

with

$$q_\ell^{(0)}(z)^{-1} = \frac{\cot(z^{1/2}\ell)}{2z^{1/2}},\quad z\in\rho(H_\ell^{(0)})\cap\rho(H_{\ell,D}^{(0)}),\ \mathrm{Im}(z^{1/2})\geq 0,\ \ell\in\mathbb{N}, \tag{2.1.43}$$

and

$$\psi_\ell^{(0)}(z,x) = \frac{\cos(z^{1/2}x)}{\cos(z^{1/2}\ell)},\quad z\in\rho(H_\ell^{(0)})\cap\rho(H_{\ell,D}^{(0)}),\ \mathrm{Im}(z^{1/2})\geq 0,\ \ell\in\mathbb{N}. \tag{2.1.44}$$

2.2. Convergence Properties of Resolvents

In this section, we study various convergence properties of the resolvents of the periodic Sturm–Liouville differential operators introduced in the previous section. The results obtained here are analogous to the results obtained in [**7**] for the case of Dirichlet boundary conditions. The novelty of our approach for the case of periodic boundary conditions lies in our use of Krein's resolvent formula (2.1.37). After applying Krein's formula and the known convergence properties of the Dirichlet operators from [**7**], we are left to investigate convergence of rank one operators. These rank one operators can be addressed explicitly through a combination of techniques from Section 1.2. Later, we show the convergence properties developed in this section have consequences for the infinite volume limit of the underlying spectral shift functions.

We begin by recalling the following convergence results for resolvents of the free Dirichlet operators $H_{\ell,D}^{(0)}$, $\ell\in\mathbb{N}$.

Lemma 2.7 (Lemma 3.1 in [**7**]). *If* $H^{(0)}_{\ell,D}$, $\ell \in \mathbb{N}$, *denotes the free Dirichlet operator defined in* (2.1.27) *and* $H^{(0)}$ *denotes (minus) the free Laplacian defined in* (2.1.14), *then the sequence* $\left\{H^{(0)}_{\ell,D} \oplus_\ell 0\right\}_{\ell=1}^{\infty}$ *converges to* $H^{(0)}$ *in the strong resolvent sense. That is, for each fixed* $z \in \mathbb{C}\backslash[0,\infty)$,

$$\operatorname*{s-lim}_{\ell\to\infty}\left(\left[H^{(0)}_{\ell,D} \oplus_\ell 0\right] - zI_{L^2(\mathbb{R};dx)}\right)^{-1} = R^{(0)}(z). \tag{2.2.1}$$

Lemma 2.8 (Lemmata 3.1 and 3.2 in [**7**]). *If* $R^{(0)}_{\ell,D}(\,\cdot\,)$, $\ell \in \mathbb{N}$, *denotes the resolvent of the free Dirichlet operator defined in* (2.1.27) *and* $R^{(0)}(\,\cdot\,)$ *denotes the resolvent of (minus) the free Laplacian defined in* (2.1.14), *then for each fixed* $z \in \mathbb{C}\backslash[0,\infty)$, *the following convergence results hold in* $\mathcal{B}_2(L^2(\mathbb{R};dx))$:

$$\lim_{\ell\to\infty}\left\|\left[u_\ell R^{(0)}_{\ell,D}(z) \oplus_\ell 0\right] - uR^{(0)}(z)\right\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} = 0, \tag{2.2.2}$$

$$\lim_{\ell\to\infty}\left\|\left[\overline{R^{(0)}_{\ell,D}(z)v_\ell} \oplus_\ell 0\right] - \overline{R^{(0)}(z)v}\right\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} = 0, \tag{2.2.3}$$

and the following convergence result holds in $\mathcal{B}_1(L^2(\mathbb{R};dx))$:

$$\lim_{\ell\to\infty}\left\|\left[\overline{u_\ell R^{(0)}_{\ell,D}(z)v_\ell} \oplus_\ell 0\right] - \overline{uR^{(0)}(z)v}\right\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} = 0. \tag{2.2.4}$$

By applying Krein's resolvent formula (2.1.41), we obtain our first new result, the following extension of Lemma 2.7 to the case of periodic boundary conditions at the endpoints of $[-\ell,\ell]$.

Theorem 2.9. *If* $H^{(0)}_{\ell}$, $\ell \in \mathbb{N}$, *denotes the free periodic operator defined in* (2.1.16) *and* $H^{(0)}$ *denotes (minus) the free Laplacian defined in* (2.1.14), *then the sequence* $\left\{H^{(0)}_{\ell} \oplus_\ell 0\right\}_{\ell=1}^{\infty}$ *converges to* $H^{(0)}$ *in the strong resolvent sense. That is, for each fixed* $z \in \mathbb{C}\backslash[0,\infty)$,

$$\operatorname*{s-lim}_{\ell\to\infty}\left(\left[H^{(0)}_{\ell} \oplus_\ell 0\right] - zI_{L^2(\mathbb{R};dx)}\right)^{-1} = R^{(0)}(z). \tag{2.2.5}$$

PROOF. We begin by introducing some notation. For each $f \in L^2(\mathbb{R}; dx)$ and $\ell \in \mathbb{N}$, we define the function $f_{<\ell} \in L^2((-\ell,\ell); dx)$ by the requirement that

$$f_{<\ell}(x) = f(x) \text{ for a.e. } x \in (-\ell,\ell). \tag{2.2.6}$$

It suffices to prove (2.2.5) for just one $z \in \mathbb{C}\backslash[0,\infty)$, which we take to be $z = -1$. The result then follows for arbitrary $z \in \mathbb{C}\backslash[0,\infty)$ by the application of the resolvent identity (cf., e.g., [**18**, Exercise 7.8]):

$$\begin{aligned}(T_2 - z_0 I_{\mathcal{H}})^{-1} - (T_1 - z_0 I_{\mathcal{H}})^{-1} &= (T_2 - z I_{\mathcal{H}})(T_2 - z_0 I_{\mathcal{H}})^{-1} \\ \times \big[(T_2 - z I_{\mathcal{H}})^{-1} - (T_1 - z I_{\mathcal{H}})^{-1}\big]&(T_1 - z I_{\mathcal{H}})(T_1 - z_0 I_{\mathcal{H}})^{-1}, \\ &z, z_0 \in \rho(T_1) \cap \rho(T_2),\end{aligned} \tag{2.2.7}$$

which holds for any two linear operators T_1 and T_2 in an arbitrary Hilbert space $\mathcal{H}$ with $\rho(T_1) \cap \rho(T_2) \neq \emptyset$. Therefore, we will show

$$\lim_{\ell\to\infty} \bigg\| \Big(\Big[H_\ell^{(0)} \oplus_\ell 0 \Big] + I_{L^2(\mathbb{R};dx)} \Big)^{-1} f - R^{(0)}(-1) f \bigg\|_{L^2(\mathbb{R};dx)} = 0, \quad f \in L^2(\mathbb{R}; dx). \tag{2.2.8}$$

For each $f \in L^2(\mathbb{R}; dx)$, we compute

$$\begin{aligned}&\bigg\| \Big(\Big[H_\ell^{(0)} \oplus_\ell 0 \Big] + I_{L^2(\mathbb{R};dx)} \Big)^{-1} f - R^{(0)}(-1) f \bigg\|_{L^2(\mathbb{R};dx)} \\ &\quad = \bigg\| \Big[\Big(H_\ell^{(0)} + I_{L^2((-\ell,\ell);dx)} \Big)^{-1} \oplus_\ell I_{L^2(\mathbb{R}\backslash(-\ell,\ell);dx)} \Big] f - R^{(0)}(-1) f \bigg\|_{L^2(\mathbb{R};dx)} \\ &\quad = \bigg\| \Big\{ \Big[\Big(H_{\ell,D}^{(0)} + I_{L^2((-\ell,\ell);dx)} \Big)^{-1} + P_\ell^{(0)}(-1) \Big] \oplus_\ell I_{L^2(\mathbb{R}\backslash(-\ell,\ell);dx)} \Big\} f - R^{(0)}(-1) f \bigg\|_{L^2(\mathbb{R};dx)} \\ &\quad \leq \bigg\| \Big(\Big[H_{\ell,D}^{(0)} \oplus_\ell 0 \Big] + I_{L^2(\mathbb{R};dx)} \Big)^{-1} f - R^{(0)}(-1) f \bigg\|_{L^2(\mathbb{R};dx)} + \Big\| \big[P_\ell^{(0)}(-1) \oplus_\ell 0 \big] f \Big\|_{L^2(\mathbb{R};dx)}, \\ &\qquad \ell \in \mathbb{N}.\end{aligned} \tag{2.2.9}$$

By Lemma 2.7,

$$\lim_{\ell\to 0} \bigg\| \Big(\Big[H_{\ell,D}^{(0)} \oplus_\ell 0 \Big] + I_{L^2(\mathbb{R};dx)} \Big)^{-1} f - R^{(0)}(-1) f \bigg\|_{L^2(\mathbb{R};dx)} = 0, \tag{2.2.10}$$

so in view of the inequality in (2.2.9), the claim in (2.2.8) reduces to showing that

$$\lim_{\ell\to\infty}\Big\|\Big[P_\ell^{(0)}(-1)\oplus_\ell 0\Big]f\Big\|_{L^2(\mathbb{R};dx)} = 0, \quad f\in L^2(\mathbb{R};dx), \tag{2.2.11}$$

that is, the sequence of operators $\Big\{P_\ell^{(0)}(-1)\oplus_\ell 0\Big\}_{\ell=1}^{\infty}$ converges strongly to the zero operator in $L^2(\mathbb{R};dx)$. To this end, we claim that the sequence of operators $\Big\{P_\ell^{(0)}(-1)\oplus_\ell 0\Big\}_{\ell=1}^{\infty}$ is uniformly bounded in $\mathcal{B}(L^2(\mathbb{R};dx))$. Indeed, by (1.2.21), (2.1.42), and (2.1.44),

$$\begin{aligned}\Big\|P_\ell^{(0)}(-1)\oplus_\ell 0\Big\|_{\mathcal{B}(L^2(\mathbb{R};dx))} &= \Big\|P_\ell^{(0)}(-1)\Big\|_{\mathcal{B}(L^2((-\ell,\ell);dx))}\\ &= \frac{\coth(\ell)}{2}\big\|\psi_\ell^{(0)}(-1,\,\cdot\,)\big\|^2_{L^2((-\ell,\ell);dx)}\\ &= \coth(\ell)\big[\tanh(\ell)+\ell\,\mathrm{sech}^2(\ell)\big]\\ &\leq 1+2\,\mathrm{csch}(2), \quad \ell\in\mathbb{N}.\end{aligned} \tag{2.2.12}$$

Therefore, by [**18**, Exercise 4.28], it suffices to prove the convergence in (2.2.11) for all f from a dense subspace of $L^2(\mathbb{R};dx)$, which we take to be $L^2(\mathbb{R};dx)\cap L^1(\mathbb{R};dx)$. Let $f\in L^2(\mathbb{R};dx)\cap L^1(\mathbb{R};dx)$ and note that

$$\begin{aligned}&\Big\|\Big[P_\ell^{(0)}(-1)\oplus_\ell 0\Big]f\Big\|^2_{L^2(\mathbb{R};dx)}\\ &\quad=\Big\|P_\ell^{(0)}(-1)f_{<\ell}\Big\|^2_{L^2((-\ell,\ell);dx)}\\ &\quad=\frac{\coth^2(\ell)}{4}\Big|\big(\psi_\ell^{(0)}(-1,\,\cdot\,),f_{<\ell}\big)_{L^2((-\ell,\ell);dx)}\Big|^2\big\|\psi_\ell^{(0)}(-1,\,\cdot\,)\big\|^2_{L^2((-\ell,\ell);dx)}\\ &\quad=\frac{\coth^2(\ell)}{2}\big[\tanh(\ell)+\ell\,\mathrm{sech}^2(\ell)\big]\Big|\big(\psi_\ell^{(0)}(-1,\,\cdot\,),f_{<\ell}\big)_{L^2((-\ell,\ell);dx)}\Big|^2\\ &\quad\leq\frac{\coth(1)}{2}\big[1+2\,\mathrm{csch}(2)\big]\Big|\big(\psi_\ell^{(0)}(-1,\,\cdot\,),f_{<\ell}\big)_{L^2((-\ell,\ell);dx)}\Big|^2, \quad \ell\in\mathbb{N}.\end{aligned} \tag{2.2.13}$$

