INDUCED PATH NUMBER FOR THE COMPLEMENTARY PRISM OF A GRID GRAPH

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ABSTRACT

The induced path number $\rho(G)$ of a graph G is defined as the minimum number of subsets into which the vertex set of G can be partitioned so that each subset induces a path. A complementary prism of a graph G that we will refer to as $CP(G)$ is the graph formed from the disjoint union of G and $\overline{G}$ and adding the edges between the corresponding vertices of G and $\overline{G}$. These new edges are called prism edges. The graph $grid(n,m)$ is the Cartesian product of P_n with P_m. In this thesis we will give an overview of a selection of important results in determining $\rho(G)$ of various graphs, we will then provide proofs for determining the exact value of $\rho(CP(grid(n,m)))$ for specific values of n and m.

DEDICATION

Dedicated to my parents for always giving me support, to my children Florence and Claire for inspiring me, and to my wife Elizabeth who made it all possible.

ACKNOWLEDGEMENTS

This work would not have been possible without the help and advice of many people. I would like to thank the Department of Mathematics and the Graduate School at UTC for giving me the opportunity to continue my studies. I would like to thank the Thesis committee for their input. I would also like to thank Dr. Walters for his invaluable revisions.

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CHAPTER 1

INTRODUCTION

1.1 Preliminary Definitions

In this thesis, we follow the notation of G. Chartrand and L. Lesniak, *Graphs & Digraphs: Sixth Edition*, Chapman & Hall, London, 2016. [**3**]. All graphs referenced are simple graphs, where $V(G)$ denotes the vertex set of a graph G, and $E(G)$ denotes the edge set of a graph G. The *complement* $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and we write $H \subseteq G$. For a nonempy set S of $V(G)$, the subgraph $G[S]$ of G, *induced* by S, has S as its vertex set and two vertices u and v are adjacent in $G[S]$ if and only if u and v are adjacent in G, and a subgraph H of G is called an induced subgraph if there is a nonempty subset S of $V(G)$ such that $H = G[S]$. For an integer $n \geq 1$, the *path* P_n is a graph of order n and size $n-1$ whose vertices are labeled by $v_1, v_2, ..., v_n$ and whose edges are $v_i v_{i+1}$ for $i = 1, 2, ..., n-1$. The *subpath* for a path P is a path from v_k to v_l with edges $v_i v_{i+1}$ for $k \leq i \leq l-1$. The *induced path number* $\rho(G)$ of a graph G is defined as the minimum number of subsets into which the vertex set of G can be partitioned so that each subset induces a path. A *complementary prism* of a graph G, denoted as $CP(G)$, is the graph formed from the disjoint union of G and $\overline{G}$ and adding the edges between the corresponding vertices of G and $\overline{G}$. These added edges are called **prism edges**. The graph $grid(n,m)$ is the Cartesian product of P_n with P_m. We are interested in finding $\rho(CP(grid(n,m)))$ for various values of n, m.

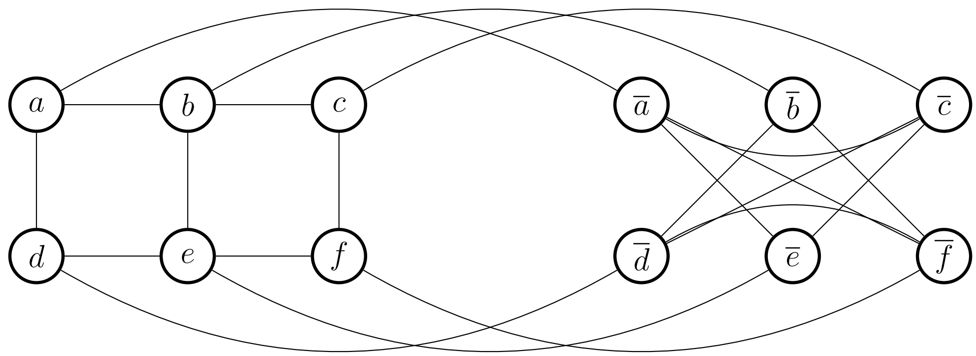

FIGURE 1.1

$CP(grid(2,3))$
All edges of $CP(grid(2,3))$ are shown

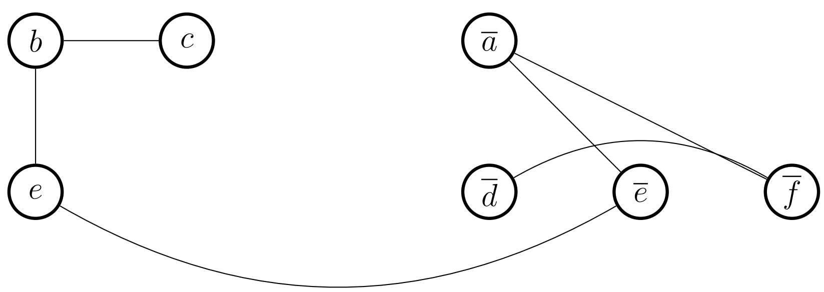

FIGURE 1.2

Path Induced by the vertices $\overline{a}, \overline{d}, \overline{e}, \overline{f}, e, b, c$

