ON THE $\delta$-CONJECTURE FOR GRAPHS WITH MINIMUM DEGREE $|G| - 4$

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ABSTRACT

For a graph $G$ of order $n$, the minimum rank of $G$ is defined to be the minimum rank among all $n \times n$ symmetric matrices whose $ij$-entry is nonzero precisely when $\{i, j\}$ is an edge of $G$. The delta conjecture proposes a relationship between the minimum rank and the minimum degree of a given graph. We prove that the delta conjecture holds for several classes of graphs; in particular, we show this relationship holds for many graphs $G$ whose minimum degree is $|G| - 4$. We then consider some implications of these results related to other problems involving minimum rank.
DEDICATION

To Linda, without your love, encouragement, and unwavering support, it would not have been possible to complete this work.
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CHAPTER 1
INTRODUCTION

Given a simple undirected graph $G$, we consider a collection of symmetric matrices whose off-diagonal zero-nonzero pattern reflects adjacency in $G$. There has been considerable work done to determine the possible ranks of matrices associated with a given graph. This topic has become quite attractive to researchers lately, thanks in part to its applications in both computer science and physics. Of particular interest is the \textit{minimum rank problem} (MRP) which asks for the smallest possible rank of a matrix whose zero-nonzero pattern describes a graph. While mathematicians in the field of linear algebra and matrix theory have already found the minimum rank of several classes of graphs, a complete solution seems far from being formed.

Historically, the MRP derives its origins from the \textit{Inverse Eigenvalue Problem} (IEP) which concerns the reconstruction of a matrix given information about its spectrum. Research on the IEP eventually led to the \textit{maximum multiplicity problem} (MMP), which concerns determining the maximum multiplicity of an eigenvalue for a given class of matrices, and the closely related \textit{maximum nullity problem} (MNP), which concerns determining the maximum nullity of a given class of matrices. The relationship between the MRP and MNP is clear: if we restrict the class of matrices to those real symmetric matrices whose zero-nonzero pattern describes a given graph $G$, the solutions to the two problems are equivalent in the sense that knowing one gives the answer to the other. Indeed, if $G$ is a graph and $A$ is an $n \times n$ real symmetric matrix whose zero-nonzero pattern describes $G$, we have (by the rank-nullity theorem)
rankA + nullityA = n or equivalently rankA = n – nullityA.

Evidently, as rank is minimized, nullity is maximized.

In 2006, an American Institute of Mathematics (AIM) workshop led to many new directions for research concerning the MRP. In particular, a relationship between the minimum degree of a graph and the maximum nullity of a real symmetric matrix which describes that graph was conjectured. Over the last decade, this so-called Delta Conjecture has been proved to hold for several classes of graphs including, but not restricted to, all bipartite graphs, all graphs whose minimum degree is at most 3, and all graphs whose minimum degree is at least \(|G| – 2\), where \(|G|\) denotes the number of vertices in \(G\). The general problem, however, remains open.

Another product of the aforementioned 2006 AIM workshop was a proposed upper bound for the sum of the minimum rank of a graph and that of its complement in terms of the size (i.e. number of vertices) of the graph. Just as with the delta conjecture, the Graph Complement Conjecture (GCC) has been shown to hold for many classes of graphs, but the general problem remains unresolved.

The primary goal of this thesis is to extend the classes of graphs for which it is known that the delta conjecture holds. In particular, we show that for most graphs \(G\) whose complement has maximum degree at most 3 (equivalent to minimum degree at least \(|G| – 4\)), the conjecture holds. The techniques employed to prove the main result are also used to prove an upper bound for the minimum rank of other classes of graphs. The remainder of this thesis is organized as follows:

In Chapter 2, we provide the preliminary definitions from linear algebra and graph theory which are needed to prove the results included in Chapters 4 and 5. In Chapter 3, we present a brief survey of the literature related to the minimum rank problem, the delta conjecture, and the graph complement conjecture. In Chapter 4, we prove the main results concerning graphs with
minimum degree at least $|G| - 4$. It is shown that for graphs with minimum degree $|G| - 4$, if the complement of $G$ is not 3-regular, triangle-free and square-free, then the delta conjecture holds. Finally, in Chapter 5, we consider both applications and extensions of the techniques and results seen in Chapter 4. We also provide limitations, a conclusion, and some possible future directions for research.
CHAPTER 2
PRELIMINARY DEFINITIONS AND EXAMPLES

All graphs we consider are simple (no loops or multiple edges) and undirected. A graph $G = (V, E)$ is an ordered pair consisting of a nonempty set $V$, called the vertex set, and a set $E$ consisting of 2-element subsets of $V$, called the edge set. If a two-element subset of $V$, $\{u, v\}$, is in $E$, it is called an edge of the graph $G$. We will often let $V = \{1, 2, \ldots, n\}$, and we will often write $uv$ to denote the edge $\{u, v\}$. Also, in the case where we are discussing multiple graphs, it will be helpful to write $G = (V_G, E_G)$ in place of $G = (V, E)$.

The order of a graph, denoted by $|G|$, is the number of vertices in $G$. That is, if $G = (V, E)$, then $|G|$ is the cardinality of $V$. In many instances, it is helpful to visualize graphs via a collection of points (vertices) and line segments (edges) (see Figure 2.1).

![Figure 2.1](image)

A graph on 7 vertices

The complement of a graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$ where $\overline{E}$ is the set of all edges which are not in $E$ (see Figure 2.2).
Let $G = (V_G, E_G)$ be a graph. A graph, $H = (V_H, E_H)$ is a **subgraph** of $G$ if both $V_H \subseteq V_G$ and $E_H \subseteq E_G$. In the case that $E_H = \{ uv \in E_G: u, v \in V_H \}$, we say $H$ is an **induced subgraph** of $G$.

If $v$ is a vertex of $G$, we define $G - v$ to be the subgraph of $G$ induced by $V_G - \{v\}$. More generally, if $V \subsetneq V_G$, then $G - V$ is the subgraph induced by $V_G - V$.

The graph $H$ is a subgraph of $G$, but not an induced subgraph.
Figure 2.3b

The graph $H'$ is an induced subgraph of $G$

Let $G = (V, E)$. We say two vertices $u, v \in V$ are adjacent (or neighbors) and write $u \sim v$ if $uv \in E$; otherwise, we say that $u$ and $v$ are nonadjacent (or non-neighbors). A pendant of $G$ is a vertex with only one neighbor and an isolated vertex of $G$ is a vertex with no neighbors. Suppose $v$ is a vertex of $G$. The neighborhood of $v$, denoted $N_G(v)$ (or $N(v)$ when there is no possibility of confusion), is defined to be the set of all vertices which are adjacent to $v$, i.e.

$$N_G(v) := \{u \in V: v \sim u\}.$$ 

The closed neighborhood of $v$ is $N(v) \cup \{v\}$, and is denoted by $N_G[v]$ (or $N[v]$). For example, in the graph in Figure 2.3 above, $N_G(1) = \{2, 4, 5, 6, 7, 8\}$ and $N[2] = \{1, 2, 3\}$. The degree of $v$, denoted by $\deg_G(v)$ (or simply $\deg(v)$), is the cardinality of $N(v)$.

Given a graph $G$, we let $\delta(G)$ denote the minimum degree among all vertices of $G$, i.e.,

$$\delta(G) = \min\{\deg(v): v \in V\},$$

and we let $\Delta(G)$ denote the maximum degree among all vertices of $G$, i.e.,

$$\Delta(G) = \max\{\deg(v): v \in V\}.$$ 

Observe that $\delta(G) + \Delta(\overline{G}) = |G| - 1$. 

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Example 2.1. For the graph shown in Figure 2.4 below, \( \text{deg}(1) = 6, \, \delta(G) = 2 \) (achieved by vertex 4), and \( \Delta(G) = 6 \) (achieved by vertex 1).

![Figure 2.4](image)

A graph with \( \delta(G) = 2 \) and \( \Delta(G) = 6 \)

Let \( u \) and \( v \) be vertices of a graph \( G \). We say that \( u \) and \( v \) are twin (or duplicate) vertices if they have the same closed neighborhoods, i.e., if \( N[u] = N[v] \) (see Figure 2.5). Observe that, according to the definition, twin vertices are necessarily adjacent. By duplicating a vertex \( v \), we mean that we are constructing a new graph \( G' \) from \( G \) by adding to \( G \) a vertex \( v' \) and joining it via edges precisely to \( v \) and all the neighbors of \( v \) (so that \( v \) and \( v' \) are twins in \( G' \)).

![Figure 2.5](image)

Vertices 1 and 2 are twin vertices
We now define some common classes of graphs.

A **path on \( n \) vertices** is a graph \( P_n = (V, E) \) where, for some labeling of the vertices, \( V = \{v_1, \ldots, v_n\} \) and \( E = \{v_iv_{i+1}: i = 1, \ldots, n-1\} \). That is, a path is a finite sequence of vertices in which each successive vertex is joined to the previous one by an edge. The **length** of the path \( P_n \) is \( n-1 \), the number of edges in the path.

An **\( n \)-cycle** is a graph \( C_n = (V, E) \) where, for some labeling of the vertices, \( V = \{v_1, \ldots, v_n\} \) and \( E = \{v_iv_{i+1}: i = 1, \ldots, n-1\} \cup \{v_1v_n\} \).

![Figure 2.6](image)

The path \( P_4 \) (left) and the cycle \( C_5 \) (right)

A graph is **connected** if between any two vertices of the graph there is a path; otherwise, we say the graph is **disconnected**. We take the graph on one vertex to be connected by default.

An **acyclic graph** is a graph which does not contain any cycle as a subgraph. If an acyclic graph is connected, it is called a **tree**; otherwise, it is called a **forest**. Clearly every path is a tree.

A graph is **unicyclic** if it is connected and contains exactly one cycle as a subgraph. One particularly interesting example of a unicyclic graph is the \( n \)-sun. An **\( n \)-sun**, denoted \( H_n \), is the graph obtained by appending pendants to each vertex of \( C_n \). A **partial \( n \)-sun** is a unicyclic subgraph of an \( n \)-sun containing at least one pendant vertex. (See Figure 2.7.)
A graph is **complete** (or a **clique**) if every pair of vertices is adjacent. A complete graph on \( n \) vertices is denoted by \( K_n \).

A graph is **bipartite** if \( V \) can be partitioned into two disjoint subsets \( V_1 \) and \( V_2 \) so that every edge in \( E \) connects a vertex in \( V_1 \) to a vertex in \( V_2 \).

**Example 2.2.** Consider the graph \( G \) in Figure 2.9. The vertex set for \( G \) can be partitioned into two sets: \( V_1 = \{x_1, x_2, x_3, x_4\} \) and \( V_2 = \{y_1, y_2, y_3\} \). Since every edge of \( G \) connects a vertex of \( V_1 \) and a vertex of \( V_2 \), the graph \( G \) is bipartite.
A bipartite graph $G = (V, E)$ with partition $\{V_1, V_2\}$ of $V$ is a complete bipartite graph if every pair of vertices $v$ in $V_1$ and $u$ in $V_2$ is adjacent. If $|V_1| = p$ and $|V_2| = q$, then this graph is denoted by $K_{p,q}$. In the case that $p = 1, q > 1$, we will call the vertex which is joined to the other $q$ vertices the center of $K_{1,q}$. Figure 2.10 below shows a complete bipartite graph.