Clearly,

$$\bigg|\chi_{(-\ell,\ell)}(x)\frac{\cosh(x)}{\cosh(\ell)}f(x)\bigg| \leq |f(x)| \text{ for a.e. } x\in\mathbb{R},\ \ell\in\mathbb{N}, \tag{2.2.14}$$

and

$$\lim_{\ell\to\infty} \chi_{(-\ell,\ell)}(x)\frac{\cosh(x)}{\cosh(\ell)} f(x) = 0 \text{ for a.e. } x \in \mathbb{R}, \tag{2.2.15}$$

so an application of Lebesgue's Dominated Convergence Theorem (cf., e.g., [**13**, Theorem 1.8]) implies

$$\begin{aligned} \lim_{\ell\to\infty} \big(\psi_\ell^{(0)}(-1,\,\cdot\,), f_{<\ell}\big)_{L^2((-\ell,\ell);dx)} &= \lim_{\ell\to\infty} \int_{-\ell}^{\ell} \psi_\ell(-1,x) f_{<\ell}(x)\, dx \\ &= \lim_{\ell\to\infty} \int_{-\infty}^{\infty} \chi_{(-\ell,\ell)}(x)\frac{\cosh(x)}{\cosh(\ell)} f(x)\, dx \\ &= 0. \end{aligned} \tag{2.2.16}$$

Finally, (2.2.16) and the inequality in (2.2.13) imply

$$\lim_{\ell\to\infty} \Big\| \Big[P_\ell^{(0)}(-1) \oplus_\ell 0\Big] f \Big\|_{L^2(\mathbb{R};dx)}^2 = 0, \quad f \in L^2(\mathbb{R}; dx) \cap L^1(\mathbb{R}; dx). \tag{2.2.17}$$

□

Another application of Krein's resolvent formula (2.1.41) yields the second new result in this section, the following extensions of (2.2.2) and (2.2.3) to the case of periodic boundary conditions at the endpoints of $[-\ell,\ell]$.

THEOREM 2.10. *If* $R_\ell^{(0)}(\,\cdot\,)$, $\ell \in \mathbb{N}$, *denotes the resolvent of the free periodic operator defined in* (2.1.16) *and* $R^{(0)}(\,\cdot\,)$ *denotes the resolvent of (minus) the free Laplacian defined in* (2.1.14), *then for each fixed* $z \in \mathbb{C}\backslash[0,\infty)$, *the following convergence results hold in* $\mathcal{B}_2(L^2(\mathbb{R};dx))$:

$$\lim_{\ell\to\infty} \Big\| \Big[u_\ell R_\ell^{(0)}(z) \oplus_\ell 0\Big] - u R^{(0)}(z) \Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} = 0, \tag{2.2.18}$$

$$\lim_{\ell\to\infty} \Big\| \Big[\overline{R_\ell^{(0)}(z) v_\ell} \oplus_\ell 0\Big] - \overline{R^{(0)}(z) v} \Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} = 0. \tag{2.2.19}$$

Proof. We begin by noting that it suffices to prove (2.2.18) for one $z \in \mathbb{C}\backslash[0,\infty)$. To see this, suppose that

$$\lim_{\ell\to\infty} \Big\| \Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big] - uR^{(0)}(z_0)\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} = 0 \tag{2.2.20}$$

for some fixed $z_0 \in \mathbb{C}\backslash[0,\infty)$. We claim that (2.2.20) actually implies (2.2.18) for all $z \in \mathbb{C}\backslash[0,\infty)$. Indeed, for any $z \in \mathbb{C}\backslash[0,\infty)$, the *first resolvent identity* (1.1.12) implies

$$\begin{aligned}
&\Big[u_\ell R_\ell^{(0)}(z) \oplus_\ell 0\Big] - uR^{(0)}(z) \\
&\quad = \Big[u_\ell\Big(R_\ell^{(0)}(z_0) + (z-z_0)R_\ell^{(0)}(z_0)R_\ell^{(0)}(z)\Big) \oplus_\ell 0\Big] - u\Big(R^{(0)}(z_0) + (z-z_0)R^{(0)}(z_0)R^{(0)}(z)\Big) \\
&\quad = \Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big] - uR^{(0)}(z_0) + (z-z_0)\Big\{\Big[u_\ell R_\ell^{(0)}(z_0)R_\ell^{(0)}(z) \oplus_\ell 0\Big] - uR^{(0)}(z_0)R^{(0)}(z)\Big\} \\
&\quad = \Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big] - uR^{(0)}(z_0) \\
&\qquad + (z-z_0)\Big\{\Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big]\Big[R_\ell^{(0)}(z) \oplus_\ell (-z^{-1})I_{L^2(\mathbb{R}\backslash(-\ell,\ell);dx)}\Big] - uR^{(0)}(z_0)R^{(0)}(z)\Big\} \\
&\quad = \Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big] - uR^{(0)}(z_0) \\
&\qquad + (z-z_0)\Big\{\Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big]\Big(\big[H_\ell^{(0)} \oplus_\ell 0\big] - zI_{L^2(\mathbb{R};dx)}\Big)^{-1} - uR^{(0)}(z_0)R^{(0)}(z)\Big\}, \quad \ell \in \mathbb{N},
\end{aligned} \tag{2.2.21}$$

so that

$$\begin{aligned}
&\Big\| \Big[u_\ell R_\ell^{(0)}(z) \oplus_\ell 0\Big] - uR^{(0)}(z)\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} \\
&\quad \leq \Big\| \Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big] - uR^{(0)}(z_0)\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} \\
&\qquad + |z-z_0| \Big\| \Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big]\Big(\big[H_\ell^{(0)} \oplus_\ell 0\big] - zI_{L^2(\mathbb{R};dx)}\Big)^{-1} - uR^{(0)}(z_0)R^{(0)}(z)\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))}, \\
&\qquad \ell \in \mathbb{N}.
\end{aligned} \tag{2.2.22}$$

The first term on the right-hand side of the inequality in (2.2.22) goes to zero as $\ell \to \infty$ by (2.2.20). To show the norm in the second term on the right-hand side of the inequality in (2.2.22) goes to zero as $\ell \to \infty$, we apply Grümm's Theorem (i.e., Theorem 1.26) with the choices $p=2$ and

$$A = uR^{(0)}(z_0), \quad A_\ell = \Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big], \quad \ell \in \mathbb{N},$$
$$B = R^{(0)}(z), \qquad B_\ell = \Big(\Big[H_\ell^{(0)} \oplus_\ell 0\Big] - zI_{L^2(\mathbb{R};dx)}\Big)^{-1}, \quad \ell \in \mathbb{N}. \tag{2.2.23}$$

If $\operatorname{dist}(\zeta,\Omega)$ denotes the distance from a point $\zeta \in \mathbb{C}$ to a subset $\Omega \subset \mathbb{C}$, then

$$\begin{aligned}
\big\|B_\ell\big\|_{\mathcal{B}(L^2(\mathbb{R};dx))} &= \big\|R_\ell^{(0)}(z) \oplus_\ell (-z^{-1}) I_{L^2(\mathbb{R}\setminus(-\ell,\ell);dx)}\big\|_{\mathcal{B}(L^2(\mathbb{R};dx))} \\
&\leq \big\|R_\ell^{(0)}(z)\big\|_{\mathcal{B}(L^2((-\ell,\ell);dx))} + \big\|(-z^{-1}) I_{L^2(\mathbb{R}\setminus(-\ell,\ell);dx)}\big\|_{\mathcal{B}(L^2(\mathbb{R}\setminus(-\ell,\ell);dx))} \\
&= \operatorname{dist}\big(z, \sigma\big(H_\ell^{(0)}\big)\big)^{-1} + |z|^{-1} \\
&\leq \operatorname{dist}(z,[0,\infty))^{-1} + |z|^{-1}, \quad \ell \in \mathbb{N},
\end{aligned} \tag{2.2.24}$$

where (1.1.14) has been applied, shows that

$$\sup_{\ell\in\mathbb{N}} \big\|B_\ell\big\|_{\mathcal{B}(L^2(\mathbb{R};dx))} < \infty. \tag{2.2.25}$$

In (2.2.24), we have used a standard norm estimate for the resolvent of a self-adjoint operator (cf., e.g., [**17**, Problem 3.5 on p. 111]). Moreover, Theorem 2.9 implies $\text{s-lim}_{\ell\to\infty} B_\ell = B$. Therefore, the choices in (2.2.23) satisfy the hypotheses of Grümm's Theorem. Consequently, one infers that

$$\lim_{\ell\to\infty} \Big\| \Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big] \Big(\big[H_\ell^{(0)} \oplus_\ell 0\big] - zI_{L^2(\mathbb{R};dx)}\Big)^{-1} - uR^{(0)}(z_0)R^{(0)}(z)\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} = 0, \tag{2.2.26}$$

and (2.2.22) implies

$$\lim_{\ell\to\infty} \Big\| \Big[u_\ell R_\ell^{(0)}(z) \oplus_\ell 0\Big] - uR^{(0)}(z)\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} = 0. \tag{2.2.27}$$

We will now prove that (2.2.20) holds for $z_0=-1$. We compute, by Krein's resolvent formula (2.1.41),

$$\begin{aligned}
\Big[u_\ell R_\ell^{(0)}(-1) \oplus_\ell 0\Big] - uR^{(0)}(-1) &= \Big[u_\ell\big(R_{\ell,D}^{(0)}(-1) + P_\ell^{(0)}(-1)\big) \oplus_\ell 0\Big] - uR^{(0)}(-1) \\
&= \Big[u_\ell R_{\ell,D}^{(0)}(-1) \oplus_\ell 0\Big] - uR^{(0)}(-1) + \Big[u_\ell P_\ell^{(0)}(-1) \oplus_\ell 0\Big], \quad \ell \in \mathbb{N}.
\end{aligned} \tag{2.2.28}$$

By the triangle inequality,

$$\Big\| \Big[u_\ell R_\ell^{(0)}(-1) \oplus_\ell 0\Big] - uR^{(0)}(-1)\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} \leq \Big\| \Big[u_\ell R_{\ell,D}^{(0)}(-1) \oplus_\ell 0\Big] - uR^{(0)}(-1)\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} + \Big\| u_\ell P_\ell^{(0)}(-1) \oplus_\ell 0\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))}, \quad \ell \in \mathbb{N}. \tag{2.2.29}$$

By (2.2.2), the first term on the right-hand side of the inequality in (2.2.29) converges to zero as $\ell \to \infty$. Thus, it suffices to show

$$\lim_{\ell\to\infty} \Big\| u_\ell P_\ell^{(0)}(-1) \oplus_\ell 0\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} = 0. \tag{2.2.30}$$