If a graph is disconnected, a connected component of the graph is a maximally connected induced subgraph. A vertex is said to be a cut vertex if its deletion results in a graph with an increased number of connected components, i.e. $v$ is a cut vertex of $G$ if $G - v$ has more connected components than $G$ (see Figure 2.11).
If $u$ and $v$ are vertices of a graph, their **distance**, denoted $d(u,v)$, is the minimum length among all paths connecting $u$ and $v$. If there is no such path connecting $u$ and $v$, i.e. they belong to different connected components, we say their distance is infinite.

![Figure 2.11a](image1.png)

A connected graph with cut vertex at vertex 4

![Figure 2.11b](image2.png)

A disconnected graph with two connected components resulting from the deletion of vertex 4 from the graph in Figure 2.11a

There are graph operations which can be used to obtain new graphs from given ones. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where $V_1$ and $V_2$ are disjoint. The **union** of $G_1$ and $G_2$, denoted $G_1 \cup G_2$, is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The **join** of $G_1$ and $G_2$ is the graph $G_1 \vee G_2$ obtained from $G_1 \cup G_2$ by adding all edges joining a vertex of $G_1$ with a vertex of $G_2$. Lastly, the **Cartesian product** of $G_1$ and $G_2$, denoted $G_1 \square G_2$, is the graph with vertex set $V_1 \times V_2$ where two distinct vertices $(u, v)$ and $(x, y)$ are adjacent if either (1) $u = x$ and $vy \in E_2$ or (2) $v = y$ and $ux \in E_1$. 

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Recall that an $n \times n$ matrix $A$ is called symmetric if $A = A^T$. Let $A$ be a real symmetric matrix. We say that $A$ is positive semidefinite if $x^T A x \geq 0$ for all vectors $x \in \mathbb{R}^n$. In the case that $x^T A x > 0$ for all nonzero $x \in \mathbb{R}^n$, we say that $A$ is positive definite.

Let $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$, and let $A = [a_{ij}]$ be an $n \times n$ real symmetric matrix $A$. We will write $G(A) = G$ and say that the matrix $A$ describes the graph $G$ in the case that $a_{ij}$ is nonzero if and only if $v_i v_j$ is an edge of $G$ whenever $i \neq j$. Note that the diagonal entries of $A$ play no part in determining whether or not $G(A) = G$.

**Example 2.4.** Let $G$ be the graph below:

![Figure 2.13](image)

A graph on four vertices
Then the matrix
\[ A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 2 & 5 & 3 & 5 \\ 0 & 3 & 0 & 3 \\ 2 & 5 & 3 & 5 \end{bmatrix} \]
satisfies \( G(A) = G \).

Let \( V = (v_1, \ldots, v_n) \) be an \( n \)-tuple of vectors in \( \mathbb{R}^d \), and let \( A = [v_1 \ldots v_n] \) be the \( d \times n \) matrix whose \( j \)th column is \( v_j \). Then the matrix \( A^T A \) is called the **Gram matrix** associated with \( V \). Evidently the Gram matrix of \( V \) is positive semidefinite (since \( x^T A^T Ax = \|Ax\| \geq 0 \) for all \( x \in \mathbb{R}^d \), where \( \|v\| \) denotes the Euclidean norm of the vector \( v \)). If \( G \) is a graph on \( n \) vertices and \( B \) is the Gram matrix associated with \( V \), we say that \( V \) is an **orthogonal representation** (or a **faithful orthogonal representation**) of the graph \( G \) if \( G(B) = G \). Alternatively, an orthogonal representation of \( G = (V, E) \) is a mapping \( v \mapsto v \) from \( V \) to \( \mathbb{R}^d \) so that \( v \perp u \) precisely when \( v \) and \( u \) are nonadjacent. The **rank of an orthogonal representation** \( V = (v_1 , \ldots, v_n) \) is the rank of the matrix \( A = [v_1 \ldots v_n] \) which, in turn, is equal to the rank of the Gram matrix \( A^T A \).

**Example 2.5.** Let \( G = C_4 \), the cycle on the four vertices \( v_1, v_2, v_3, \) and \( v_4 \). Let \( V = (v_1, v_2, v_3, v_4) \) where \( v_1 = (1,1,0,0)^T, v_2 = (2,0,3,0)^T, v_3 = (0,0,1,1)^T, v_4 = (0,-1,0,1)^T \), and let \( A = [v_1 \ v_2 \ v_3 \ v_4] \). Then the Gram matrix associated with \( V \) is
\[ B = \begin{bmatrix} 2 & 2 & 0 & -1 \\ 2 & 13 & 3 & 0 \\ 0 & 3 & 2 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix}. \]

It is easily verified that \( G(B) = C_4 \) and that \( v_1, v_2, v_3, \) and \( v_4 \) are linearly independent. Thus, \( V \) is an orthogonal representation of \( C_4 \) of rank 4.
Let $S_n$ denote the set of all $n \times n$ real symmetric matrices, and let $P_n$ denote the set of all $n \times n$ real positive semidefinite matrices. Given a graph $G$ whose order is $n$, the **minimum rank** of $G$, denoted by $mr(G)$, is the quantity

$$mr(G) = \min\{\text{rank} A : A \in S_n \text{ and } G(A) = G\}.$$ 

If, in addition, we require that $A$ be positive semidefinite, we call this quantity the **minimum positive semidefinite rank** of $G$ and denote it by $mr_+(G)$, i.e.

$$mr_+(G) = \min\{\text{rank} A : A \in P_n \text{ and } G(A) = G\}.$$ 

Since every positive semidefinite matrix can be viewed as the Gram matrix of a suitable $m$-tuple of vertices, the minimum positive semidefinite rank of $G$ corresponds to the minimum rank among all orthogonal representations of $G$. We observe that by definition of an orthogonal representation, the zero vector is allowed to be one of the vectors in $V$ if $G$ has isolated vertices. It is sometimes useful to not allow this to occur. The **minimum vector rank** of $G$, denoted $mvr(G)$ is the minimum rank among all orthogonal representations of $G$ which do not include the zero vector.

The **minimum rank problem** (MRP) asks for the minimum rank of a given graph $G$ while the **minimum positive semidefinite rank problem** (MRP$_+$) asks for the minimum positive semidefinite rank of a given graph $G$.

The **maximum nullity** of $G$, denoted by $M(G)$ is the quantity

$$M(G) = \max\{\text{nullity} A : A \in S_n \text{ and } G(A) = G\},$$

and if we require that $A$ be positive semidefinite, we call this quantity the **maximum positive semidefinite nullity** of $G$ and denote it by $M_+(G)$, i.e.

$$M_+(G) = \max\{\text{nullity} A : A \in P_n \text{ and } G(A) = G\}.$$ 

It is immediately clear that $mr(G) + M(G) = mr_+(G) + M_+(G) = n$. Furthermore, $mr(G) \leq mr_+(G)$ and $M_+(G) \leq M(G)$. 

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We conclude this section by defining two open problems related to the MRP. Let $G$ be a graph. The $\delta$-conjecture states that $\delta(G) \leq M(G)$, that is,

$$\text{mr}(G) \leq |G| - \delta(G),$$

or equivalently,

$$\text{mr}(G) \leq \Delta(G) + 1.$$ Just as we have both the MRP and the MRP$^+$, we have the $\delta$-conjecture and the $\delta^+$-conjecture. As one might expect, this states that $\delta(G) \leq M^+_+(G)$ and this is equivalent to both $\text{mr}_+(G) \leq |G| - \delta(G)$ and $\text{mr}_+(G) \leq \Delta(G) + 1$. Of course, the $\delta^+$-conjecture implies the $\delta$-conjecture. Lastly, the graph complement conjecture (GCC) states that

$$\text{mr}(G) + \text{mr}(\overline{G}) \leq |G| + 2,$$

and similarly, the GCC$^+$ states that

$$\text{mr}^+(G) + \text{mr}^+(\overline{G}) \leq |G| + 2.$$ Clearly the GCC$^+$ implies the GCC.
CHAPTER 3
SURVEY OF THE LITERATURE

The minimum rank problem has gained considerable attention since it was first studied by Nylen [14] in 1996. Over the last 20 years, considerable progress has been made. In this chapter, we consider some known results concerning minimum rank, the delta conjecture, and the graph complement conjecture.

**Path Cover Number and Zero Forcing Number**

While studying the MRP, researchers have found it helpful to introduce “graph parameters” which can sometimes provide upper and lowers bounds for the minimum rank of a graph. For example, it is well-known that for any graph \( G \), \( \text{mr}(G) \leq |G| - 1 \) (and so \( M(G) \geq 1 \)) [7].

When Nylen first studied the MRP, he restricted his attention to graphs of the form \( G = T \) where \( T \) is a tree, ultimately finding a recursive formula for \( \text{mr}(T) \). In 1999, Johnson and Leal Duarte [11] improved on this result, providing a more efficient method for computing \( \text{mr}(T) \) using the notion of the “path cover number.”

**Definition 3.1.** Let \( G \) be a graph. The **path cover number** of \( G \), denoted by \( P(G) \) is the minimum number of vertex disjoint paths, occurring as induced subgraphs of \( G \), which cover every vertex of \( G \).
**Example 3.2.** The path cover number of any path is 1. The path cover number of any $n$-cycle is 2. The path cover number of the 5-sun is 3 (see Figure 3.1).

![Figure 3.1](image)

A 5-sun with a path covering which achieves $P(H_5) = 3$

**Theorem 3.3 [11].** For a tree $T$, $M(T) = P(T)$ or equivalently, $mr(T) = |T| - P(T)$.

From Theorem 3.3, it is immediately clear that $P_n$ has minimum rank $n - 1$. The fact that paths are the only graphs with this property is a consequence of a theorem from Fiedler [8].

**Theorem 3.4 [7, Corollary 1.5].** $mr(G) = |G| - 1$ if and only if $G = P_{|G|}$.

Barioli, Fallat, and Hogben [2] investigated the potential relationship between the minimum rank and the path cover number of a graph. It was already known that $M(G) \neq P(G)$ for some graphs which are not trees. As an example, consider $K_4$, the complete graph on 4 vertices. Clearly $mr(K_4) = 1$ so that $M(K_4) = 3$. On the other hand, $P(K_4) = 2 < M(K_4)$. The next natural question was then: does $P(G) \leq M(G)$ hold in general? If so, then $|G| - P(G)$ would serve as an upper bound for the minimum rank of a graph $G$. Unfortunately, this turns out not to be the case. A consequence of a result proved by Van der Holst in 2003 [15] is that $P(G) \leq M_4(G)$ does not
hold in general. A year later, Barioli, Fallat, and Hogben [2] showed by example (via the 5-sun) that $P(G) \leq M(G)$ did not hold either.

**Theorem 3.5** [15]. *Let G be a graph. Then $\mr_4(G) = |G| – 1$ if and only if $G = T$ is a tree.*

**Example 3.6.** Consider $K_{1,3}$, the star on four vertices. By Theorem 3.5, $\mr_4(K_{1,3}) = 3$ so that $M_4(K_{1,3}) = 1$. On the other hand, $P(K_{1,3}) = 2 > M_4(K_{1,3})$.

Because the path cover number serves as neither an upper nor lower bound for the maximum nullity of a graph, it has fallen out of favor with researchers interested in the MRP.

In 2007, an AIM special working group [1] introduced a new graph parameter: the *zero forcing number*. In order to define the zero forcing number of a graph, we introduce some preliminary definitions.