To this end, note that

$$\begin{aligned}
&\Big\| u_\ell P_\ell^{(0)}(-1) \oplus_\ell 0\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} \\
&\quad = \Big\| u_\ell P_\ell^{(0)}(-1)\Big\|_{\mathcal{B}_2(L^2((-\ell,\ell);dx))} \\
&\quad = \frac{\coth(\ell)}{2}\Big\| u_\ell \big(\psi_\ell^{(0)}(-1,\,\cdot\,),\,\cdot\,\big)_{L^2((-\ell,\ell);dx)} \psi_\ell^{(0)}(-1,\,\cdot\,)\Big\|_{\mathcal{B}_2(L^2((-\ell,\ell);dx))} \\
&\quad = \frac{\coth(\ell)}{2}\Big\| \big(\psi_\ell^{(0)}(-1,\,\cdot\,),\,\cdot\,\big)_{L^2((-\ell,\ell);dx)} u_\ell(\,\cdot\,)\psi_\ell^{(0)}(-1,\,\cdot\,)\Big\|_{\mathcal{B}_2(L^2((-\ell,\ell);dx))} \\
&\quad = \frac{\coth(\ell)}{2}\Big\| \psi_\ell^{(0)}(-1,\,\cdot\,)\Big\|_{L^2((-\ell,\ell);dx)} \Big\| u_\ell(\,\cdot\,)\psi_\ell^{(0)}(-1,\,\cdot\,)\Big\|_{L^2((-\ell,\ell);dx)}, \quad \ell \in \mathbb{N},
\end{aligned} \tag{2.2.31}$$

where we have used the fact that

$$\big\| A \oplus_\ell 0\big\|_{\mathcal{B}_p(L^2(\mathbb{R};dx))} = \big\|A\big\|_{\mathcal{B}_p(L^2((-\ell,\ell);dx))}, \quad A \in \mathcal{B}_p(L^2((-\ell,\ell);dx)), \quad \ell \in \mathbb{N}. \tag{2.2.32}$$

By an elementary calculation,

$$\Big\| \psi_\ell^{(0)}(-1,\,\cdot\,)\Big\|_{L^2((-\ell,\ell);dx)} = \Big[\tanh(\ell) + \ell\,\mathrm{sech}^2(\ell)\Big]^{\frac{1}{2}}, \quad \ell \in \mathbb{N}.$$

Thus,

$$\lim_{\ell\to\infty} \frac{\coth(\ell)}{2}\Big\| \psi_\ell^{(0)}(-1,\,\cdot\,)\Big\|_{L^2((-\ell,\ell);dx)} = \frac{1}{2}. \tag{2.2.33}$$

In addition,

$$\Big\| u_\ell(\,\cdot\,)\psi_\ell^{(0)}(-1,\,\cdot\,)\Big\|^2_{L^2((-\ell,\ell);dx)} = \int_{-\ell}^{\ell} |V(x)| \bigg|\frac{\cosh(x)}{\cosh(\ell)}\bigg|^2 dx$$

$$= \int_{-\infty}^{\infty} |V(x)|\chi_{(-\ell,\ell)}(x)\left|\frac{\cosh(x)}{\cosh(\ell)}\right|^2 dx, \quad \ell \in \mathbb{N}.$$

In order to apply Lebesgue's Dominated Convergence Theorem, we note that

$$|V(x)|\chi_{(-\ell,\ell)}(x)\left|\frac{\cosh(x)}{\cosh(\ell)}\right|^2 \leq |V(x)| \text{ for a.e. } x \in \mathbb{R},\ \ell \in \mathbb{N}, \tag{2.2.34}$$

and

$$\lim_{\ell\to\infty}\left|V(x)\chi_{(-\ell,\ell)}(x)\left(\frac{\cosh(x)}{\cosh(\ell)}\right)^2\right| = 0 \text{ for a.e. } x \in \mathbb{R}. \tag{2.2.35}$$

By (2.1.2), the hypotheses of Lebesgue's Dominated Convergence Theorem are met and

$$\lim_{\ell\to\infty}\left\|u_\ell(\,\cdot\,)\psi_\ell^{(0)}(-1,\,\cdot\,)\right\|_{L^2((-\ell,\ell);dx)}^2 = 0. \tag{2.2.36}$$

Therefore, by (2.2.33)

$$\lim_{\ell\to\infty}\frac{\coth(\ell)}{2}\left\|\psi_\ell^{(0)}(-1,\,\cdot\,)\right\|_{L^2((-\ell,\ell);dx)}\left\|u_\ell(\,\cdot\,)\psi_\ell^{(0)}(-1,\,\cdot\,)\right\|_{L^2((-\ell,\ell);dx)} = \frac{1}{2}\cdot 0 = 0,$$

and (2.2.30) follows.

The claim in (2.2.19) actually follows for all $z \in \mathbb{C}\backslash[0,\infty)$ by an adjoint argument. Indeed, one notes that

$$\begin{aligned} \overline{R_\ell^{(0)}(z)v_\ell} &= \big(v_\ell R_\ell^{(0)}(\overline{z})\big)^*, \\ \overline{R^{(0)}(z)v} &= \big(vR^{(0)}(\overline{z})\big)^*, \quad z \in \mathbb{C}\backslash[0,\infty), \end{aligned} \tag{2.2.37}$$

and since $\|A^*\|_{\mathcal{B}_2(\mathcal{H})} = \|A\|_{\mathcal{B}_2(\mathcal{H})}$, $A \in \mathcal{B}_2(\mathcal{H})$, one infers that

$$\begin{aligned} &\left\|\left[\,\overline{R_\ell^{(0)}(z)v_\ell} \oplus_\ell 0\right] - \overline{R^{(0)}(z)v}\,\right\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} \\ &\quad = \left\|\left[v_\ell R_\ell^{(0)}(\overline{z}) \oplus_\ell 0\right] - vR^{(0)}(\overline{z})\right\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))}, \quad z \in \mathbb{C}\backslash[0,\infty). \end{aligned} \tag{2.2.38}$$

By repeating the proof of (2.2.18) with u_ℓ and u replaced by v_ℓ and v, respectively, one infers that

$$\lim_{\ell\to\infty}\left\|\left[v_\ell R_\ell^{(0)}(\overline{z}) \oplus_\ell 0\right] - vR^{(0)}(\overline{z})\right\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} = 0, \quad z \in \mathbb{C}\backslash[0,\infty), \tag{2.2.39}$$

and (2.2.19) follows. □

By combining Krein's resolvent formula (2.1.41) with Theorem 2.10, we obtain the following convergence result.

THEOREM 2.11. *If* $R_\ell^{(0)}(\,\cdot\,)$, $\ell \in \mathbb{N}$, *denotes the resolvent of the free periodic operator defined in* (2.1.16) *and* $R^{(0)}(\,\cdot\,)$ *denotes the resolvent of (minus) the free Laplacian defined in* (2.1.14)*, then for each fixed* $z \in \mathbb{C}\backslash[0,\infty)$*, the following convergence result holds in* $\mathcal{B}_1(L^2(\mathbb{R};dx))$*:*

$$\lim_{\ell\to\infty} \Big\| \Big[\, \overline{u_\ell R_\ell^{(0)}(z) v_\ell} \oplus_\ell 0 \Big] - \overline{u R^{(0)}(z) v}\, \Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} = 0. \tag{2.2.40}$$

PROOF. It suffices to prove (2.2.40) for one $z \in \mathbb{C}\backslash[0,\infty)$. To see this, suppose that

$$\lim_{\ell\to\infty} \Big\| \Big[\, \overline{u_\ell R_\ell^{(0)}(z_0) v_\ell} \oplus_\ell 0 \Big] - \overline{u R^{(0)}(z_0) v}\, \Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} = 0 \tag{2.2.41}$$

for some fixed $z_0 \in \mathbb{C}\backslash[0,\infty)$. If $z \in \mathbb{C}\backslash[0,\infty)$, then (2.2.41) actually implies (2.2.40). Indeed, by the *first resolvent identity* (1.1.12),

$$\begin{aligned}
&\Big[\, \overline{u_\ell R_\ell^{(0)}(z) v_\ell} \oplus_\ell 0 \Big] - \overline{u R^{(0)}(z) v} \\
&\quad = \Big[\, \overline{u_\ell \big(R_\ell^{(0)}(z_0) + (z-z_0) R_\ell^{(0)}(z_0) R_\ell^{(0)}(z)\big) v_\ell} \oplus_\ell 0 \Big] \\
&\qquad - \overline{u \big(R^{(0)}(z_0) + (z-z_0) R^{(0)}(z_0) R^{(0)}(z)\big) v} \\
&\quad = \Big[\, \overline{u_\ell R_\ell^{(0)}(z_0) v_\ell} \oplus_\ell 0 \Big] + (z-z_0) \Big[\, \overline{u_\ell R_\ell^{(0)}(z_0) R_\ell^{(0)}(z) v_\ell} \oplus_\ell 0 \Big] \\
&\qquad - \overline{u R^{(0)}(z_0) v} - (z-z_0) \overline{u R^{(0)}(z_0) R^{(0)}(z) v} \\
&\quad = \Big[\, \overline{u_\ell R_\ell^{(0)}(z_0) v_\ell} \oplus_\ell 0 \Big] - \overline{u R^{(0)}(z_0) v} \\
&\qquad + (z-z_0) \Big\{ \Big[\, \overline{u_\ell R_\ell^{(0)}(z_0) R_\ell^{(0)}(z) v_\ell} \oplus_\ell 0 \Big] - \overline{u R^{(0)}(z_0) R^{(0)}(z) v} \Big\} \\
&\quad = \Big[\, \overline{u_\ell R_\ell^{(0)}(z_0) v_\ell} \oplus_\ell 0 \Big] - \overline{u R^{(0)}(z_0) v}
\end{aligned}$$

$$+ (z - z_0)\Big\{ \Big[u_\ell R_\ell^{(0)}(z_0)\overline{R_\ell^{(0)}(z)v_\ell} \oplus_\ell 0\Big] - uR^{(0)}(z_0)\overline{R^{(0)}(z)v}\Big\}$$

$$= \Big[\overline{u_\ell R_\ell^{(0)}(z_0)v_\ell} \oplus_\ell 0\Big] - \overline{uR^{(0)}(z_0)v}$$

$$+ (z - z_0)\Big\{ \Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big]\Big[\overline{R_\ell^{(0)}(z)v_\ell} \oplus_\ell 0\Big] - uR^{(0)}(z_0)\overline{R^{(0)}(z)v}\Big\}, \quad \ell \in \mathbb{N}. \quad (2.2.42)$$

As a result, one obtains the following estimate:

$$\Big\| \Big[\overline{u_\ell R_\ell^{(0)}(z)v_\ell} \oplus_\ell 0\Big] - \overline{uR^{(0)}(z)v}\Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))}$$

$$\leq \Big\| \Big[\overline{u_\ell R_\ell^{(0)}(z_0)v_\ell} \oplus_\ell 0\Big] - \overline{uR^{(0)}(z_0)v}\Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} \quad (2.2.43)$$

$$+ |z - z_0| \Big\| \Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big]\Big[\overline{R_\ell^{(0)}(z)v_\ell} \oplus_\ell 0\Big] - uR^{(0)}(z_0)\overline{R^{(0)}(z)v}\Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))}, \quad \ell \in \mathbb{N}.$$

In light of (2.2.41), the first term on the right-hand side of the inequality in (2.2.43) goes to zero as $\ell \to \infty$. By (2.2.18) and (2.2.19), one has

$$\lim_{\ell\to\infty} \Big\| \Big[u_\ell R_\ell^{(0)}(z_0) \oplus_\ell 0\Big] - uR^{(0)}(z_0)\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} = 0,$$

$$\lim_{\ell\to\infty} \Big\| \Big[\overline{R_\ell^{(0)}(z)v_\ell} \oplus_\ell 0\Big] - \overline{R^{(0)}(z)v}\Big\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} = 0, \quad (2.2.44)$$

so the second term on the right-hand side of the inequality in (2.2.43) goes to zero as $\ell \to \infty$ by a direct application of Lemma 1.24. Finally, (2.2.40) follows from (2.2.43) by the Squeeze Theorem.

To show that (2.2.40) holds for $z = -1$, we apply (2.1.41) and compute

$$\Big[\overline{u_\ell R_\ell^{(0)}(-1)v_\ell} \oplus_\ell 0\Big] - \overline{uR^{(0)}(-1)v}$$

$$= \Big[\overline{u_\ell\big(R_{\ell,D}^{(0)}(-1) + P_\ell^{(0)}(-1)\big)v_\ell} \oplus_\ell 0\Big] - \overline{uR^{(0)}(-1)v}$$

$$= \Big[\overline{u_\ell R_{\ell,D}^{(0)}(-1)v_\ell + u_\ell P_\ell^{(0)}(-1)v_\ell} \oplus_\ell 0\Big] - \overline{uR^{(0)}(-1)v}$$

$$= \Big[\Big(\overline{u_\ell R_{\ell,D}^{(0)}(-1)v_\ell} + \overline{u_\ell P_\ell^{(0)}(-1)v_\ell}\Big) \oplus_\ell 0\Big] - \overline{uR^{(0)}(-1)v}$$

$$= \Big[\overline{u_\ell R_{\ell,D}^{(0)}(-1)v_\ell} \oplus_\ell 0\Big] - \overline{uR^{(0)}(-1)v} + \Big[\overline{u_\ell P_\ell^{(0)}(-1)v_\ell} \oplus_\ell 0\Big], \quad \ell \in \mathbb{N}, \quad (2.2.45)$$

where the splitting of the closure in the third equality is justified by the fact that $u_\ell R_{\ell,D}^{(0)}(-1)v_\ell$ and $u_\ell P_\ell^{(0)}(-1)v_\ell$ are bounded on the dense subspace $\operatorname{dom}(v_\ell) \subset L^2((-\ell,\ell);dx)$ for each $\ell \in \mathbb{N}$. By (2.2.45),

$$\begin{aligned}
&\left\| \left[\overline{u_\ell R_\ell^{(0)}(-1)v_\ell} \oplus_\ell 0\right] - \overline{uR^{(0)}(-1)v} \right\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} \\
&\quad \leq \left\| \left[\overline{u_\ell R_{\ell,D}^{(0)}(-1)v_\ell} \oplus_\ell 0\right] - \overline{uR^{(0)}(-1)v} \right\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} \\
&\qquad + \left\| \overline{u_\ell P_\ell^{(0)}(-1)v_\ell} \oplus_\ell 0 \right\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))}, \quad \ell \in \mathbb{N}.
\end{aligned} \tag{2.2.46}$$

In light of (2.2.4) and (2.2.46), to prove (2.2.40) for $z = -1$, it suffices to show

$$\lim_{\ell\to\infty} \left\| \overline{u_\ell P_\ell^{(0)}(-1)v_\ell} \oplus_\ell 0 \right\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} = 0 \tag{2.2.47}$$

The closure $\overline{u_\ell P_\ell^{(0)}(-1)v_\ell}$ can be computed explicitly. Fix $\ell \in \mathbb{N}$. If $f \in \operatorname{dom}(v_\ell)$, then

$$\begin{aligned}
\overline{u_\ell P_\ell^{(0)}(-1)v_\ell} f &= u_\ell P_\ell^{(0)}(-1)v_\ell f \\
&= u_\ell \frac{\coth(\ell)}{2} \big(\psi_\ell^{(0)}(-1,\,\cdot\,), v_\ell f\big)_{L^2((-\ell,\ell);dx)} \psi_\ell^{(0)}(-1,\,\cdot\,) \\
&= \left(\frac{\coth(\ell)}{2} v_\ell \psi_\ell^{(0)}(-1,\,\cdot\,), f\right)_{L^2((-\ell,\ell);dx)} u_\ell \psi_\ell^{(0)}(-1,\,\cdot\,) \\
&= \big(\Psi_\ell^{(0)}, f\big)_{L^2((-\ell,\ell);dx)} \Phi_\ell^{(0)},
\end{aligned}$$

where

$$\begin{aligned}
&\Psi_\ell^{(0)} := \frac{\coth(\ell)}{2} v_\ell \psi_\ell^{(0)}(-1,\,\cdot\,) \in L^2((-\ell,\ell);dx), \\
&\Phi_\ell^{(0)} := u_\ell \psi_\ell^{(0)}(-1,\,\cdot\,) \in L^2((-\ell,\ell);dx), \quad \ell \in \mathbb{N}.
\end{aligned} \tag{2.2.48}$$

Therefore, the bounded operator

$$\overline{u_\ell P_\ell^{(0)}(-1)v_\ell} \in \mathcal{B}(L^2((-\ell,\ell);dx))$$

coincides with the bounded rank one operator

$$\big(\Psi_\ell^{(0)},\,\cdot\,\big)_{L^2((-\ell,\ell);dx)} \Phi_\ell^{(0)} \in \mathcal{B}(L^2((-\ell,\ell);dx))$$

on the dense subspace $\operatorname{dom}(v_\ell) \subset L^2((-\ell,\ell);dx))$. Thus, by continuity of bounded operators,

$$\overline{u_\ell P_\ell^{(0)}(-1)v_\ell} = \big(\Psi_\ell^{(0)}, \,\cdot\,\big)_{L^2((-\ell,\ell);dx)}\Phi_\ell^{(0)}, \quad \ell \in \mathbb{N}. \tag{2.2.49}$$

By (2.2.49), (1.2.18), and (2.2.32),

$$\begin{aligned}\Big\| \overline{u_\ell P_\ell^{(0)}(-1)v_\ell} \oplus_\ell 0 \Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} &= \Big\| \overline{u_\ell P_\ell^{(0)}(-1)v_\ell} \Big\|_{\mathcal{B}_1(L^2((-\ell,\ell);dx))} \\ &= \big\|\Psi_\ell^{(0)}\big\|_{L^2((-\ell,\ell);dx)}\big\|\Phi_\ell^{(0)}\big\|_{L^2((-\ell,\ell);dx)} \\ &= \frac{\coth(\ell)}{2}\int_{-\ell}^{\ell}|V(x)|\bigg(\frac{\cosh(x)}{\cosh(\ell)}\bigg)^2 dx \\ &= \frac{\coth(\ell)}{2}\int_{-\infty}^{\infty}|V(x)|\chi_{(-\ell,\ell)}(x)\bigg(\frac{\cosh(x)}{\cosh(\ell)}\bigg)^2 dx, \quad \ell \in \mathbb{N}.\end{aligned} \tag{2.2.50}$$

Finally,

$$\lim_{\ell\to\infty}\int_{-\infty}^{\infty}|V(x)|\chi_{(-\ell,\ell)}(x)\bigg(\frac{\cosh(x)}{\cosh(\ell)}\bigg)^2 dx = 0$$

by Lebesgue's Dominated Convergence Theorem (cf. (2.2.34), (2.2.35)), and (2.2.47) follows from (2.2.50). □

The following upper bound was derived in [**7**].

Lemma 2.12 (Lemma 3.8 in [**7**]). *If $R_{\ell,D}^{(0)}(\,\cdot\,)$, $\ell \in \mathbb{N}$, denotes the resolvent of the free Dirichlet operator defined in* (2.1.27)*, then there exists $E_D < 0$ and a constant $C_D > 0$ such that*

$$\Big\| \overline{u_\ell R_{\ell,D}^{(0)}(z)v_\ell} \oplus_\ell 0 \Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} \leq C_D|z|^{-\frac{1}{2}}, \quad z \in (-\infty, E_D], \quad \ell \in \mathbb{N}. \tag{2.2.51}$$

Moreover, E_D and C_D are independent of ℓ.

By applying Krein's resolvent formula, we can extend Lemma 2.12 to $R_\ell^{(0)}(\,\cdot\,)$.

THEOREM 2.13. *If $R_\ell^{(0)}(\cdot)$, $\ell \in \mathbb{N}$, denotes the resolvent of the free periodic operator defined in* (2.1.16), *then there exists $E_P < 0$ and a constant $C_P > 0$ such that*

$$\left\| \overline{u_\ell R_\ell^{(0)}(z) v_\ell} \oplus_\ell 0 \right\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} \leq C_P |z|^{-\frac{1}{2}}, \quad z \in (-\infty, E_P], \quad \ell \in \mathbb{N}. \tag{2.2.52}$$

PROOF. By Krein's resolvent formula (2.1.41), with $z = -k^2$, $k > 0$, and (2.2.51),

$$\begin{aligned} &\left\| \overline{u_\ell R_\ell^{(0)}(-k^2) v_\ell} \oplus_\ell 0 \right\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} \\ &\quad \leq \left\| \overline{u_\ell R_{\ell,D}^{(0)}(-k^2) v_\ell} \oplus_\ell 0 \right\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} + \left\| \overline{u_\ell P_\ell^{(0)}(-k^2) v_\ell} \oplus_\ell 0 \right\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} \\ &\quad \leq C_D |z|^{-\frac{1}{2}} + \left\| \overline{u_\ell P_\ell^{(0)}(-k^2) v_\ell} \oplus_\ell 0 \right\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))}, \quad z = -k^2 \in (-\infty, E_D], \quad \ell \in \mathbb{N}. \end{aligned} \tag{2.2.53}$$

On the other hand,

$$\begin{aligned} &\left\| \overline{u_\ell P_\ell^{(0)}(-k^2) v_\ell} \oplus_\ell 0 \right\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} \\ &\quad = \frac{\coth(k\ell)}{2k} \int_{-\infty}^{\infty} |V(x)| \chi_{(-\ell,\ell)}(x) \left(\frac{\cosh(kx)}{\cosh(k\ell)} \right)^2 dx \\ &\quad \leq \frac{\coth(1)}{2k} M, \quad k \geq 1, \quad \ell \in \mathbb{N}. \end{aligned} \tag{2.2.54}$$

Hence, by (2.2.53) and (2.2.54),

$$\begin{aligned} &\left\| \overline{u_\ell R_\ell^{(0)}(-k^2) v_\ell} \oplus_\ell 0 \right\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} \\ &\quad \leq \left[C_D + \frac{\coth(1)}{2} M \right] \cdot |z|^{-\frac{1}{2}}, \quad z = -k^2 \in (-\infty, E_P], \quad \ell \in \mathbb{N}, \end{aligned} \tag{2.2.55}$$

where $E_P := \min\{E_D, -1\}$. Hence, (2.2.52) holds with

$$C_P := C_D + 2^{-1} \coth(1) M. \tag{2.2.56}$$

□

CHAPTER 3

THE KREIN SPECTRAL SHIFT FUNCTION AND VAGUE CONVERGENCE IN THE INFINITE VOLUME LIMIT $\ell \to \infty$

In this chapter, we introduce the Krein spectral shift functions for the pairs $(H, H^{(0)})$ and $(H_\ell, H_\ell^{(0)})$. Section 3.1 recalls the definition and basic abstract theory of the spectral shift function. In Section 3.2, we prove that H_ℓ and $H_\ell^{(0)}$ are resolvent comparable. Finally, in Section 3.3, we state and prove the main results of this thesis, vague convergence of the spectral shift functions for $(H_\ell, H_\ell^{(0)})$ to the spectral shift function for the pair $(H, H^{(0)})$ in the limit $\ell \to \infty$, thus answering Question 1.1 in the affirmative.