**Definition 3.7** [1, Definition 2.1]. *Let G be a graph where each vertex of G is colored either white or black. We define the following color-change rule: If u is a black vertex of G with exactly one neighbor v colored white, change the color of v to black. In this case, we say that u forces v (see Figure 3.2(a) and (b)).*

![Figure 3.2](image)

*Figure 3.2*

A demonstration of the color-change rule where, initially, only $u$ is colored black
**Definition 3.8** [1, Definition 2.1]. Given a coloring of $G$ where each vertex is colored either white or black, the **derived coloring** of $G$ is the result of applying the color-change rule until it is no longer applicable. Given an initial coloring of $G$, its derived coloring is unique [1].

**Example 3.9.** An initial coloring of a graph and its derived coloring are seen in Figure 3.2a and Figure 3.2c, respectively.

**Definition 3.10** [1, Definition 2.1]. A **zero forcing set** for a graph $G = (V, E)$ is a subset of vertices $Z$ of $V$ such that, if initially, the vertices in $Z$ are colored black and all other vertices of $G$ are colored white, then the derived coloring of $G$ will be one in which each vertex is black.

**Example 3.10.** If $G$ is the graph in Figure 3.2, we see that $Z = \{u\}$ is *not* a zero forcing set for $G$. On the other hand, if $Z' = \{u, x\}$, then it is easily verified that $Z'$ is a zero forcing set for $G$.

**Definition 3.11** [1, Definition 2.1]. The **zero forcing number** of graph $G$, denoted $Z(G)$ is the minimum cardinality among all zero forcing sets for $G$, i.e.,

$$Z(G) = \min\{|Z| : Z \text{ is a zero forcing set for } G\}.$$ 

**Example 3.12.** $Z(P_n) = 1$ and $Z(C_n) = 2$. To see this, let $P_n$ be given and color exactly one of the pendants black. For the $n$-cycle ($n \geq 3$), it is clear that $Z(C_n) > 1$. Now pick any two adjacent vertices and color them black while coloring the rest white. In both cases, the derived coloring will have all of the vertices black.

One important result concerning the zero forcing number is the following:

**Theorem 3.13** [1, Proposition 2.4]. Given a graph $G$ and a zero forcing set $Z$ for $G$, $M(G) \leq |Z|$, and thus $M(G) \leq Z(G)$. Consequently, $mr(G) \geq |G| - Z(G)$. 

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The fact that \(|G| - Z(G)\) serves as a lower bound for \(mr(G)\) has been used to great effect, leading to the determination of the minimum rank of a number of classes of graphs [1].

**Orthogonal Representations**

We now turn our attention to the minimum positive semidefinite rank problem. Orthogonal representations have been an invaluable tool for calculating the minimum positive semidefinite rank for several classes of graphs. We provide some of the more notable results.

**Theorem 3.14** [1, Proposition 3.15]. If \(n \geq 5\), then \(mr(\overline{C_n}) = 3\).

**Theorem 3.15** [1, Theorem 3.16]. For any tree \(T\), \(mr_+(\overline{T}) \leq 3\).

**Theorem 3.16** [10, Corollary 3.4]. Let \(H\) be a unicyclic graph. Then \(mr(\overline{H}) \leq 4\).

The next two results give a method for determining the \(mr_+(G)\) for graphs \(G\) which can be constructed from graphs whose minimum positive semidefinite rank is known.

**Theorem 3.17** [10, Theorem 2.1]. Let \(Y = (V_Y, E_Y)\) be a graph of order at least two such that there is an orthogonal representation in \(\mathbb{R}^d\), \(d \geq 3\) satisfying

\[
v \notin \text{span}(u) \text{ for each pair of distinct vertices } v, u \in V_Y.\tag{3.1}
\]

Let \(X\) be a graph that can be constructed by starting with \(Y\) and adding one vertex at a time, such that the newly added vertex is adjacent to all prior vertices except at most one vertex. Then there exists a \(d\)-dimensional orthogonal representation of \(X\) satisfying (3.1); in particular, \(mr_+(X) \leq d\).
It is easy to see that $Y = \overline{P_2}$ has an orthogonal representation in $\mathbb{R}^3$ satisfying (3.1) and that $X = \overline{P_n}$ ($n \geq 3$) can be constructed from $Y$ as described in Theorem 3.17. Consequently, $\overline{P_n}$ has an orthogonal representation in $\mathbb{R}^3$ satisfying (3.1). We will use this fact to prove Proposition 4.5.

**Theorem 3.18 [10, Theorem 2.2].** Let $Y = (V_Y, E_Y)$ be a graph such that the order of $Y$ is at least two, $Y$ does not contain $K_4$ as a subgraph, and there is an orthogonal representation of $Y$ in $\mathbb{R}^4$ satisfying

\begin{align*}
  &v \not\in \text{span}(u) \text{ for } v \neq u \\
  &\dim \text{span}(u, v, w) = 3 \quad \text{for all distinct } u, v, w \text{ such that } v \not\sim u
\end{align*}

for all vertices in $V_Y$. Let $X$ be a graph that can be constructed by starting with $Y$ and adding one vertex at a time, such that the newly added vertex is adjacent to all prior vertices except at most two nonadjacent vertices. Then there is an orthogonal representation of $X$ satisfying (3.2) and (3.3); in particular, $\text{mr}_+(X) \leq 4$.

We also consider two results concerning the minimum vector rank and its relationship to the minimum positive semidefinite rank. Both results will be used in Chapter 4.

**Theorem 3.19 [9].** Let $G$ be a graph with $m$ isolated vertices ($m \geq 0$). Then

$$\text{mvr}(G) = \text{mr}_+(G) + m.$$ 

That is, the difference between $\text{mvr}(G)$ and $\text{mr}_+(G)$ is exactly the number of isolated vertices of $G$.

**Theorem 3.20 [9, Proposition 2.6].** Let $G_1$ and $G_2$ be graphs. Then

$$\text{mr}_+(G_1 \lor G_2) = \max\{\text{mvr}(G_1), \text{mvr}(G_2)\}.$$ 

This result can be extended to the join of any number of graphs, i.e., if $G_1, \ldots, G_n$ are graphs, then

$$\text{mr}_+(G_1 \lor \cdots \lor G_n) = \max\{\text{mvr}(G_1), \ldots, \text{mvr}(G_n)\}.$$
The δ-Conjecture and the Graph Complement Conjecture

We conclude this chapter with two open problems related to the MRP. Let $G$ be a graph. The δ-conjecture suggests that $\delta(G)$, the minimum degree of $G$, serves as a lower bound for $M(G)$. Maehara proposed a stronger version of the δ-conjecture (what we now refer to as the $\delta_+\text{-conjecture}$) in 1987 [12]. After the introduction of the MRP, the δ-conjecture gained renewed significance. Since it was first proposed (in its modern form) at a 2006 AIM workshop, researchers have shown the conjecture holds for several classes of graphs (see [5] and [13].) We present one result concerning the δ-conjecture which we build upon in Chapter 4.

**Theorem 3.21** [5, Proposition 4.3]. Let $G$ be a graph of order $n$. If $\delta(G) \leq 3$ or $\delta(G) \geq |G| - 2$, then $\delta(G) \leq M(G)$ or equivalently, $mr(G) \leq \Delta(G) + 1$.

The Graph Complement Conjecture proposes an upper bound for the sum of the minimum ranks of a graph and its complement. More specifically, it states that for any graph $G$,

$$mr(G) + mr(G) \leq |G| + 2.$$  

The GCC (and its positive semidefinite analogue, the GCC+) were first proposed during the aforementioned 2006 AIM workshop concerning minimum rank research. As with the δ-conjecture, significant progress in the resolution of the GCC has been made. We present some of the results related to the GCC and the GCC+.

**Theorem 3.22** [2, Corollary 2.8 and Corollary 2.11]. If $G$ is a graph of order at most 10, then the GCC holds for $G$. If $G$ is a graph of order at most 8, then the GCC+ holds for $G$.

**Theorem 3.23** [10, Corollary 3.6]. Let $G$ be a graph. If $mr(G) \leq 4$ or $mr(G) \leq 4$, then the GCC holds for $G$. If $mr(G) \leq 4$ or $mr(G) \leq 4$, then the GCC+ holds for $G$.  

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CHAPTER 4
THE $\delta_+$-CONJECTURE FOR GRAPHS WITH $\delta(G) \geq |G| - 4$

The goal of this chapter is extend the classes of graphs for which we know the $\delta_+$-conjecture holds. There are a number of observations which we will use repeatedly throughout this chapter.

**Observation 4.1.** Let $G$ be a graph and let $d$ be the smallest positive integer such that there exists an orthogonal representation for $G$ in $\mathbb{R}^d$. Then $d = mr_+(G)$.

Observation 4.1 is a consequence of the fact that a Gram matrix is positive semidefinite and every positive semidefinite matrix is the Gram matrix of some suitable $n$-tuple of vectors.

**Observation 4.2.** Let $G = (V, E)$ be a graph on two or more vertices. If $u$ and $v$ are twin vertices of $G$ and $v$ is not isolated in $G - u$, then $u$ and $v$ may be assigned to the same vector in any orthogonal representation of $G$. Consequently, $mr_+(G) = mr_+(G - u)$.

For completeness, we prove Observation 4.2.

**Proof.** Because $G - u$ is an induced subgraph of $G$, it is clear that $mr_+(G - u) \leq mr_+(G)$ [6]. Let $V$ be an orthogonal representation of $G - u$ with rank $d = mr_+(G - u)$. Since $v$ is not an isolated vertex of $G - u$, we note that $V(v) \neq 0$. Define $V^*: V \to \mathbb{R}^d$ by the following mapping:

$$V^*(x) = \begin{cases} V(x) & \text{for } x \in V - \{u\} \\ V(v) & \text{for } x = u \end{cases}$$
By construction, it is clear that for all \( x, y \in (V - \{u\}) \), \( V^*(x) \perp V^*(y) \) if and only if \( x \) and \( y \) are nonadjacent. We must show that \( V^*(u) \perp V^*(x) \) if and only if \( u \) and \( x \) are nonadjacent. If \( x = v \), we observe that \( u \) and \( v \) are adjacent and that \( V^*(u) \) and \( V^*(v) \) are not orthogonal (since \( V^*(u) = V^*(v) \) and \( V^*(v) \neq 0 \)). Now suppose \( x \) is different from \( v \). Since \( u \) and \( v \) are twins, \( u \) and \( x \) are nonadjacent if and only if \( v \) and \( x \) are nonadjacent, i.e., if and only if \( V^*(v) \perp V^*(x) \), i.e., if and only if \( V^*(u) \perp V^*(x) \). We conclude that \( V^* \) is an orthogonal representation of \( G \) with rank \( d \). Hence, \( m_r(G) \leq d = m_r(G - u) \leq m_r(G) \), from which the claim follows. □

**Observation 4.3.** Let \( V \) be a subspace of \( \mathbb{R}^d \), and let \( U_1, \ldots, U_m \) be a finite collection of proper subspaces of \( V \). Then \( V \neq \bigcup_{i=1}^m U_i \).