3.1. Abstract Theory of the Krein Spectral Shift Function

A Krein spectral shift function may be introduced for any pair of *resolvent comparable* self-adjoint operators (A, B) in an abstract Hilbert space $\mathcal{H}$. In this section, we provide a brief introduction to the abstract theory of the Krein spectral shift function. The basic properties of the spectral shift function are recalled without proof in Theorem 3.3. For a complete discussion of Krein's spectral shift function, we refer to [**19**, Chapter 8, §7].

Definition 3.1. *Two self-adjoint operators A and B acting in a Hilbert space $\mathcal{H}$ are said to be resolvent comparable if and only if*

$$\left[(A - z_0 I_{\mathcal{H}})^{-1} - (B - z_0 I_{\mathcal{H}})^{-1}\right] \in \mathcal{B}_1(\mathcal{H}) \tag{3.1.1}$$

for some $z_0 \in \rho(A) \cap \rho(B)$.

By (2.2.7), the containment in (3.1.1) actually implies

$$\big[(A - zI_{\mathcal{H}})^{-1} - (B - zI_{\mathcal{H}})^{-1}\big] \in \mathcal{B}_1(\mathcal{H}), \quad z \in \rho(A) \cap \rho(B). \tag{3.1.2}$$

That is, the trace class containment of the resolvent difference for *one* $z_0 \in \rho(A) \cap \rho(B)$ actually implies the trace class containment of the resolvent difference in (3.1.2) for *all* $z \in \rho(A) \cap \rho(B)$.

In order to introduce the Krein spectral shift function for the pair (A, B) in a manner well-suited for applications to Schrödinger operators, we follow the discussion of the spectral shift function in [**7**, Appendix A] and assume henceforth that A and B are resolvent comparable, the operator A is bounded from below in $\mathcal{H}$:

$$A \geq \gamma I_{\mathcal{H}}, \tag{3.1.3}$$

for some $\gamma \in \mathbb{R}$, and that B can be written as a quadratic form sum (cf., e.g., [**17**, §6.5]) of the operator A and a self-adjoint operator W in $\mathcal{H}$,

$$B = A +_{\mathfrak{q}} W, \tag{3.1.4}$$

where "$+_{\mathfrak{q}}$" denotes the quadratic form sum of two operators. We assume that W may be factored as

$$W = W_1 W_2, \tag{3.1.5}$$

with

$$\operatorname{dom}(W_j) \supseteq \operatorname{dom}(|A|^{1/2}), \quad j \in \{1, 2\}, \tag{3.1.6}$$

with $|A| = (A^*A)^{1/2}$ denoting the absolute value of A (cf., e.g., [**17**, §4.3]), and

$$\overline{W_2(A - zI_{\mathcal{H}})^{-1}W_1} \in \mathcal{B}_1(\mathcal{H}), \quad z \in \rho(A). \tag{3.1.7}$$

The assumptions in (3.1.4)–(3.1.7) allow one to express the resolvent of B in terms of the resolvent of A and the factors W_1 and W_2 via *Kato's resolvent equation*

$$\begin{aligned}(B - zI_{\mathcal{H}})^{-1} &= (A - zI_{\mathcal{H}})^{-1} \\ &\quad - \overline{(A - zI_{\mathcal{H}})^{-1}W_1}\Big[I_{\mathcal{H}} + \overline{W_2(A - zI_{\mathcal{H}})^{-1}W_1}\Big]^{-1} W_2(A - zI_{\mathcal{H}})^{-1}, \\ &\qquad z \in \rho(B) \cap \rho(A),\end{aligned} \tag{3.1.8}$$

and the operator B is bounded from below.

To define Krein's spectral shift function for the pairs (A, B), we first introduce the class of functions $\mathfrak{K}(\mathbb{R})$.

DEFINITION 3.2. *A function $f : \mathbb{R} \to \mathbb{C}$ belongs to the class $\mathfrak{K}(\mathbb{R})$ if and only if f possesses two locally bounded derivatives, f' and f'', with*

$$\big(\lambda^2 f'(\lambda)\big)' = \mathcal{O}\big(|\lambda|^{-1-\varepsilon}\big) \quad as \quad |\lambda| \to \infty, \tag{3.1.9}$$

for some $\varepsilon = \varepsilon(f) > 0$, and

$$\lim_{\lambda \to -\infty} f(\lambda) = \lim_{\lambda \to \infty} f(\lambda) \quad and \quad \lim_{\lambda \to -\infty} \lambda^2 f'(\lambda) = \lim_{\lambda \to \infty} \lambda^2 f'(\lambda). \tag{3.1.10}$$

Since every function in $C_0^\infty(\mathbb{R})$ vanishes identically as $|x| \to \infty$, it is clear that $C_0^\infty(\mathbb{R}) \subset \mathfrak{K}(\mathbb{R})$. The Krein spectral shift function is introduced in the following theorem.

THEOREM 3.3 (Theorem 8.7.1 in [**19**]). *If A and B are resolvent comparable self-adjoint operators in $\mathcal{H}$ which satisfy* (3.1.3)–(3.1.7)*, then*

$$[f(B) - f(A)] \in \mathcal{B}_1(\mathcal{H}), \quad f \in \mathfrak{K}(\mathbb{R}), \tag{3.1.11}$$

and there exists a unique (a.e.) real-valued Lebesgue measurable function $\xi(\,\cdot\,; B, A)$ *such that*

$$\operatorname{tr}_{\mathcal{H}}(f(B) - f(A)) = \int_{-\infty}^{\infty} f'(\lambda)\, \xi(\lambda; B, A)\, d\lambda, \quad f \in \mathfrak{K}(\mathbb{R}), \tag{3.1.12}$$

with

$$\int_{-\infty}^{\infty} \frac{|\xi(\lambda; B, A)|}{1 + \lambda^2}\, d\lambda < \infty \quad \text{and} \quad \xi(\lambda; B, A) = 0 \text{ for a.e. } \lambda \in (-\infty, \lambda_0), \tag{3.1.13}$$

where $\lambda_0 = \min(\sigma(A) \cup \sigma(B))$.

DEFINITION 3.4. *The function* $\xi(\,\cdot\,; B, A)$ *is the Krein spectral shift function for the pair* (A, B)*, and* (3.1.12) *is called Krein's trace formula.*

EXAMPLE 3.5. *By direct calculation, one may show that for each* $z \in \mathbb{C}\backslash\mathbb{R}$*, the function* $g_z(\lambda) = (\lambda - z)^{-1}$ *belongs to* $\mathfrak{K}(\mathbb{R})$*. Taking* $f = g_z$*,* $z \in \mathbb{C}\backslash\mathbb{R}$*, in* (3.1.12)*, one infers that*

$$\operatorname{tr}_{\mathcal{H}}\left((B - zI_{\mathcal{H}})^{-1} - (A - zI_{\mathcal{H}})^{-1}\right) = -\int_{-\infty}^{\infty} \frac{\xi(\lambda; B, A)}{(\lambda - z)^2}\, d\lambda, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.1.14}$$

3.2. The Krein Spectral Shift Function for the Pairs $(H, H^{(0)})$ and $(H_\ell, H_\ell^{(0)})$

Under the integrability assumption in (2.1.2), it is known that H and $H^{(0)}$ form a resolvent comparable pair (cf., e.g., [**17**, Lemma 9.34]). Moreover, $H_{\ell,D}$ and $H_{\ell,D}^{(0)}$ are resolvent comparable for each $\ell \in \mathbb{N}$, a fact which is applied repeatedly in [**3**] and [**7**]. We summarize these results in the following lemma for future reference.

LEMMA 3.6 ([**3**], [**7**], and Lemma 9.34 in [**17**]). *The operators* H *and* $H^{(0)}$ *are resolvent comparable, that is*

$$\left[R(z) - R^{(0)}(z)\right] \in \mathcal{B}_1(L^2(\mathbb{R}; dx)), \quad z \in \rho(H) \cap \rho(H^{(0)}), \tag{3.2.1}$$

and for each $\ell \in \mathbb{N}$, the operators $H_{\ell,D}$ and $H_{\ell,D}^{(0)}$ are resolvent comparable, that is

$$\big[R_{\ell,D}(z) - R_{\ell,D}^{(0)}(z)\big] \in \mathcal{B}_1(L^2((-\ell,\ell);dx)), \quad z \in \rho(H_{\ell,D}) \cap \rho(H_{\ell,D}^{(0)}),\ \ell \in \mathbb{N}. \tag{3.2.2}$$

Employing (3.2.2) and Krein's resolvent formula (2.1.41), one shows that H_ℓ and $H_\ell^{(0)}$ are resolvent comparable for each $\ell \in \mathbb{N}$.

Lemma 3.7. *For each $\ell \in \mathbb{N}$, the operators H_ℓ and $H_\ell^{(0)}$ are resolvent comparable, that is*

$$\big[R_\ell(z) - R_\ell^{(0)}(z)\big] \in \mathcal{B}_1(L^2((-\ell,\ell);dx)), \quad z \in \rho(H_\ell) \cap \rho(H_\ell^{(0)}),\ \ell \in \mathbb{N}. \tag{3.2.3}$$

Proof. Let $\ell \in \mathbb{N}$ and $z \in \rho(H_\ell) \cap \rho(H_\ell^{(0)})$ be fixed. By Krein's resolvent formula (2.1.37) and (2.1.41),

$$\begin{aligned} R_\ell(z) - R_\ell^{(0)}(z) &= R_{\ell,D}(z) + P_\ell(z) - \big[R_{\ell,D}^{(0)}(z) + P_\ell^{(0)}(z)\big] \\ &= \big[R_{\ell,D}(z) - R_{\ell,D}^{(0)}(z)\big] + \big[P_\ell(z) - P_\ell^{(0)}(z)\big]. \end{aligned} \tag{3.2.4}$$

By (3.2.2) the first term after the second equality in (3.2.4) belongs to $\mathcal{B}_1(L^2((-\ell,\ell);dx))$. Moreover, $P_\ell(z) - P_\ell^{(0)}(z)$ is finite rank, therefore,

$$\big[P_\ell(z) - P_\ell^{(0)}(z)\big] \in \mathcal{B}_1(L^2((-\ell,\ell);dx)). \tag{3.2.5}$$

Since $\mathcal{B}_1(L^2((-\ell,\ell);dx))$ is a vector space, (3.2.3) follows from (3.2.4). $\square$

In addition to (3.2.1), one infers that (3.1.3)–(3.1.7) are satisfied by the pair $(A,B) = (H^{(0)}, H)$ with W as the operator of multiplication by V, W_1 the operator of multiplication by v, and W_2 the operator of multiplication by u. Therefore, there exists a unique real-valued Krein spectral shift function $\xi(\cdot\,; H, H^{(0)})$ which satisfies

$$\int_{-\infty}^{\infty} \frac{|\xi(\lambda; H, H^{(0)})|}{1+\lambda^2}\, d\lambda < \infty \quad \text{and} \quad \xi(\lambda; H, H^{(0)}) = 0 \ \text{ for a.e. } \lambda \in (-\infty, \lambda_\infty), \tag{3.2.6}$$

(cf. (2.1.25)), such that the following trace formula holds:

$$\operatorname{tr}_{L^2(\mathbb{R};dx)}\big(f(H)-f(H^{(0)})\big)=\int_{-\infty}^{\infty}\xi(\lambda;H,H^{(0)})f'(\lambda)\,d\lambda,\quad f\in\mathfrak{K}(\mathbb{R}). \tag{3.2.7}$$