**Observation 4.4.** Let \( V \) be a subspace of \( \mathbb{R}^d \) and let \( A \) and \( B \) be subspaces of \( V \). Then \( A \subset B \) if and only if \( B^\perp \subset A^\perp \). In particular, for any three pairwise independent vectors \( u, v, w \) of \( \mathbb{R}^4 \),

\[
\dim \text{span}\{u, v, w\} = 3 \text{ if and only if } u^\perp \cap v^\perp \not\subset w^\perp.
\]

Recall that the \( \delta \)-conjecture is known to hold for all graphs \( G \) satisfying \( \delta(G) \geq |G| - 2 \). From the theorems of Chapter 3, this result can be extended as follows:

**Proposition 4.5.** Let \( G \) be a graph satisfying \( \delta(G) = |G| - 3 \). Then the \( \delta_+ \)-conjecture holds for \( G \).

**Proof.** We first show that the complement of a cycle satisfies the \( \delta_+ \)-conjecture. Suppose \( G = \overline{C_n} \).

If \( n \leq 4 \), the result holds trivially; therefore, assume \( n \geq 5 \). Then \( \overline{G} = (V, E) \) where \( V = \{v_1, \ldots, v_n\} \) and \( E = \{v_i v_{i+1} : i = 1, \ldots, n-1\} \cup \{v_1 v_n\} \). Let \( G' \) be the subgraph of \( G \) induced by \( v_1, \ldots, v_{n-2} \). Note that \( \overline{G'} = P_{n-2} \). By the remarks following Theorem 3.17, there is an orthogonal representation of \( G' \) in \( \mathbb{R}^3 \) satisfying \( v_i \notin \text{span}\{v_j\} \) for distinct \( i \) and \( j \) with \( 1 \leq i, j \leq n - 2 \). We show how to pick
vectors $v_{n-1}$ and $v_n$ so that $V = (v_1, \ldots, v_n)$ is an orthogonal representation of $G$ in $\mathbb{R}^3$. Observe that $v_{n-2}^\perp \not\subseteq \text{span}\{v_1, v_i\}$ for any $i \leq n - 2$. (This is clear for $i = n - 2$. For $i$ different from $n - 2$, note that since both spaces are two-dimensional, the only way for this relation to be false is to have equality. In that case, $v_{n-2}$ would be adjacent to $v_1$ in $\overline{G}$, which is false.) Choose $v_{n-1}$ so that $v_{n-1} \not\in v_{n-2}^\perp \cap v_i^\perp$ for all $i \leq n - 2$. (To see that such a choice is possible, use the linear independence for every collection of two of these vectors, Observation 4.3, and the fact that $v_{n-2}^\perp \not\subseteq \text{span}\{v_1, v_i\}$). Choose $v_n \in v_{n-1}^\perp \cap v_1^\perp$. If $v_{n-1}^\perp \cap v_1^\perp \subseteq v_i^\perp$ for some $i \neq 1, n - 1$, (i.e., if we have an unwanted orthogonality) then $v_{n-1} \in \text{span}\{v_1, v_i\}$, which is not the case. It follows that $V = (v_1, \ldots, v_n)$ is the desired orthogonal representation of $G$.

We now consider the general case. If $\delta(G) = |G| - 3$ (which is equivalent to $\Delta(\overline{G}) = 2$), $\overline{G}$ is the (disjoint) union of cycles and paths. Therefore $G$ is the join of graphs $G_1, \ldots, G_m$ where $\overline{G}_i$ is either a cycle or a path. By the previous paragraph and Theorem 3.15, $\text{mr}_+(G_i) \leq 3$ for each $i$. Furthermore, by Theorem 3.20,

$$\text{mr}_+(G) = \max\{\text{mvr}(G_i): i = 1, \ldots, m\}.$$ 

Note that $\text{mvr}(G_i) = \text{mr}_+(G_i)$ unless $G_i$ contains an isolated vertex by Theorem 3.19. However, because we assumed $\delta(G) = |G| - 3$, if $G_i$ contains an isolated vertex, then $|G_i| \leq 3$ and so $\text{mvr}(G_i) \leq 3$. All together, we have $\text{mr}_+(G) \leq 3 = |G| - \delta(G)$, that is, the $\delta_+$-conjecture holds for $G$. □

**Graphs with $\delta(G) = |G| - 4$**

We now turn our attention solely on graphs $G$ with $\delta(G) = |G| - 4$. Our main goal will be to show that many such graphs have an orthogonal representation in $\mathbb{R}^4$, and therefore, satisfy the $\delta_+$-conjecture. We note the following:
Lemma 4.6. Let $G$ be a graph with no twin vertices, and let $G'$ be a graph obtained from $G$ solely by duplicating vertices. If $G$ satisfies the $\delta_*$-conjecture, then so does $G'$.

Proof. It suffices to prove the lemma in the case that $G'$ is obtained from $G$ via the addition of a single duplicate vertex. Assume $G$ satisfies the $\delta_*$-conjecture, i.e. $\delta(G) \leq M_*(G)$. It is clear that

$$\delta(G) \leq \delta(G') \leq \delta(G) + 1.$$

We consider two cases.

First, if the vertex we are duplicating is not an isolated vertex, then by Observation 4.2 and the fact that, for any graph $H$, $\operatorname{mr}_*(H) + M_*(H) = |H|$, the addition of this vertex to $G$ results in a graph with maximum nullity exactly one more than that of $G$, i.e. $M_*(G') = 1 + M_*(G)$. Therefore,

$$\delta(G') \leq \delta(G) + 1 \leq M_*(G) + 1 = M_*(G').$$

Hence, $G'$ satisfies the $\delta_*$-conjecture.

On the other hand, if we are duplicating an isolated vertex, then we know $\delta(G) = 0$ and $M_*(G) \geq 2$ (since isolated vertices contribute nothing to the minimum rank). By duplicating an isolated vertex, we obtain a path of length 1. This path will contribute 1 to the minimum rank of the resulting graph and consequently, $M_*(G') = M_*(G) - 1$. Therefore,

$$\delta(G') \leq \delta(G) + 1 \leq 1 \leq M_*(G) - 1 = M_*(G').$$

Again we find $G'$ satisfies the $\delta_*$-conjecture. □

Because we will mostly be working with graph complements when constructing orthogonal representations, it will be helpful to provide an equivalent definition of twins in this context. We observe that vertices $u$ and $v$ are twins in $G$ if and only if they share the same neighborhood in $\overline{G}$. (The proof of this fact is trivial.)

Suppose $G$ is a graph such that $\delta(G) = |G| - 4$ or equivalently $\Delta(\overline{G}) = 3$. We will refer to such graphs as $\mathbf{A(3)}$ co-graphs. Additionally, assume $G$ has no twins (which is valid by Lemma
Order the vertices of $G$ randomly, $v_1, \ldots, v_n$. If we have $m$ vectors, $v_1, \ldots, v_m$ chosen ($m < n$), in $\mathbb{R}^4$ so that

$$v_i \text{ is orthogonal to } v_j \text{ precisely when } v_i \text{ and } v_j \text{ are adjacent in } \overline{G},$$

we ask: is it always possible to choose $v_{m+1} \in \mathbb{R}^4$ so that $v_1, \ldots, v_{m+1}$ satisfy (*)? The answer is clearly no if we have chosen $v_j \in \text{span}\{v_i\}$ for $i, j \leq m$, as an orthogonal representation without twins will never have such vectors [9]. However, even in the case in which we do not allow any of the $m$ vectors to be multiples of one another, it may still be impossible to choose an appropriate vector $v_{m+1}$ as shown in the next example.

**Example 4.7.** Consider the $\Delta(3)$ co-graph $G$ whose complement is shown in Figure 4.1 below. Furthermore, assume the vertices of $G$ have been ordered as in the figure. Suppose $v_1, v_2, v_3,$ and $v_4$ have been chosen as follows:

$$v_1 = (1,0,0,0)^T, \quad v_2 = (1,1,0,0)^T, \quad v_3 = (0,0,1,0)^T, \quad v_4 = (2,1,0,0)^T.$$ 

It is easy to check that $v_1, \ldots, v_4$ satisfy (*) and every collection of two of these vectors is linearly independent. Since $v_5$ is adjacent to both $v_1$ and $v_4$ and nonadjacent to $v_2$ and $v_3$ (all with respect to $\overline{G}$), in order to satisfy (*), we must choose $v_5$ so that $v_5$ is orthogonal to both $v_1$ and $v_4$, but not to $v_2$ or $v_3$. However, because $v_2$ is linearly dependent on $v_1$ and $v_4$ ($v_2 = v_4 - v_1$), $v_5$ cannot be orthogonal to both $v_1$ and $v_4$ without also being orthogonal to $v_2$. 
The complement of the graph $G$

Perhaps the easiest way to avoid the problem seen in Example 4.7 is to insist not only that no vector chosen is a multiple of another, but also that any nonempty collection of three or fewer chosen vectors be linearly independent. Unfortunately, this is not always possible if we choose the vertices randomly. For example, suppose $G$ is a C-$\Delta(3)$ graph with at least seven vertices which we order $v_1, \ldots, v_n$ ($n \geq 7$). Furthermore, suppose $v_6$ is adjacent to $v_1, v_2,$ and $v_3$ in $\overline{G}$ (and nothing else). Assume vectors $v_1, \ldots, v_5 \in \mathbb{R}^4$ have been chosen to satisfy (*) and so that any nonempty collection of three or fewer of these vectors is linearly independent. At this point, there is essentially a unique choice for $v_6$ since $v_1^\perp \cap v_2^\perp \cap v_3^\perp$ is a one-dimensional subspace by Observation 4.4. As we illustrate in Example 4.8 below, this leads to another potential problem.

**Example 4.8.** Suppose $H$ is a graph whose complement is shown in Figure 4.2 below with vertices ordered as in the figure. Furthermore, suppose the vectors $v_1, \ldots, v_5$ have been chosen with

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}.$$

Observe that the above vectors satisfy (*) and any nonempty collection of at most three of these is linearly independent. We note that $v_1^\perp \cap v_2^\perp \cap v_3^\perp = \text{span}\{e_4\} \subset \text{span}\{v_4, v_5\}$. Therefore, in order
to choose $v_6$ so that $v_1, \ldots, v_6$ satisfy (*), we must give up our condition concerning linear independence for any choice of three vectors. In this case, $v_6$ is necessarily linearly dependent on $v_4$ and $v_5$. It is now impossible to pick $v_7$ so that $v_1, \ldots, v_7$ satisfy (*) via an argument similar to the one given in Example 4.7, that is, $v_7$ cannot be orthogonal to $v_4$ and $v_5$ without also being orthogonal to $v_6$.

![Figure 4.2](image)

The complement of the graph $H$

We observe that by choosing a vertex of degree 3 in the complement “too early”, we are forced to pick a vector in a one-dimensional space. Our strategy will be to avoid or delay as much as possible making such a choice. To that end, we introduce a definition.

**Definition 4.9.** Let $G = (V, E)$ be a connected graph on $n$ vertices. A **non-disconnecting sequence** of $G$ is a finite sequence of vertices, $v_1, \ldots, v_m$ with $m < n$, so that $G - \{v_1, \ldots, v_i\}$ is a connected graph for each $i \in \{1, \ldots, m\}$. The graph $G - \{v_1, \ldots, v_m\}$ is called the **remainder** of the non-disconnecting sequence.
**Definition 4.10.** Let $G$ be a graph, let $H$ be a subgraph of $G$, and let $v$ be a vertex of $G$. If $v$ is also a vertex of $H$, we define $d(v, H) = 0$. If $v$ is not a vertex of $H$, we define

$$d(v, H) = \min\{d(v, u) : u \text{ is a vertex of } H\}.$$ 

The quantity $d(v, H)$ is called the **distance between** $v$ **and** $H$.

**Lemma 4.11.** Let $G = (V_G, E_G)$ be a connected graph and let $H = (V_H, E_H)$ be a connected subgraph so that $V_H \subset V_G$. Let $u$ be a vertex of $G$ of maximal distance from $H$. Then $G - u$ is connected.

**Proof.** We observe that by assumption, $u$ satisfies $d(u, H) = \max\{d(x, H) : x \in V_G\}$. In particular, $u$ is not a vertex of $H$. (To see this, note that since we assumed $V_H \subset V_G$, $d(u, H) > 0$). Suppose for contradiction that $G - u$ is disconnected. Let $G_1, G_2$ be distinct connected components of $G - u$ so that $H$ is contained in $G_1$. Let $w$ be a vertex of $G_2$. Because $G - u$ is disconnected, it follows that any path in $G$ connecting $w$ with a vertex of $H$ necessarily contains $u$. But this implies that for every vertex $v$ of $H$, in $G$ we have $d(w, v) > d(u, v)$. So if $v_0$ is a vertex of $H$ satisfying $d(w, v_0) = d(w, H)$, then $d(w, v_0) > d(u, v_0) \geq d(u, H)$. Consequently, $d(w, H) > d(u, H)$, contradicting that $u$ is a vertex of maximal distance from $H$. Hence, $G - u$ is connected. □

**Corollary 4.12.** Let $G = (V_G, E_G)$ be a connected graph of order $n$ and let $H = (V_H, E_H)$ be a connected subgraph of $G$ of order $m$ so that $V_H \subset V_G$. Then there is a non-disconnecting sequence $v_1, \ldots, v_{n-m}$ of $G$ whose remainder contains $H$ as a (not necessarily induced) subgraph.

**Proof.** From Lemma 4.11, we see that for $v_1$ a vertex of maximal distance from $H$ in $G$, $G - v_1$ is connected. Furthermore, because $V_H \subset V_G$, it follows that $v_1 \notin V_H$. If $V_H = V_G - \{v_1\}$, the result follows. Otherwise, let $v_2$ be a vertex of maximal distance from $H$ in $G - v_1$. Another application of Lemma 4.11 tells us that $G - \{v_1, v_2\}$ is connected and $v_2 \notin V_H$. We repeat this process, obtaining
$v_1, \ldots, v_{n-m}$. By construction, this sequence is a non-disconnecting sequence of $G$ with the property that $v_i \notin V_H$ for each $i \in \{1, \ldots, n-m\}$. Since $|H| = m$, it follows that the remainder of this non-disconnecting sequence contains $H$ as a subgraph. □

We consider two special cases which will be important. First, recall that a graph is called \textit{m-regular} if every vertex of the graph has degree $m$. Based on Corollary 4.12, we make the following observations:

**Observation 4.13.** Let $v$ be a vertex of a connected graph $G$ on $n$ vertices. Then there is a non-disconnecting sequence $v_1, \ldots, v_{n-1}$ of $G$ whose remainder is the graph on the single node $v$. In this case, it makes sense to let $v_n = v$, and, with an abuse of terminology, to say $v_1, \ldots, v_n$ is also a non-disconnecting sequence of $G$.

**Observation 4.14.** Let $G$ be a connected graph on $n$ vertices. If $G$ is \textit{m-regular} ($m < n-1$), then there is a non-disconnecting sequence $v_1, \ldots, v_{n-(m+1)}$ of $G$ whose remainder contains $K_{1,m}$ as a (not necessarily induced) subgraph. Furthermore, any vertex of $G$ may be chosen to be the center of this $K_{1,m}$.

\textbf{Δ(3) Co-graphs with Connected Complement}

We are now ready to show that certain classes of graphs $G$ with $\delta(G) = |G| - 4$ satisfy the $\delta_4$-conjecture. We first consider $\Delta(3)$ co-graphs with connected complement.

**Proposition 4.15.** Let $G = (V, E)$ be a graph on $n$ vertices with $\overline{G}$ not 3-regular. Assume $G$ also satisfies the following conditions:

\begin{enumerate}
  \item $\delta(G) = n - 4$ (or equivalently, $\Delta(\overline{G}) = 3$);
\end{enumerate}
(2) \( \overline{G} \) is connected.

Then \( G \) has an orthogonal representation in \( \mathbb{R}^4 \), and thus, \( G \) satisfies the \( \delta_s \)-conjecture.

**Proof.** Because of Lemma 4.6, we may assume \( G \) has no twin vertices. Consider \( \overline{G} \) which by (2) is connected. Since \( \overline{G} \) is not 3-regular and \( \Delta(\overline{G}) = 3 \), we choose \( v \in V \) such that \( \deg_{\overline{G}}(v) < 3 \). By Observation 4.13, there is a non-disconnecting sequence of \( \overline{G}, v_1, \ldots, v_n \in V \) such that \( v = v_n \). Note that for each \( i \in \{2, \ldots, n\} \), \( v_i \) cannot be adjacent to more than two vertices from \( \{v_1, \ldots, v_{i-1}\} \) in \( \overline{G} \) (see Figure 4.3). We will now choose vectors in \( \mathbb{R}^4 \) so that \( \mathbf{v}_i \) corresponds to \( v_i \) and \( \mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_n) \) is an orthogonal representation of \( G \).

![Figure 4.3](image)

If \( v_a, v_b, \) and \( v_c \) appear in the non-disconnecting sequence before \( v_i \), then \( G - \{v_1, \ldots, v_{\max\{a,b,c\}}\} \) is disconnected which is impossible.

Let \( \mathbf{v}_1 \) be any nonzero vector in \( \mathbb{R}^4 \).

Suppose vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_{m-1} \) \((2 \leq m \leq n)\) have been chosen so that

\[ (H1) \quad \mathbf{v}_i \perp \mathbf{v}_j \text{ if and only if } v_i \text{ and } v_j \text{ are neighbors in } \overline{G}; \text{ and} \]

\[ (H2) \quad \text{every nonempty set of at most three of these vectors is linearly independent.} \]
We show how to pick \( v_m \) so that \( v_1, \ldots, v_m \) also satisfy (H1) and (H2). Because \( v_1, \ldots, v_n \) is a non-disconnecting sequence and we chose \( v_n \) to have degree strictly less than 3 in \( \overline{G} \), there are three cases to consider:

**Case 1.** \( v_m \) is not adjacent (in \( \overline{G} \)) to any previous vertex in the sequence. In this case, in order to choose \( v_m \in \mathbb{R}^4 \) so that \( v_1, \ldots, v_m \) satisfy (H1) and (H2), we must have \( v_m \not\in v_i^\perp \) for each \( i \in \{1, \ldots, m - 1\} \) and \( v_m \not\in \text{span}\{v_j, v_k\} \) for \( j, k \in \{1, \ldots, m - 1\} \). By Observation 4.3, it is certainly possible to pick \( v_m \) satisfying these two requirements.

**Case 2.** \( v_m \) is adjacent (in \( \overline{G} \)) to exactly one previous vertex in the sequence, say \( v_p \). In order to choose \( v_m \in \mathbb{R}^4 \) so that \( v_1, \ldots, v_m \) satisfy (H1) and (H2), we must have \( v_m \in v_p^\perp \) such that \( v_m \not\in v_i^\perp \) for \( i \in \{1, \ldots, p, \ldots, m - 1\} \) and \( v_m \not\in \text{span}\{v_j, v_k\} \) for \( j, k \in \{1, \ldots, m - 1\} \). We prove that such a choice is indeed possible. Observe that for each \( i \), (H2) tell us that \( v_p^\perp = v_i^\perp \) implies \( p = i \). Consequently, for \( i \neq p \), \( v_p^\perp \cap v_i^\perp \not\subseteq v_p^\perp \). Furthermore, for each combination of \( j \) and \( k \), \( \text{span}\{v_j, v_k\} \) is at most 2-dimensional while \( v_p^\perp \) is 3-dimensional. Therefore, \( v_p^\perp \cap (\text{span}\{v_j, v_k\}) \not\subseteq v_p^\perp \). Again, by Observation 4.3, it is possible to choose the desired \( v_m \).

**Case 3.** \( v_m \) is adjacent (in \( \overline{G} \)) to two previous vertices in the sequence, say \( v_p \) and \( v_q \). In order to choose \( v_m \in \mathbb{R}^4 \) so that \( v_1, \ldots, v_m \) satisfy (H1) and (H2), we must have \( v_m \in v_p^\perp \cap v_q^\perp \) so that \( v_m \not\in v_i^\perp \) for each \( i \neq p, q \) and \( v_m \not\in \text{span}\{v_j, v_k\} \) for \( j, k \in \{1, \ldots, m - 1\} \). Again, we show that such a choice is possible. If \( v_p^\perp \cap v_q^\perp \subseteq v_i^\perp \) for some \( i \neq p, q \), then by Observation 4.4, \( v_i \in \text{span}\{v_p, v_q\} \), contrary to (H2). Therefore, we have

\[
(v_p^\perp \cap v_q^\perp) \cap v_i^\perp \not\subseteq v_p^\perp \cap v_q^\perp
\]
for each \( i \neq p, q \). Furthermore, \( v_p^\perp \cap v_q^\perp \subseteq \text{span}\{v_j, v_k\} \) if and only if \( j \) and \( k \) are distinct and \( v_p^\perp \cap v_q^\perp = \text{span}\{v_j, v_k\} \) (since both the left- and right-hand sides of the relation above are 2-dimensional). In this case, we see that \( v_p \) and \( v_q \) share three neighbors: \( v_m, v_j, \) and \( v_k \) in \( \overline{G} \). By (1), these are the only neighbors of \( v_p \) and \( v_q \) in \( \overline{G} \). Thus, \( v_p \) and \( v_q \) are twins in \( G \), contradicting our assumption that \( G \) has no twins. Therefore, we may conclude

\[
( v_p^\perp \cap v_q^\perp ) \cap (\text{span}\{v_j, v_k\}) \nsubseteq v_p^\perp \cap v_q^\perp
\]

for each combination of \( j \) and \( k \). One final application of Observation 4.3 shows it is possible to choose the desired \( v_m \).

We have shown that in all cases, it is possible to choose \( v_m \in \mathbb{R}^4 \) so that \( v_1, \ldots, v_m \) satisfies (H1) and (H2). It follows that we may choose a sequence of vectors \( v_1, \ldots, v_n \in \mathbb{R}^4 \) so that \( v_i \) and \( v_j \) are orthogonal if and only if \( v_i \) and \( v_j \) are adjacent in \( \overline{G} \). Consequently, \( V = (v_1, \ldots, v_n) \) is an orthogonal representation for \( G \), and the proof is complete. □

We now investigate the case in which \( \overline{G} \) is a 3-regular graph. We show that, in most cases, \( \Delta(3) \) co-graphs whose complement is 3-regular satisfy the \( \delta_+ \)-conjecture. Recall a graph is called \textbf{triangle-free} if it does not contain any 3-cycle as a subgraph. Similarly, a graph is called \textbf{square-free} if it does not contain any 4-cycle as a subgraph.