In particular, the resolvents of H and $H^{(0)}$ are connected via Kato's resolvent equation (3.1.8),

$$R(z)=R^{(0)}(z)-\overline{R^{(0)}(z)v}\Big[I_{L^2(\mathbb{R};dx)}+\overline{uR^{(0)}(z)v}\Big]^{-1}uR^{(0)}(z),\quad z\in\rho(H)\cap\rho(H^{(0)}). \tag{3.2.8}$$

Similarly, by Lemma 3.7, $(H_\ell, H_\ell^{(0)})$ is a pair of resolvent comparable semibounded self-adjoint operators for each $\ell\in\mathbb{N}$, and the pair $(A,B)=(H_\ell^{(0)},H_\ell)$ satisfies (3.1.3)–(3.1.7) with W as the operator of multiplication by V_ℓ, W_1 the operator of multiplication by v_ℓ, and W_2 the operator of multiplication by u_ℓ. Therefore, for each $\ell\in\mathbb{N}$, there exists a unique real-valued Krein spectral shift function $\xi(\cdot;H_\ell,H_\ell^{(0)})$ which satisfies

$$\int_{-\infty}^{\infty}\frac{|\xi(\lambda;H_\ell,H_\ell^{(0)})|}{1+\lambda^2}\,d\lambda<\infty\quad\text{and}\quad\xi(\lambda;H_\ell,H_\ell^{(0)})=0\ \text{ for a.e. }\lambda\in(-\infty,\lambda_\infty), \tag{3.2.9}$$

(cf. (2.1.25)) such that the following trace formula holds:

$$\operatorname{tr}_{L^2((-\ell,\ell);dx)}\big(f(H_\ell)-f(H_\ell^{(0)})\big)=\int_{-\infty}^{\infty}\xi(\lambda;H_\ell,H_\ell^{(0)})f'(\lambda)\,d\lambda,\\ f\in\mathfrak{K}(\mathbb{R}),\ \ell\in\mathbb{N}. \tag{3.2.10}$$

In particular, the resolvents of H_ℓ and $H_\ell^{(0)}$ are connected via Kato's resolvent equation (3.1.8),

$$R_\ell(z)=R_\ell^{(0)}(z)-\overline{R_\ell^{(0)}(z)v_\ell}\Big[I_{L^2((-\ell,\ell);dx)}+\overline{u_\ell R_\ell^{(0)}(z)v_\ell}\Big]^{-1}u_\ell R_\ell^{(0)}(z),\\ z\in\rho(H_\ell)\cap\rho(H_\ell^{(0)}),\ \ell\in\mathbb{N}. \tag{3.2.11}$$

3.3. Vague Convergence of $\xi(\,\cdot\,;H_\ell,H_\ell^{(0)})$ to $\xi(\,\cdot\,;H,H^{(0)})$ as $\ell\to\infty$

With the new results from Chapter 2, namely Theorem 2.9, Theorem 2.10, and Theorem 2.11, we may now verify the conditions of Hypothesis B.1 and apply Theorem B.2 and

Corollary B.3 to obtain the principal new result of this thesis, weak and vague convergence results for $\xi(\,\cdot\,; H_\ell, H_\ell^{(0)})$ as $\ell \to \infty$.

THEOREM 3.8. *If* $H_\ell^{(0)}$ *denotes the free periodic operator defined in* (2.1.16)*,* H_ℓ *denotes the perturbed periodic operator defined in* (2.1.12)*,* $H^{(0)}$ *denotes (minus) the free Laplacian defined in* (2.1.14)*, and* H *is defined as in* (2.1.5)*, then*

$$\lim_{\ell\to\infty} \int_{\mathbb{R}} \frac{\xi(\lambda; H_\ell, H_\ell^{(0)})}{\lambda^2+1} f(\lambda)\, d\lambda = \int_{\mathbb{R}} \frac{\xi(\lambda; H, H^{(0)})}{\lambda^2+1} f(\lambda)\, d\lambda, \quad f \in C_b(\mathbb{R}). \tag{3.3.1}$$

PROOF. It suffices to check that conditions (i)–$(viii)$ in Hypothesis B.1 are satisfied under the correspondences:

$$\begin{aligned}
&v \text{ corresponds to } V_1^*, && u \text{ corresponds to } V_2, \\
&v_\ell \text{ corresponds to } V_{1,\ell}^*, && u_\ell \text{ corresponds to } V_{2,\ell}, \quad \ell \in \mathbb{N}, \\
&H^{(0)} \text{ corresponds to } A^{(0)}, && H_\ell^{(0)} \text{ corresponds to } A_\ell^{(0)},\ \ell \in \mathbb{N},
\end{aligned} \tag{3.3.2}$$

and to then apply Theorem B.2.

Condition (i) is satisfied because $L^2(\mathbb{R}; dx) = L^2((-\ell,\ell); dx) \oplus_\ell L^2(\mathbb{R}\backslash(-\ell,\ell); dx)$. This is the decomposition required. Condition (ii) is satisfied because $H^{(0)}$ is a self-adjoint operator in $L^2(\mathbb{R}; dx)$, and for each $\ell \in \mathbb{N}$, $H_\ell^{(0)}$ is a self-adjoint operator in $L^2((-\ell,\ell); dx)$. Also, the operator $H^{(0)}$ is bounded from below in $L^2(\mathbb{R}; dx)$ and $H_\ell^{(0)}$ is bounded from below in $L^2((-\ell,\ell); dx)$. In fact, both $H^{(0)}$ and $H_\ell^{(0)}$ are nonnegative operators in their respective Hilbert spaces. Condition (iii) is satisfied because u and v are closed operators in $L^2(\mathbb{R}; dx)$, and for each $\ell \in \mathbb{N}$, u_ℓ and v_ℓ are closed operators in $L^2((-\ell,\ell); dx)$ such that

$$\operatorname{dom}(u) \cap \operatorname{dom}(v) \supseteq \operatorname{dom}(|H^{(0)}|^{1/2}) = H^1(\mathbb{R}),$$

$$\operatorname{dom}(u_\ell) \cap \operatorname{dom}(v_\ell) \supseteq \operatorname{dom}(|H_\ell^{(0)}|^{1/2}) = \operatorname{dom}(Q_{H_\ell^{(0)}}), \quad \ell \in \mathbb{N},$$

with

$$V = u^* v \text{ a self-adjoint operator in } L^2(\mathbb{R}; dx),$$

and for each $\ell \in \mathbb{N}$,

$$V_\ell = u_\ell^* v_\ell \text{ is a self-adjoint operator in } L^2((-\ell, \ell); dx).$$

Condition (iv) is satisfied because

$$\overline{u(H^{(0)} - zI_{L^2(\mathbb{R};dx)})^{-1}v^*},\ \overline{u_\ell(H_\ell^{(0)} - zI_{L^2((-\ell,\ell);dx)})^{-1}v_\ell^*} \oplus_\ell 0 \in \mathcal{B}_1(L^2(\mathbb{R}; dx)), \quad \ell \in \mathbb{N}, \tag{3.3.3}$$

$$u(H^{(0)} - zI_{L^2(\mathbb{R};dx)})^{-1},\ u_\ell(H_\ell^{(0)} - zI_{L^2((-\ell,\ell);dx)})^{-1} \oplus_\ell 0 \in \mathcal{B}_2(L^2(\mathbb{R}; dx)), \quad \ell \in \mathbb{N}, \tag{3.3.4}$$

$$\overline{(H^{(0)} - zI_{L^2(\mathbb{R};dx)})^{-1}v^*},\ \overline{(H_\ell^{(0)} - zI_{L^2((-\ell,\ell);dx)})^{-1}v_\ell^*} \oplus_\ell 0 \in \mathcal{B}_2(L^2(\mathbb{R}; dx)), \quad \ell \in \mathbb{N}, \tag{3.3.5}$$

for some (and hence for all) $z \in \mathbb{C}\backslash\mathbb{R}$. The containment results for $H^{(0)}$ in (3.3.3)–(3.3.5) are known from [**7**]. For $H_\ell^{(0)}$, the containments follow from the analogous containments for $H_{\ell,D}^{(0)}$, also known from [**7**], via the Krein resolvent formula. In addition,

$$\lim_{z\downarrow-\infty} \Big\| \Big[\overline{u(H^{(0)} - zI_{L^2(\mathbb{R};dx)})^{-1}v^*} \Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} = 0, \tag{3.3.6}$$

$$\lim_{z\downarrow-\infty} \Big\| \overline{u_\ell(H_\ell^{(0)} - zI_{L^2((-\ell,\ell);dx)})^{-1}v_\ell^*} \oplus_\ell 0 \Big\|_{\mathcal{B}_1((L^2(\mathbb{R};dx))} = 0, \quad \ell \in \mathbb{N}. \tag{3.3.7}$$

The condition in (3.3.6) follows from [**7**, Lemma 3.8], while the condition in (3.3.7) follows from Theorem 2.13.

Condition (v) is is satisfied for some (and hence for all) $z \in \mathbb{C}\backslash\mathbb{R}$ by Theorem 2.9. Equalities (B.12) and (B.13) hold in condition (vi) by Theorem 2.10, and equality (B.11) follows from Theorem 2.11. Thus, condition (vi) is satisfied. Condition (vii) is clearly satisfied since u_ℓ, v_ℓ, u, and v are real-valued functions:

$$(v_\ell f, u_\ell g)_{L^2((-\ell,\ell);dx)} = \int_{-\ell}^{\ell} \overline{v_\ell(x) f(x)} u_\ell(x) g(x)\, dx$$

$$= \int_{-\ell}^{\ell} \overline{u_\ell(x) f(x)} v_\ell(x) g(x)\, dx$$
$$= (u_\ell f, v_\ell g)_{L^2((-\ell,\ell);dx)}, \quad f, g \in \mathrm{dom}(u_\ell) \cap \mathrm{dom}(v_\ell), \tag{3.3.8}$$

and a similar calculation applies to u and v.

If $V_\pm$ denote the positive and negative parts of V, then $V \in L^1(\mathbb{R}; dx)$ implies $V_\pm \in L^1(\mathbb{R}; dx)$. By [**17**, Lemma 9.33], $V_\pm$ are infinitesimally form bounded with respect to $H^{(0)}$. If $\ell \in \mathbb{N}$ and $V_{\ell,\pm}$ denote the positive and negative parts of V_ℓ, then $V \in L^1(\mathbb{R}; dx)$ implies $V_{\ell,\pm} \in L^1((-\ell, \ell); dx)$. A straightforward modification of the proof of [**17**, Lemma 9.33], simply replacing $\mathbb{R}$ by $(-\ell, \ell)$, implies that $V_{\ell,\pm}$ are infinitesimally form bounded with respect to $H_\ell^{(0)}$. Therefore, condition $(viii)$ is satisfied.

Having verified the conditions of Hypothesis B.1, the result of the theorem now follows by applying Theorem B.2. □

By applying Corollary B.3, we obtain the following vague convergence result for spectral shift functions.