**Proposition 4.16.** Let \( G = (V, E) \) be a graph on \( n \) vertices so that \( \overline{G} \) is 3-regular but not triangle-free. If \( \overline{G} \) is connected, then \( G \) has an orthogonal representation in \( \mathbb{R}^4 \), and thus, \( G \) satisfies the \( \delta_+ \)-conjecture.
Proof. Again we may assume $G$ has no twin vertices. Furthermore, we may assume $n \geq 5$ (otherwise the result follows trivially). By Observation 4.14 and the assumption that $\overline{G}$ is 3-regular but not triangle-free, there is a non-disconnecting sequence $v_1, \ldots, v_{n-4}$ of $\overline{G}$ whose remainder contains $K_{1,3}$ as a subgraph, but not an induced subgraph (here is where we use the “not triangle-free” assumption).

Now consider the vertices in the remainder. Choose one of these which is adjacent to the other three (there is at least one) and call it an “inner vertex.” Call the remaining three “outer vertices.” Because $K_{1,3}$ is a non-induced subgraph of the remainder, there are three distinct cases: (i) exactly one pair of outer vertices is adjacent in $\overline{G}$, (ii) exactly two pairs of outer vertices are adjacent in $\overline{G}$, and (iii) each outer vertex is adjacent to every other outer vertex.

In case (i), label the inner vertex $v_{n-1}$, the adjacent outer vertices $v_{n-3}$ and $v_{n-2}$ (in any order), and the remaining vertex $v_n$ (see Figure 4.4a). Observe that $v_1, \ldots, v_{n-1}$ is also non-disconnecting sequence of $\overline{G}$.

![Figure 4.4](image)

Sample labeling in the case that exactly one pair of “outer vertices” is adjacent in $\overline{G}$
Choose nonzero vectors \( v_1, \ldots, v_{n-3} \in \mathbb{R}^4 \) just as in the proof of Proposition 4.15. Note that, because of the 3-regularity of \( \overrightarrow{G} \), \( v_{n-2} \) is necessarily adjacent to two vertices in the sequence, \( v_{n-3} \) and say, \( v_p \), where \( p < n \). Also, in addition to \( v_{n-1}, v_n \) is adjacent to two vertices in the sequence, say \( v_a \) and \( v_b \) (where \( a, b < n-3 \), and we one of \( a \) or \( b \) may equal \( p \)) (see Figure 4.4b). We know (via the proof of Proposition 4.15) it is possible to choose nonzero \( v_{n-2} \in v_p \perp \cap v_{n-3} \perp \) with \( v_{n-2} \notin v_i \perp \) (\( i < n-3, i \neq p \)), and \( v_{n-2} \notin \text{span}\{v_j, v_k\} \) for any \( j, k \in \{1, \ldots, n-3\} \). We now show that in addition to these requirements, we may pick \( v_{n-2} \notin \text{span}\{v_{n-3}\} + (v_a \perp \cap v_b \perp \cap v_i \perp) \) for every \( i \leq n-3, i \neq a, b \). To do so, it suffices to show that

\[
v_p \perp \cap v_{n-3} \perp \notin \text{span}\{v_{n-3}\} + (v_a \perp \cap v_b \perp \cap v_i \perp) \tag{4.1}\]

for each such \( i \). By (H2), \( \dim (v_p \perp \cap v_{n-3} \perp) = 2 \) and \( \dim (v_a \perp \cap v_b \perp \cap v_i \perp) = 1 \) for each such \( i \). Hence, the left-hand side of (4.1) is 2-dimensional and the right-hand side is at most 2-dimensional. Therefore, the only way for (4.1) to be false is if \( v_p \perp \cap v_{n-3} \perp = \text{span}\{v_{n-3}\} + (v_a \perp \cap v_b \perp \cap v_i \perp) \). Since this is clearly not so (\( v_{n-3} \) is in one but not the other), (4.1) holds and it is possible to choose the desired vector \( v_{n-2} \). As a consequence of this choice, we claim

\[
v_{n-2} \notin \text{span}\{v_{n-3}\} + (v_a \perp \cap v_b \perp \cap v_{n-2} \perp).\]

If this is not the case, then there exist nonzero \( x, y \in \mathbb{R} \) so that \( v_{n-2} = x v_{n-3} + y u \) where \( u \in v_{n-2} \perp \). However, this implies

\[
\|v_{n-2}\|^2 = x(v_{n-3} \cdot v_{n-2}) + y(u \cdot v_{n-2}) = 0,
\]

since both \( v_{n-3} \) and \( u \) are orthogonal to \( v_{n-2} \). This is impossible since we chose \( v_{n-2} \) to be nonzero. We conclude our choice for \( v_{n-2} \) satisfies

\[
v_{n-2} \notin \text{span}\{v_{n-3}\} + (v_a \perp \cap v_b \perp \cap v_i \perp) \quad \text{for all } i \leq n-2, i \neq a, b. \tag{4.2}\]
Next, we choose $v_{n-1} \in v_{n-3}^\perp \cap v_{n-2}^\perp$ with $v_{n-1} \notin v_i^\perp \ (i < n-3)$, $v_{n-1} \notin \text{span}\{v_j, v_k\}$ for any $j, k \in \{1, \ldots, n-2\}$, and additionally, $v_{n-1} \notin \text{span}\{v_a, v_b, v_i\}$ for $i \leq n-2$, $i \neq a, b$. We already know we can satisfy the first two of these requirements; it remains to show that we can satisfy the additional one. It suffices to prove that

$$v_{n-3}^\perp \cap v_{n-2}^\perp \notin \text{span}\{v_a, v_b, v_i\}$$

for any $i \leq n-2$, $i \neq a, b$. Assume for contradiction that this is not the case. By Observation 4.4, this implies

$$v_a^\perp \cap v_b^\perp \cap v_i^\perp \subset \text{span}\{v_{n-3}, v_{n-2}\} \tag{4.3}$$

for some $i \neq a, b$. Because $v_{n-3} \notin v_a^\perp \cap v_b^\perp \cap v_i^\perp$ (otherwise $v_{n-3}$ would have too many neighbors), (4.3) is equivalent to

$$v_{n-2} \in [\text{span}\{v_{n-3}\} + (v_a^\perp \cap v_b^\perp \cap v_i^\perp)]$$

for some $i \neq a, b$. But this contradicts (4.2)! Hence, $v_{n-3}^\perp \cap v_{n-2}^\perp \notin \text{span}\{v_a, v_b, v_i\}$, and so another application of Observation 4.3 tells us it is possible to pick the desired $v_{n-1}$. Observe that by this choice of $v_{n-1}$,

$$\dim (\text{span}\{v_a, v_b, v_{n-1}, v_i\}) = 4 \quad \text{for all } i \neq a, b, n-1. \tag{4.4}$$

Finally, choose $v_n \in v_a^\perp \cap v_b^\perp \cap v_{n-1}^\perp$. By (4.4), Observation 4.3, and Observation 4.4, we find $v_n \notin v_i^\perp$ for $i \neq a, b, n-1$. With $n$ vectors picked, let $V = (v_1, \ldots, v_n)$. By the way $v_1, \ldots, v_n$ were chosen, it follows that $v_i \perp v_j$ if and only if $v_i$ is adjacent to $v_j$ in $\overline{G}$. Hence, $V$ is an orthogonal representation of $G$.

For case (ii), an almost identical proof holds. In this case, again let $v_{n-1}$ be the inner vertex, and now let $v_{n-3}$ be the outer vertex which is adjacent to the other two. Replacing $b$ with $n-3$ gives the result.
Finally, case (iii) is impossible. Indeed, in this case, $G$ would have to be the empty graph on 4 vertices contrary to our hypothesis that $G$ is of order at least 5.

Hence, in all possible cases, $G$ has an orthogonal representation in $\mathbb{R}^4$, and it follows that $G$ satisfies the $\delta_+$-conjecture. □

There are of course graphs which are both 3-regular and triangle-free. A graph on an even number of vertices $n \geq 6$ is called a Möbius ladder if it can be constructed from an $n$-cycle by adding edges connecting “opposite pairs” of vertices (see Figure 4.5). We denote such graphs by $M_n$. Note that a Möbius ladder is both 3-regular and triangle-free (see Figure 4.5). Thus, Proposition 4.15 and Proposition 4.16 tell us nothing about the minimum rank of the graph whose complement is a Möbius ladder. However, the next proposition shows that we can handle this class of graphs as well.

Proposition 4.17. Let $G = (V, E)$ be a graph on $n$ vertices so that $\overline{G}$ is both 3-regular and triangle-free, but not square-free. If $\overline{G}$ is connected, then $G$ has an orthogonal representation in $\mathbb{R}^4$, and thus, $G$ satisfies the $\delta_+$-conjecture.
Proof. As usual, we may assume $G$ has no twin vertices. Choose any 4-cycle in $\overline{G}$, and choose any of its vertices, say $v$. By Observation 4.14, there is a non-disconnecting sequence $v_1, \ldots, v_{n-4}$ of $\overline{G}$ whose remainder contains $K_{1,3}$ as a subgraph. Furthermore, we may assume $v$ is the center of this $K_{1,3}$. Consequently, $v$ is the “inner vertex” among the four vertices in the remainder, and two of the outer vertices will lie along this 4-cycle in $\overline{G}$. In this case, relabel the inner vertex $v_{n-1}$, and label the two outer vertices which lie on the 4-cycle $v_n$ and $v_{n-2}$. Finally, label the remaining outer vertex (which does not lie on the aforementioned 4-cycle) $v_{n-3}$ (see Figure 4.6).

![Figure 4.6](image)

Sample labeling where three of the remaining vertices lie on a 4-cycle

Call the remaining vertex of this 4-cycle $v_a$. Also, let the remaining neighbor of $v_n$ be $v_b$ and let the remaining neighbor of $v_{n-2}$ be $v_p$ (see Figure 4.6). We note that $v_b$ and $v_p$ cannot be equal as this would imply $v_{n-2}$ and $v_n$ are twins in $G$. We also observe that $v_{n-3}$ and $v_a$ cannot be adjacent as this would imply $v_a$ and $v_{n-1}$ are twins. Now proceed as in the proof of Proposition 4.16. There are two key differences, namely:
In showing that we may choose \( v_{n-2} \not\in [\text{span}\{v_{n-3}\} + (v_a^\perp \cap v_b^\perp \cap v_{i}^\perp)] \) for any \( i \neq a, b \), we must now show that \( v_p^\perp \cap v_a^\perp \not\in [\text{span}\{v_{n-3}\} + (v_a^\perp \cap v_b^\perp \cap v_{n-2}^\perp)] \) for any such \( i \). This is true because otherwise, \( v_{n-3} \) would be adjacent to \( v_a \), which we just noted is not the case.

In showing that after choosing \( v_{n-2} \not\in [\text{span}\{v_{n-3}\} + (v_a^\perp \cap v_b^\perp \cap v_{n-2}^\perp)] \), the reasoning will be different. If \( v_{n-2} \not\in [\text{span}\{v_{n-3}\} + (v_a^\perp \cap v_b^\perp \cap v_{n-2}^\perp)] \), then there are nonzero \( x, y \in \mathbb{R} \) and \( w \in v_a^\perp \cap v_b^\perp \cap v_{n-2}^\perp \) so that \( v_{n-2} = xv_{n-3} +yw \). Therefore

\[
0 = v_{n-2} \cdot v_a = x(v_{n-3} \cdot v_a) + y(w \cdot v_a) = x(v_{n-3} \cdot v_a) + 0,
\]

i.e. \( v_{n-3} \cdot v_a = 0 \). This then implies \( v_a \) and \( v_{n-3} \) are adjacent, which, as we noted earlier, cannot be the case.