Corollary 3.9. *Assume the hypotheses of Theorem* 3.8. *If* $g \in C_0(\mathbb{R})$, *then*

$$\lim_{\ell \to \infty} \int_{\mathbb{R}} \xi(\lambda; H_\ell, H_\ell^{(0)})\, g(\lambda)\, d\lambda = \int_{\mathbb{R}} \xi(\lambda; H, H^{(0)})\, g(\lambda)\, d\lambda. \tag{3.3.9}$$

3.4. Conclusion and Possible Future Work

By applying the Krein resolvent formula to relate $R_\ell^{(0)}(z)$ to $R_{\ell,D}^{(0)}(z)$, we are able to rely on the convergence results from [**7**] for the Birman–Schwinger-type operators for $R_{\ell,D}^{(0)}(z)$ (viz., (2.2.1), (2.2.2), (2.2.3), and (2.2.4)) to reduce the proofs of the analogous convergence results for the Birman–Schwinger-type operators for $R_\ell^{(0)}(z)$ down to analyzing convergence properties of rank one operators. This simplifies the analysis, as it is possible to compute the $\mathcal{B}_p$-norm of a rank one operator by applying Proposition 1.22. In turn, the limits that result can be computed by applying Lebesgue's Dominated Convergence Theorem, once suitable upper bounds are established. This approach is systematically used throughout the proofs of Theorem 2.9, 2.10, 2.11, and 2.13. Ultimately, these results imply that the conditions of Hypothesis B.1 are fulfilled. Convergence of the underlying spectral shift functions follows upon applying Theorem B.2 and Corollary B.3. Therefore, we obtain an affirmative answer to Question 1.1 and obtain the first vague convergence results for a case of coupled self-adjoint boundary conditions (viz., periodic boundary conditions).

Building on the work of this thesis, the next natural step would be to try to extend Theorem 3.8 and Corollary 3.9 to all coupled self-adjoint boundary conditions, if possible. Krein's resolvent formula could once again be used to relate the Birman–Schwinger operators for restrictions with arbitrary coupled self-adjoint boundary conditions to $H_{\ell,D}^{(0)}$. Unlike the periodic case, one will encounter rank two terms in the most general situation, and the coefficients of these rank two terms depend on the interval parameter ℓ in a nontrivial way. In particular, without a detailed investigation, it is not obvious that one could control these terms in the limit $\ell \to \infty$. On the other hand, in the special case when $R_{1,2} = 0$ in (1.0.10), one only obtains a rank one term in the Krein resolvent formula, which would simplify the investigation. A detailed analysis of the case $R_{1,2} = 0$ is a good starting point for future work.

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APPENDIX A

PROOF OF KREIN'S RESOLVENT FORMULA

In this appendix, we provide a complete proof of Lemma 2.5, Krein's resolvent formula, which relates the resolvent operator of the periodic extension H_ℓ to the Dirichlet extension $H_{\ell,D}$ via the rank one term $P_\ell(\,\cdot\,)$. A proof of Krein's resolvent formula for arbitrary self-adjoint boundary conditions is given in [**4**]. The proof provided below is based on the idea from [**4**] and is specifically tailored to periodic boundary conditions.

Proof of Lemma 2.5: Fix $\ell \in \mathbb{N}$ for the remainder of this proof. First, we claim that $q_\ell(z) \neq 0$ if $z \in \rho(H_\ell) \cap \rho(H_{\ell,D})$. To see this, suppose by way of contradiction that $z \in \rho(H_\ell) \cap \rho(H_{\ell,D})$ and $q_\ell(z) = 0$. Then, by the very definition of $q_\ell(z)$, it follows that

$$\psi_\ell'(z, -\ell) = \psi_\ell'(z, \ell). \tag{A.1}$$

On the other hand, it is clear from (2.1.33) that

$$\psi_\ell(z, -\ell) = \psi_\ell(z, \ell), \tag{A.2}$$

and $\psi_\ell(z, \,\cdot\,) \in \operatorname{dom}(H_{\ell,\max})$ since it is a linear combination of $\psi_{j,\ell}(z, \,\cdot\,)$, $j \in \{1, 2\}$. Therefore, (A.1) and (A.2) imply $\psi_\ell(z, \,\cdot\,) \in \operatorname{dom}(H_\ell)$, and then

$$H_\ell \psi_\ell(z, \,\cdot\,) = H_{\ell,\max}\psi_\ell(z, \,\cdot\,) = H_{\ell,\max}\psi_{2,\ell}(z, \,\cdot\,) + H_{\ell,\max}\psi_{1,\ell}(z, \,\cdot\,) = z\psi_\ell(z, \,\cdot\,), \tag{A.3}$$

which implies z is an eigenvalue of H_ℓ and $\psi_\ell(z, \,\cdot\,)$ is a corresponding eigenfunction. Note that $\psi_\ell(z, \,\cdot\,)$ is not identically zero since it is the sum of two linearly independent functions. This is a contradiction to the assumption $z \in \rho(H_\ell)$, so it must be that $q_\ell(z) \neq 0$.

Now, let $z \in \rho(H_\ell) \cap \rho(H_{\ell,D})$ and $f \in L^2((-\ell, \ell); dx)$ arbitrary. Define the operator T by

$$T = R_{\ell,D}(z) - q_\ell(z)^{-1}(\psi_\ell(\overline{z}, \,\cdot\,), \,\cdot\,)_{L^2((-\ell,\ell);dx)}\psi_\ell(z, \,\cdot\,). \tag{A.4}$$

We will prove $T = R_\ell(z)$. To this end, define

$$g_f = Tf = R_{\ell,D}(z)f - q_\ell(z)^{-1}(\psi_\ell(\overline{z}, \,\cdot\,), f)_{L^2((-\ell,\ell);dx)}\psi_\ell(z, \,\cdot\,). \tag{A.5}$$

We claim that $g_f \in \operatorname{dom}(H_\ell)$. Since g_f is the difference of two functions in $\operatorname{dom}(H_{\max})$, it suffices to show that g_f satisfies the periodic boundary conditions:

$$g_f(-\ell) = g_f(\ell) \quad \text{and} \quad g_f'(-\ell) = g_f'(\ell). \tag{A.6}$$

Since $R_{\ell,D}(z)f \in \operatorname{dom}(H_{\ell,D})$, the function $R_{\ell,D}(z)f$ satisfies Dirichlet boundary conditions:

$$[R_{\ell,D}(z)f](-\ell) = [R_{\ell,D}(z)f](\ell) = 0. \tag{A.7}$$

Thus, one computes for g_f:

$$\begin{aligned} g_f(-\ell) &= [R_{\ell,D}(z)f](-\ell) - q_\ell(z)^{-1}(\psi_\ell(\overline{z},\,\cdot\,), f)_{L^2((-\ell,\ell);dx)}\psi_\ell(z,-\ell) \\ &= -q_\ell(z)^{-1}(\psi_\ell(\overline{z},\,\cdot\,), f)_{L^2((-\ell,\ell);dx)}, \end{aligned} \tag{A.8}$$

and, similarly,

$$g_f(\ell) = -q_\ell(z)^{-1}(\psi_\ell(\overline{z},\,\cdot\,), f)_{L^2((-\ell,\ell);dx)}. \tag{A.9}$$

Obviously, the first equality in (A.6) follows. In order to show the second equality in (A.6), we establish two preliminary identities for the boundary values of the derivative of $R_{\ell,D}f$:

$$\begin{aligned} [R_{\ell,D}(z)f]'(-\ell) &= (\psi_{2,\ell}(\overline{z},\,\cdot\,), f)_{L^2((-\ell,\ell);dx)}, \\ [R_{\ell,D}(z)f]'(\ell) &= -(\psi_{1,\ell}(\overline{z},\,\cdot\,), f)_{L^2((-\ell,\ell);dx)}. \end{aligned} \tag{A.10}$$

Using $\psi_{j,\ell}(z,\,\cdot\,)$, $j \in \{1,2\}$, to construct the Green's function for $R_{\ell,D}(z)$ (cf., e.g., [**15**, Proposition 15.13]), one has

$$\begin{aligned} [R_{\ell,D}(z)f](x) = W(z)^{-1}\bigg[&\psi_{2,\ell}(z,x)\int_{-\ell}^{x}\psi_{1,\ell}(z,x')f(x')\,dx' \\ &+ \psi_{1,\ell}(z,x)\int_{x}^{\ell}\psi_{2,\ell}(z,x')f(x')\,dx'\bigg], \quad x \in [-\ell,\ell], \end{aligned} \tag{A.11}$$

where

$$W(z) = \psi_{2,\ell}(z,x)\psi_{1,\ell}'(z,x) - \psi_{1,\ell}(z,x)\psi_{2,\ell}'(z,x), \quad x \in [-\ell,\ell], \tag{A.12}$$

denotes the Wronskian of $\psi_{2,\ell}(z,\,\cdot\,)$ and $\psi_{1,\ell}(z,\,\cdot\,)$ and is constant. Separately choosing $x=-\ell$ and $x=\ell$ in (A.12) reveals

$$W(z)=\psi_{1,\ell}'(z,-\ell)=-\psi_{2,\ell}'(z,\ell). \tag{A.13}$$

Then (A.10) follows by taking the derivative in (A.11) and applying (A.13) and using the property $\overline{\psi_{j,\ell}(z,\,\cdot\,)}=\psi_{j,\ell}(\overline{z},\,\cdot\,)$, $j\in\{1,2\}$, which follows from the fact that V is real-valued. Having now shown (A.10), it is a simple matter to check the second equality in (A.6). One computes

$$\begin{aligned} g_f'(-\ell)&=(\psi_{2,\ell}(\overline{z},\,\cdot\,),f)_{L^2((-\ell,\ell);dx)}-q_\ell(z)^{-1}(\psi_\ell(\overline{z},\,\cdot\,),f)_{L^2((-\ell,\ell);dx)}\psi_\ell'(z,-\ell),\\ g_f'(\ell)&=-(\psi_{1,\ell}(\overline{z},\,\cdot\,),f)_{L^2((-\ell,\ell);dx)}-q_\ell(z)^{-1}(\psi_\ell(\overline{z},\,\cdot\,),f)_{L^2((-\ell,\ell);dx)}\psi_\ell'(z,\ell). \end{aligned} \tag{A.14}$$

Then the difference $g_f'(-\ell)-g_f'(\ell)$ is shown to be zero using (A.14) and the second equality in(A.13):

$$\begin{aligned} &g_f'(-\ell)-g_f'(\ell)\\ &\quad=(\psi_\ell(\overline{z},\,\cdot\,),f)_{L^2((-\ell,\ell);dx)}\\ &\qquad+q_\ell(z)^{-1}\big[(\psi_\ell(\overline{z},\,\cdot\,),f)_{L^2((-\ell,\ell);dx)}\psi_\ell'(z,\ell)-(\psi_\ell(\overline{z},\,\cdot\,),f)_{L^2((-\ell,\ell);dx)}\psi_\ell'(z,-\ell)\big]\\ &\quad=(\psi_\ell(\overline{z},\,\cdot\,),f)_{L^2((-\ell,\ell);dx)}-(\psi_\ell(\overline{z},\,\cdot\,),f)_{L^2((-\ell,\ell);dx)}\\ &\quad=0. \end{aligned} \tag{A.15}$$

Hence, we have shown $g_f\in\operatorname{dom}(H_\ell)$. Since $g_f\in\operatorname{dom}(H_\ell)$, it also belongs to the domain of the operator $H_\ell-zI_{L^2((-\ell,\ell);dx)}$ and one computes:

$$\begin{aligned} &(H_\ell-zI_{L^2((-\ell,\ell);dx)})g_f\\ &\quad=(H_\ell-zI_{L^2((-\ell,\ell);dx)})Tf\\ &\quad=(H_{\ell,\max}-zI_{L^2((-\ell,\ell);dx)})Tf \end{aligned}$$

$$
\begin{aligned}
&= \underbrace{(H_{\ell,\max} - zI_{L^2((-\ell,\ell);dx)})R_{\ell,D}(z)f}_{=f} \\
&\quad - q_\ell(z)^{-1}(\psi_\ell(\overline{z},\,\cdot\,), f)_{L^2((-\ell,\ell);dx)} \underbrace{(H_{\ell,\max} - zI_{L^2((-\ell,\ell);dx)})\psi_\ell(z,\,\cdot\,)}_{=0} \\
&= f. \qquad \text{(A.16)}
\end{aligned}
$$

Therefore, applying $R_\ell(z)$ throughout (A.16),

$$
Tf = R_\ell(z)f, \qquad \text{(A.17)}
$$

and since $f \in L^2((-\ell,\ell);dx)$ was arbitrary, one infers $T = R_\ell(z)$. □

APPENDIX B

ABSTRACT CRITERIA FOR VAGUE CONVERGENCE OF KREIN SPECTRAL SHIFT FUNCTIONS

In this appendix, for completeness, we recall the abstract convergence criteria developed in [**6**] for vague and weak convergence of a sequence of Krein spectral shift functions to a limiting Krein spectral shift function. We have tailored the statements of the results below to suit our applications to one-dimensional Schrödnger operators. The criteria are given in Hypothesis B.1 and the convergence results are given in Theorem B.2 and Corollary B.3.