The remainder of the proof now follows as in the first case in Proposition 4.16. □

While a Möbius ladder is both 3-regular and triangle-free, it is not square free. Hence, Proposition 4.17 ensures that the complement of such a graph satisfies the \( \delta_r \)-conjecture. On the other hand, the Petersen graph (see Figure 4.7) is 3-regular, triangle-free and square-free. Thus, we cannot deduce from the theorems above whether or not the complement of the Petersen graph satisfies the \( \delta \)-conjecture.

The Petersen graph is 3-regular, triangle-free, and square-free
We summarize Proposition 4.15, Proposition 4.16 and Proposition 4.17 as follows:

**Theorem 4.18.** Let $G = (V, E)$ be a graph on $n$ vertices so that $\bar{G}$ is not 3-regular, triangle-free, and square-free. Assume $G$ also satisfies the following conditions:

1. $\delta(G) = n - 4$ (or equivalently, $\Delta(\bar{G}) = 3$);
2. $\bar{G}$ is connected.

Then $G$ has an orthogonal representation in $\mathbb{R}^4$, and thus, $G$ satisfies the $\delta_v$-conjecture.

### $\Delta(3)$ Co-graphs with Disconnected Complement

We now consider graphs with disconnected complement. Suppose $G$ is a $\Delta(3)$ co-graph whose complement is disconnected. Then $\bar{G} = \bigcup_{i=1}^{m} H_i$ (where the $H_i$ are the connected components of $\bar{G}$) and therefore $G = \bar{H}_1 \vee \cdots \vee \bar{H}_m$. Since $G$ is a $\Delta(3)$ co-graph, so is $\bar{H}_i$ for each $i \in \{1, \ldots, m\}$.

By Theorem 3.20,

$$mr_+(G) = \max \{ mvr(\bar{H}_1), \ldots, mvr(\bar{H}_m) \}. \quad (4.5)$$

Recall that the minimum vector rank of a graph differs from the minimum positive semidefinite rank of that graph by exactly the number of its isolated vertices (Theorem 3.19). Note that because each $\bar{H}_i$ is a $\Delta(3)$ co-graph, if $\bar{H}_j$ contains an isolated vertex, then $|\bar{H}_j| \leq 4$ and consequently, $mr_+ (\bar{H}_j) \leq mvr(\bar{H}_j) \leq 4$. Because of this and (4.5), we deduce

$$mr_+(G) \leq \max \{ mr(\bar{H}_1), \ldots, mvr(\bar{H}_m), 4 \}. \quad (4.6)$$

Therefore, provided that none of the $H_i$ are 3-regular, triangle free and square-free, we can conclude via (4.6) and Theorem 4.18 that $mr_+(G) \leq 4 = |G| - \delta(G)$. We summarize:
Theorem 4.19. Let $G$ be a graph of order at least 5, and let $H_1, \ldots, H_m$ ($m \geq 1$) be the connected components of $\bar{G}$. If $\delta(G) = |G| - 4$ and none of the $H_i$ are 3-regular, triangle free and square-free, then $G$ satisfies the $\delta_s$-conjecture.

A proof of the full $\delta_s$-conjecture for graphs with minimum degree $|G| - 4$ was not forthcoming and is left for future research. In the next chapter, we consider applications of the results of this chapter as well as the techniques used to prove them.
CHAPTER 5
ADDITIONAL CONSEQUENCES AND EXTENSIONS

In this chapter, we consider some applications of the methods and results provided in Chapter 4. Effective use of Theorem 4.19 allows us to determine the minimum positive semidefinite rank of several graphs.

**Example 5.1.** It is known that the minimum rank of the 4-antiprism graph $G_8$ (see Figure 5.1) is 4 [1]. The complement of $G_8$ is connected, 3-regular, triangle-free, but not square free. Hence, by Theorem 4.19, $4 = \text{mr}(G_8) \leq \text{mr}_4(G_8) \leq 4$, so that $\text{mr}_4(G_8) = 4$.

![Figure 5.1](image)

The 4-antiprism graph $G_8$ (left) and its complement $M_8$ (right)

The complement of the 4-antiprism graph is the Möbius ladder on 8 vertices. In fact, we can determine $\text{mr}_4(M_n)$ for any even positive integer $n$.

**Proposition 5.2.** Let $n = 2k$ be an even integer with $k \geq 3$, and let $M_n$ denote the Möbius ladder on $n$ vertices. Then $\text{mr}_4(M_6) = 2$ and $\text{mr}_4(M_n) = 4$ for $n > 6$. 

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Proof. Since $\overline{M_6} = K_3 \cup K_3$, and $mr_+(K_3) = 1$, the first assertion is clear. Now suppose $n > 6$ (equivalently, $k > 3$), and recall that with an appropriate labeling of the vertices, the Möbius ladder has vertex set $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and edge set

$$E = \{v_iv_{i+1} : i = 0, 1, \ldots, n-1\} \cup \{v_iv_{i+k} : i = 0, 1, \ldots, k-1\},$$

where subscripts are read modulo $n$ (see Figure 5.2). By Theorem 4.19, $mr_+(\overline{M_n}) \leq 4$. Also, by Theorem 3.13, $mr_+(\overline{M_n}) \geq n - Z(\overline{M_n})$. Therefore, it suffices to show that $Z(\overline{M_n}) = n - 4$.

Since $M_n$ is 3-regular, $\overline{M_n}$ is $(n-4)$-regular, and so $n - 4 \leq Z(\overline{M_n})$ (it is easy to see that the minimum degree of a graph serves as a lower bound for its zero forcing number). We will demonstrate a zero-forcing set of $\overline{M_n}$ of size $n - 4$.

Step 1: Let the set $Z$ consist of $v_0$ and all but one of its neighbors in $\overline{M_n}$. Observe that $|Z| = n - 4$. Suppose we have an initial coloring where the black vertices are precisely those vertices in $Z$. Clearly $v_0$ will then force its only remaining white neighbor to become black. There are now three remaining white vertices: the neighbors of $v_0$ in $M_n$, i.e. $v_1$, $v_k$, and $v_{n-1}$.

Step 2: Consider $v_{1+k}$. Its neighbors in $M_n$ are $v_1$, $v_k$, and $v_{2+k}$. Because we assumed $k \geq 4$, we observe that $v_{1+k} \neq v_{n-1}$ and that $v_{2+k} \neq v_{n-1}$. It follows that in $\overline{M_n}$, $v_{1+k}$ is not adjacent to $v_1$ or $v_k$, but it is adjacent to $v_{n-1}$. Since all but $v_1$, $v_k$, and $v_{n-1}$ are black vertices, it follows that $v_{1+k}$ forces $v_{n-1}$. A similar argument shows that $v_{k-1}$ forces $v_1$.

Step 3: Since all but $v_k$ are now black, any neighbor of $v_k$ will force $v_k$.

Thus, $Z$ is a zero forcing set of $\overline{M_n}$, and it follows that $Z(\overline{M_n}) \leq n - 4$. Since we also have the reverse inequality, we conclude $Z(\overline{M_n}) = n - 4$, and the result follows. (Figure 5.2 demonstrates steps 1 - 3 in the case that $n = 8$.) □
The Möbius ladder on 8 vertices. The graph is 3-regular but is not square-free (it contains the 4-cycle $v_0 \rightarrow v_4 \rightarrow v_3 \rightarrow v_7 \rightarrow v_0$). Therefore, $\text{mr}_4(M_8) \leq 4$ by Theorem 4.19.

$M_8$, the complement of $M_8$. Color each of the vertices in $Z$ black, where $Z = \{v_0, v_2, v_3, v_5\}$. This is the initial coloring. Clearly $v_0$ will force $v_6$ to change color.

Now $v_5$ will force $v_7$ and $v_3$ will force $v_1$ to change color.

Since there is only one white vertex left, any of its neighbors will force it to change color, e.g. $v_2$ will force $v_4$. Since 4 is the minimum degree of $M_8$ and $|Z| = 4$, it follows that $Z(M_8) = 4$.

Consequently, $\text{mr}_4(M_8) = 4$

Figure 5.2

Demonstrating the proof of Proposition 5.2 in the case $n = 8$
We now consider other applications of the results in Chapter 4. Our next result, concerning the Positive Semidefinite Graph Complement Conjecture (GCC\(_+\)), is an immediate consequence of Theorem 3.23.

**Proposition 5.3.** Let \(G\) be a graph satisfying the criteria of Theorem 4.19. Then both \(G\) and \(\bar{G}\) satisfy the GCC\(_+\).

There are several classes of graphs which meet the criteria of Theorem 4.19.

**Example 5.4.** Any \(n\)-sun or Möbius ladder satisfies the GCC\(_+\). Furthermore, the Cartesian product of an edge and any \(n\)-cycle, a **prism graph**, satisfies the GCC\(_+\) as such a graph always contains a 4-cycle (see Figure 5.3).

![Figure 5.3](image)

The prism graph \(C_5 \square P_2\)

Using a method similar to the one employed in the proof of Proposition 4.15, we can extend Proposition 5.3. To do so, we first need a definition.
**Definition 5.5.** Let $G$ be a connected graph on $n$ vertices. A **non-disconnecting 2-sequence** is a non-disconnecting sequence $v_1, \ldots, v_n$ of $G$ so that for $3 \leq i \leq n$, $v_i$ is adjacent to at most two previous vertices in the sequence.

**Proposition 5.6.** Let $G$ be a graph on $n \geq 5$ vertices such that $\bar{G}$ is connected and after the deletion of any twin vertices in $G$, $\bar{G}$ does not contain $K_{2,3}$ as a subgraph. If there exists a non-disconnecting 2-sequence $v_1, \ldots, v_n$ of $\bar{G}$, then $G$ has an orthogonal representation in $\mathbb{R}^4$.

**Proof.** By Lemma 4.6, we may assume $G$ has no twins. By hypothesis, for $i \in \{3, \ldots, n\}$, $v_i$ cannot be adjacent to more than two vertices from $\{v_1, \ldots, v_{i-1}\}$ in $\bar{G}$. Our goal is to choose vectors $v_i$ in $\mathbb{R}^4$ so that $V = (v_1, \ldots, v_n)$ is an orthogonal representation of $G$.

Let $v_1$ be any nonzero vector in $\mathbb{R}^4$.

Now, just as in the proof of Proposition 4.15, suppose vectors $v_1, \ldots, v_{m-1}$ ($2 \leq m < n$) have been chosen to satisfy (H1) and (H2). Again, we show how to pick $v_m$ so that $v_1, \ldots, v_m$ also satisfies (H1) and (H2).

We must consider the same three cases presented in the proof of Proposition 4.15. The proofs for Case 1 ($v_m$ is adjacent to none of the previous vertices in the sequence) and Case 2 ($v_m$ is adjacent to exactly one of the previous vertices in the sequence) can be used without change. It remains to show that if $v_m$ is adjacent to exactly two previous vertices in the sequence, then $v_m$ can be chosen so that $v_1, \ldots, v_m$ satisfies (H1) and (H2). Let $v_p$ and $v_q$ ($p, q < m$) be vertices adjacent to $v_m$. We want to choose $v_m \in v_p^\perp \cap v_q^\perp$ so that $v_m \not\in v_i^\perp$ for $i \in (\{1, \ldots, m-1\} - \{p, q\})$ and $v_m \not\in \text{span}\{v_j, v_k\}$ for $j, k \in \{1, \ldots, m-1\}$. By the same argument given in the proof of Proposition 4.15,

$$\left(v_p^\perp \cap v_q^\perp\right) \cap v_i^\perp \subseteq v_p^\perp \cap v_q^\perp \quad \text{for any } i \neq p, q, i < m.$$  \hfill (5.1)
Next, assume for contradiction that \( v_p^\perp \cap v_q^\perp \subset \text{span}\{v_j, v_k\} \) for distinct \( j, k < m \). Because both the left- and right-hand side of this relation are 2-dimensional, it must be that \( v_p^\perp \cap v_q^\perp = \text{span}\{v_j, v_k\} \).