HYPOTHESIS B.1 (Hypothesis 3.1 in [**6**]). *Let* $\mathcal{H} := L^2(\mathbb{R}; dx)$.

(i) For each $\ell \in \mathbb{N}$*, decompose* $\mathcal{H}$ *according to*

$$L^2(\mathbb{R}; dx) = L^2((-\ell, \ell); dx) \oplus_\ell L^2(\mathbb{R}\backslash(-\ell, \ell); dx), \tag{B.1}$$

and write $\mathcal{H}_\ell := L^2((-\ell, \ell); dx)$ *and* $\mathcal{H}_\ell^c = L^2(\mathbb{R}\backslash(-\ell, \ell); dx)$.

(ii) Let $A^{(0)}$ *be a self-adjoint operator in* $\mathcal{H}$*, and for each* $\ell \in \mathbb{N}$*, let* $A_\ell^{(0)}$ *be self-adjoint operators in* $\mathcal{H}_\ell$*. In addition, suppose that* $A^{(0)}$ *is bounded from below in* $\mathcal{H}$*, and that for each* $\ell \in \mathbb{N}$*,* $A_\ell^{(0)}$ *is bounded from below in* $\mathcal{H}_\ell$*.*

(iii) Suppose that V_1*, and* V_2 *are closed operators in* $\mathcal{H}$*, and for each* $\ell \in \mathbb{N}$*, assume that* $V_{1,\ell}$*, and* $V_{2,\ell}$ *are closed operators in* $\mathcal{H}_\ell$ *such that*

$$\mathrm{dom}(V_1) \cap \mathrm{dom}(V_2) \supseteq \mathrm{dom}(|A^{(0)}|^{1/2}), \tag{B.2}$$

$$\mathrm{dom}(V_{1,\ell}) \cap \mathrm{dom}(V_{2,\ell}) \supseteq \mathrm{dom}(|A_\ell^{(0)}|^{1/2}), \quad \ell \in \mathbb{N}, \tag{B.3}$$

with

$$V = V_1^* V_2 \ \textit{ is a self-adjoint operator in } \mathcal{H}, \tag{B.4}$$

and for each $\ell \in \mathbb{N}$*,*

$$V_\ell = V_{1,\ell}^* V_{2,\ell} \ \textit{ is a self-adjoint operator in } \mathcal{H}_\ell. \tag{B.5}$$

(iv) Suppose

$$\overline{V_2(A^{(0)} - zI_{\mathcal{H}})^{-1}V_1^*}, \ \overline{V_{2,\ell}(A_\ell^{(0)} - zI_{\mathcal{H}_\ell})^{-1}V_{1,\ell}^*} \oplus_\ell 0 \in \mathcal{B}_1(\mathcal{H}), \quad \ell \in \mathbb{N}, \tag{B.6}$$

$$V_2(A^{(0)}-zI_{\mathcal H})^{-1},\ V_{2,\ell}(A_\ell^{(0)}-zI_{\mathcal H_\ell})^{-1}\oplus_\ell 0\in\mathcal B_2(\mathcal H),\quad \ell\in\mathbb N, \tag{B.7}$$

$$\overline{(A^{(0)}-zI_{\mathcal H})^{-1}V_1^*},\ \overline{(A_\ell^{(0)}-zI_{\mathcal H_\ell})^{-1}V_{1,\ell}^*}\oplus_\ell 0\in\mathcal B_2(\mathcal H),\quad \ell\in\mathbb N, \tag{B.8}$$

for some (and hence for all) $z\in\mathbb C\backslash\mathbb R$. In addition, assume that

$$\begin{aligned}\lim_{z\downarrow-\infty}\Big\|\Big[\overline{V_2(A^{(0)}-zI_{\mathcal H})^{-1}V_1^*}\Big\|_{\mathcal B_1(\mathcal H)}&=0,\\ \lim_{z\downarrow-\infty}\Big\|\overline{V_{2,\ell}(A_\ell^{(0)}-zI_{\mathcal H_\ell})^{-1}V_{1,\ell}^*}\oplus_\ell 0\Big\|_{\mathcal B_1(\mathcal H)}&=0,\quad \ell\in\mathbb N.\end{aligned} \tag{B.9}$$

(v) Assume that for some (and hence for all) $z\in\mathbb C\backslash\mathbb R$,

$$\operatorname*{s-lim}_{\ell\to\infty}\bigg[(A_\ell^{(0)}-zI_{\mathcal H_\ell})^{-1}\oplus_\ell\frac{-1}{z}I_{\mathcal H_\ell^c}\bigg]=(A^{(0)}-zI_{\mathcal H})^{-1}. \tag{B.10}$$

(vi) Suppose that for some (and hence for all) $z\in\mathbb C\backslash\mathbb R$,

$$\lim_{\ell\to\infty}\Big\|\Big[\overline{V_{2,\ell}(A_\ell^{(0)}-zI_{\mathcal H_\ell})^{-1}V_{1,\ell}^*}\oplus_\ell 0\Big]-\overline{V_2(A^{(0)}-zI_{\mathcal H})^{-1}V_1^*}\Big\|_{\mathcal B_1(\mathcal H)}=0, \tag{B.11}$$

$$\lim_{\ell\to\infty}\Big\|\Big[V_{2,\ell}(A_\ell^{(0)}-zI_{\mathcal H_\ell})^{-1}\oplus_\ell 0\Big]-V_2(A^{(0)}-zI_{\mathcal H})^{-1}\Big\|_{\mathcal B_2(\mathcal H)}=0, \tag{B.12}$$

$$\lim_{\ell\to\infty}\Big\|\Big[\overline{(A_\ell^{(0)}-zI_{\mathcal H_\ell})^{-1}V_{1,\ell}^*}\oplus_\ell 0\Big]-\overline{(A^{(0)}-z)^{-1}V_1^*}\Big\|_{\mathcal B_2(\mathcal H)}=0. \tag{B.13}$$

(vii) Suppose that

$$\begin{aligned}&(V_2f,V_1g)_{\mathcal H}=(V_1f,V_2g)_{\mathcal H},\quad f,g\in\operatorname{dom}(V_1)\cap\operatorname{dom}(V_2),\\ &(V_{2,\ell}f,V_{1,\ell}g)_{\mathcal H}=(V_{1,\ell}f,V_{2,\ell}g)_{\mathcal H},\quad f,g\in\operatorname{dom}(V_{1,\ell})\cap\operatorname{dom}(V_{2,\ell}),\quad \ell\in\mathbb N.\end{aligned} \tag{B.14}$$

(viii) Decomposing V and each V_ℓ, $\ell\in\mathbb N$, into their positive and negative parts,

$$V_\pm=(1/2)[|V|\pm V],\quad V_{\ell,\pm}=(1/2)[|V_\ell|\pm V_\ell],\ \ell\in\mathbb N, \tag{B.15}$$

$V_\pm$ are assumed to be infinitesimally form bounded with respect to $A^{(0)}$, and for each $\ell\in\mathbb N$, $V_{\ell,\pm}$ are assumed to be infinitesimally form bounded with respect to $A_\ell^{(0)}$.

Assuming Hypothesis B.1, the pairs $(A,A^{(0)})$ and $(A_\ell,A_\ell^{(0)})$, $\ell\in\mathbb N$, are resolvent comparable in the sense that

$$\big[(A-zI_{\mathcal H})^{-1}-(A^{(0)}-zI_{\mathcal H})^{-1}\big]\in\mathcal B_1(\mathcal H),\quad z\in\mathbb C\backslash\mathbb R, \tag{B.16}$$

and

$$\big[(A_\ell - zI_{\mathcal{H}})^{-1} - (A_\ell^{(0)} - zI_{\mathcal{H}})^{-1}\big] \in \mathcal{B}_1(\mathcal{H}_\ell), \quad z \in \mathbb{C}\backslash\mathbb{R},\ \ell \in \mathbb{N}. \tag{B.17}$$

Let $\xi(\lambda; A, A^{(0)})$ and $\xi(\lambda; A_\ell, A_\ell^{(0)})$ denote the Krein spectral shift functions for the pairs $(A, A^{(0)})$ and $(A_\ell, A_\ell^{(0)})$, respectively, which are normalized to vanish identically as $\lambda \to -\infty$. Under the assumptions in Hypothesis B.1, the following convergence results hold for the sequence of spectral shift functions $\big\{\xi(\lambda; A_\ell, A_\ell^{(0)})\big\}_{\ell=1}^{\infty}$. Recall that $C_b(\mathbb{R})$ denotes the set of bounded continuous functions on $\mathbb{R}$,

$$C_b(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \,|\, f \text{ is continuous and } |f(x)| \leq C \text{ for some } C = C(f) \geq 0\},$$

and $C_0(\mathbb{R})$ denotes the set of continuous functions on $\mathbb{R}$ with compact support,

$$\begin{aligned} C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \,|\, f &\text{ is continuous and } f(x) = 0 \\ &\text{for all } |x| \geq R \text{ for some } R = R(f) > 0\}. \end{aligned} \tag{B.18}$$

Theorem B.2 (Theorem 3.13 in [**6**]). *Assume Hypothesis* B.1*. Then*

$$\lim_{\ell\to\infty} \int_{\mathbb{R}} \frac{\xi(\lambda; A_\ell, A_\ell^{(0)})}{\lambda^2+1} f(\lambda)\, d\lambda = \int_{\mathbb{R}} \frac{\xi(\lambda; A, A^{(0)})}{\lambda^2+1} f(\lambda)\, d\lambda, \quad f \in C_b(\mathbb{R}). \tag{B.19}$$

The factor $(1+\lambda^2)^{-1}$ is essential in (B.19). Without it, the integrals need not be finite. As a consequence of Theorem B.2, $\big\{\xi(\,\cdot\,; A_\ell, A_\ell^{(0)})\big\}_{\ell=1}^{\infty}$ converges vaguely to $\xi(\,\cdot\,; A, A^{(0)})$ as $\ell \to \infty$, which is the content of the following corollary.

Corollary B.3 (Corollary 3.11 in [**6**]). *Assume Hypothesis* B.1 *and let* $g \in C_0(\mathbb{R})$*. Then*

$$\lim_{\ell\to\infty} \int_{\mathbb{R}} \xi(\lambda; A_\ell, A_\ell^{(0)})\, g(\lambda)\, d\lambda = \int_{\mathbb{R}} \xi(\lambda; A, A^{(0)})\, g(\lambda)\, d\lambda. \tag{B.20}$$

VITA

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