Therefore, \( v_j, v_k \in v_p^\perp \cap v_q^\perp \). This implies, by (H1), \( v_j \) and \( v_k \) are neighbors of \( v_p \) and \( v_q \). But then \( \overline{G} \) contains \( K_{2,3} \) as a subgraph (see Figure 5.2), contradicting our assumption that it did not. Hence,

\[
(v_p^\perp \cap v_q^\perp) \cap (\text{span}\{v_j, v_k\}) \not\subseteq v_p^\perp \cap v_q^\perp \text{ for each } j, k < m. \tag{5.2}
\]

Thus, by (5.1), (5.2), and Observation 4.3, it is possible to choose the desired \( v_m \).

Now proceed as in the proof of Proposition 4.15 to obtain \( V = (v_1, \ldots, v_n) \), the required orthogonal representation of \( G \) in \( \mathbb{R}^4 \). □

![Figure 5.4](image)

If \( v_p^\perp \cap v_q^\perp \subset \text{span}\{v_j, v_k\} \), then \( \overline{G} \) contains \( K_{2,3} \)

**Corollary 5.7.** Any graph satisfying the criteria of Proposition 5.6 satisfies the GCC\(_+\).

*Proof.* Again, this follows immediately from Theorem 3.23. □

Note that we made no assumptions about the minimum degree of \( G \) in Proposition 5.6. Indeed, we can manufacture examples of graphs which satisfy the conditions of Proposition 5.6 which have minimum degree as small as we like.
**Example 5.8.** An $n$-wheel is the graph $W_n = K_1 \lor C_n$. The vertex of a wheel which is adjacent to every other vertex is called the **hub** while any edge containing the hub is called a **spoke**. We will define a “partial wheel” to be a wheel with at least one, but not all of its spokes removed (see Figure 5.5). Partial wheels on at least 6 vertices do not contain $K_{2,3}$ as a subgraph (the only wheel which contains $K_{2,3}$ as a subgraph is $W_4$, the wheel on 5 vertices). Let $G$ be a graph on at least 5 vertices whose complement is a partial wheel. We will show $\text{mr}_4(G) \leq 4$. If $\overline{G}$ is a partial-$W_4$, then Theorem 4.19 applies and we are done. If $\overline{G}$ is a partial-$W_m$ where $m \geq 5$, then $\overline{G}$ has at least one vertex $v$ of degree 2. Let $v_1$ be the hub; its removal leaves an $m$-cycle. Choose either vertex adjacent to $v$ and let it be $v_2$. Next, let $v_3$ be the vertex adjacent to $v_2$ which is not $v$. Continue in this manner choosing vertices $v_4$, ..., $v_{m+1}$ along the cycle so that $v_i$ is adjacent to $v_{i-1}$ and so that $v_{m+1} = v$. Clearly $v_1, \ldots, v_{m+1}$ is a non-disconnecting 2-sequence of $\overline{G}$. By Proposition 5.6, $G$ has an orthogonal representation in $\mathbb{R}^4$. Furthermore, by Corollary 5.7, $G$ satisfies the GCC$_+$.  

![Figure 5.5](image)

The wheel $W_6$ (left) and two of its partial wheels (right)

**Example 5.9.** Any graph whose complement is the Cartesian product of two paths satisfies the criteria of Proposition 5.6. Again by Corollary 5.7, such graphs also satisfy the GCC$_+$. Figure 5.6 shows how to construct the necessary non-disconnecting 2-sequence of $P_d \square P_3$.  

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Our final result is a generalization of Proposition 4.15 which gives an upper bound for the minimum positive semidefinite rank for a particular class of graphs.

**Proposition 5.10.** Let $G$ be a graph on $n$ vertices such that $\overline{G}$ is connected. If $\Delta(\overline{G}) = d$ ($d < n$) and $\overline{G}$ is not $d$-regular, then $G$ has an orthogonal representation in $\mathbb{R}^{2d-2}$.

*Proof.* As usual, we assume $G$ has no twins. Since $\overline{G}$ is not $d$-regular, there is a vertex $v$ such that $\text{deg}_{\overline{G}}(v) < d$. Also, since $\overline{G}$ is connected, by Observation 4.13, there is a non-disconnecting sequence $v_1, \ldots, v_n$ such that $v_n = v$. We observe that since the sequence is non-disconnecting and we chose $\text{deg}_{\overline{G}}(v) < d$, for each $i > 1$, $v_i$ is adjacent to at most $d - 1$ vertices in $\{v_1, \ldots, v_{i-1}\}$. We now choose $v_1, \ldots, v_n$ in $\mathbb{R}^{2d-2}$ so that $V = (v_1, \ldots, v_n)$ is an orthogonal representation of $G$.

Let $v_1$ be any vector in $\mathbb{R}^{2d-2}$.

Suppose vectors $v_1, \ldots, v_{m-1}$ ($2 \leq m \leq n$) have been chosen in $\mathbb{R}^{2d-2}$ so that

(H3) $v_i \perp v_j$ if and only if $v_i$ and $v_j$ are neighbors in $\overline{G}$; and

(H4) every nonempty set of at most $d$ of these vectors is linearly independent.
We must pick \( v_m \) so that \( v_1, \ldots, v_m \) also satisfy (H3) and (H4). Suppose \( v_m \) is adjacent to \( k \) previous vertices in the sequence. We consider three cases:

**Case 1.** \( k = 0 \). In this case, choose \( v_m \) in \( \mathbb{R}^{2d-2} \) so that \( v_m \not\in v_i^\perp \) for \( i \in \{1, \ldots, m-1\} \) and \( v_m \) is not in the span of any collection of at most \( d-1 \) of the previously chosen vectors. By Observation 4.3, we see that this is certainly possible, and that this choice of \( v_m \) guarantees \( v_1, \ldots, v_m \) satisfies (H3) and (H4).

**Case 2.** \( 0 < k < d-1 \). Say that the \( k \) vertices are \( v_{i_1}, \ldots, v_{i_k} \). Choose \( v_m \in v_{i_1}^\perp \cap \cdots \cap v_{i_k}^\perp \) so that \( v_m \not\in v_i^\perp \) for \( i < m, \ i \neq i_j \) (\( j = 1, \ldots, k \)) and \( v_m \) is not in the span of any collection of at most \( d-1 \) of the previously chosen vectors. (H4) guarantees that

\[
 v_{i_1}^\perp \cap \cdots \cap v_{i_k}^\perp \not\subseteq v_i^\perp
\]

for any \( i < m, \ i \neq i_j \). Also, (H4) guarantees \( \dim(v_{i_1}^\perp \cap \cdots \cap v_{i_k}^\perp) = 2d-2-k > d-1 \). Hence, \( v_{i_1}^\perp \cap \cdots \cap v_{i_k}^\perp \) cannot be contained in the span of any collection of \( d-1 \) vectors. Thus, the desired \( v_m \) can be chosen.

**Case 3.** \( k = d-1 \), again, let the vertices be \( v_{i_1}, \ldots, v_{i_k} \). Choose \( v_m \) satisfying the same requirements as in Case 2. Again, (H4) guarantees we can avoid any unnecessary orthogonality for \( v_m \). This time, \( \dim(v_{i_1}^\perp \cap \cdots \cap v_{i_k}^\perp) = d-1 \), and so it may be that there is a collection of \( d-1 \) previously chosen vectors \( u_1, \ldots, u_{d-1} \) whose span equals \( \dim(v_{i_1}^\perp \cap \cdots \cap v_{i_k}^\perp) \). However, if this were the case, the vertices which correspond to \( u_1, \ldots, u_{d-1} \) would be twins in \( G \), contrary to our hypothesis that \( G \) has no twins. It now follows that we may pick the desired \( v_m \).
We have shown in all cases it is possible to choose \( \mathbf{v}_m \in \mathbb{R}^{2d-2} \) so that \( \mathbf{v}_1, \ldots, \mathbf{v}_m \) satisfies (H3) and (H4). Hence, we may select vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^{2d-2} \) satisfying (H3), i.e. \( \mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_n) \) is an orthogonal representation of \( G \). □

**Conclusion and Future Directions**

We have shown that the \( \delta^+ \)-conjecture holds for most graphs \( G \) satisfying \( \delta(G) = |G| - 4 \). The full proof in the case that \( \delta(G) = |G| - 4 \) was not forthcoming and is left for future research.

The next question is then, can we extend the methods used in Chapter 4 in a natural way for graphs with other values of \( \delta \)? In the case that \( \delta(G) = |G| - 4 \), in building an orthogonal representation, we required any collection of at most 3 vectors to be linearly independent. So for \( \delta(G) = |G| - 5 \), we would expect that, in order to build an orthogonal representation for this graph, we should require any collection of at most 4 vectors to be linearly independent. However, as the graph below illustrates, this constraint leads to difficulty.

![Figure 5.7](image)

A graph \( G \) with minimum degree \( \delta(G) = |G| - 5 \)
Suppose $G$ is the graph above and note that $|G| = 9$ and $\delta(G) = 4$. To satisfy the $\delta$-conjecture, we would need an orthogonal representation of $G$ in $\mathbb{R}^5$. Assume there exists such a representation, say $V = (v_1, v_2, \ldots, v_9)$, meeting the requirement that every collection of at most 4 of its vectors are linearly independent. It is easy to see that $\text{span}\{v_1, v_2, v_3\}$ and $\text{span}\{v_7, v_8, v_9\}$ are orthogonal to one another, and consequently, the rank of $V$ is, at minimum 6. This is impossible and so no such orthogonal representation exists.

We make an observation: the closed neighborhoods of the vertices $v_1$, $v_2$, and $v_3$, considered pairwise, differ by exactly one vertex. (Similar statements can be made for the three vertices $v_4$, $v_5$, and $v_6$ as well as the three vertices $v_7$, $v_8$, and $v_9$). We might consider such vertices quasi-twins. Recall now that in the case $\delta(G) = |G| - 4$, we were able to delete twin vertices and, because of this, we were able to construct representations in which every collection of at most 3 vectors was linearly independent. In a similar way, if it can be shown that a triple of quasi-twins contributes at most two to the minimum positive semidefinite rank of a graph, it may be possible to extend our method to graphs with $\delta(G) = |G| - 5$. We leave this for future research as well.
REFERENCES


VITA

Matthew Villanueva was born in Secaucus, NJ, to John and Regina Villanueva. He, along with his identical twin brother are the youngest of four children. He attended Our Lady of Mercy School and continued to Saint Peter’s Preparatory School. After graduating high school in 2007, he attended Rutgers, The State University of New Jersey, where he became interested in mathematics as well as mathematics education. He completed his Bachelor of Arts degree in mathematics in May 2011. Matthew then worked for two years as an instructor at Rutgers University before he eventually decided to return to school. In 2015, he accepted a graduate teaching assistantship at the University of Tennessee at Chattanooga. Matthew graduated with a Masters of Science degree in Mathematics in May 2017. He plans on continuing his education by pursuing a Ph.D. in mathematics.