1.2 Previous Research

Several papers have been published where exact bounds of the induced path number for various graphs have been found [**1**], [**2**], [**4**], [**7**], [**8**]. I. Broere, G. S. Domke, E. Jonck, and L.R. Markus, [**1**] made the following observations. For a path on n vertices $\rho(P_n) = 1$ and for the cycle on n vertices $\rho(C_n) = 2$, and for the complete graph on n vertices $\rho(K_n) = \lceil \frac{n}{2} \rceil$. From G. Chartrand, J. Hashimi, M. Hossain, J. McCanna, and N. Sherwani, [**2**] we certainly have $\rho(grid(n, m)) = 1$ for $n = 1$ or $m = 1$.

THEOREM 1.1. [**2**] *If* $m \geq 2$*and* $n \geq 2$ *then* $\rho(grid(n,m)) = 2$.

As examples of cases where determining $\rho(G)$ is more difficult, Broere et al. [**1**] provide the following two theorems.

THEOREM 1.2. [**1**] *If* $n \geq m$ *, then*

$$\rho(K_m \times K_n) = \begin{cases} \frac{n}{2} & \textit{if } n \textit{ is even and } n > m, \\ \frac{n}{2} + \lceil \frac{m}{4} \rceil & \textit{if } n \textit{ is even and } n{=}m \\ \frac{n-1}{2} + \lceil \frac{m}{2} \rceil & \textit{if } n \textit{ is odd} \end{cases}$$

THEOREM 1.3. [**1**] *Suppose m and n are positive integers. Then* $\rho(C_m \times C_n) \leq 3$.

Broere et al. [**1**] also give the induced path number of the complement of certain classes of graphs, including the Cartesian product of complete graphs, and various combinations of Cartesian products of paths and cycles. This area of research is still very active. In an interesting case Broere et al. were unable to give a proof of the following conjecture:

CONJECTURE 1.4. [**1**] $\rho(\overline{K_m \times K_n}) = \lceil \frac{mn}{5} \rceil$ *except when:*

1. $m = 3$ *and* $n \in (3, 5, 8, 10, 11, 13, 14, 15, ...)$

2. m*,* n *are odd;* $m, n \geq 5$ *and* $mn \equiv 0 \bmod 5$

In Cases (1) and (2) the conjecture is

$\rho(\overline{K_m \times Kn}) = \lceil \frac{mn}{5} \rceil + 1$

Papers by J.H. Hattingh, O.A. Saleh, L C. van der Merwe and T. Walters [**7**], [**8**] established Nordhaus-Gaddum type results, which give bounds on the sum of the induced path number of a graph and its complement.

THEOREM 1.5. [**7**] *For any Graph G of order n,* $\sqrt{n} \leq \rho(G) + \rho(\overline{G}) \leq \lceil \frac{3n}{2} \rceil$.

Hattingh et al. [**8**] use the above results and methods to improve the bounds under specific conditions and give results for various graphs using $\Psi(G)$ to denote $\rho(G) + \rho(\overline{G})$, such as:

Theorem 1.6. [**8**] *Let G be a graph of order $n \geq 3$ such that neither G nor $\overline{G}$ is a complete graph. Then $\Psi(G) \leq \lceil \frac{3(n-1)}{2} \rceil$. Moreover, this bound is best possible.*

Corollary 1.7. [**8**] *The upper bound in Theorem 1.6 is achieved if and only if G or $\overline{G}$ is a complete graph.*

CHAPTER 2

INDUCED PATHS IN CP(grid(n,m))

2.1 Labeling and Definitions

We now wish to calculate $\rho(CP(grid(n,m)))$ for some values of n and m, where $grid(n,m)$ is $P_n \times P_m$. It is helpful to have a good labeling system for the vertices of the graph $grid(n,m)$, which we call the grid side of $CP(grid(n,m))$, and $\overline{grid(n,m)}$, which we call the complement side of $CP(grid(n,m))$, (refer to Figure 2.1). For any $grid(n,m)$ and $\overline{grid(n,m)}$ with $n,m \geq 1$ we label the vertices of $grid(n,m)$ as an ordered pair (i,j) for $1 \leq i \leq n$ and $1 \leq j \leq m$ and $i,j \in \mathbf{N}$. Each vertex (i,j) in $grid(n,m)$ is adjacent to the following vertices $\{(i-1,j),(i+1,j),(i,j-1),(i,j+1)\}$ if they exist. In $\overline{grid(n,m)}$ the vertices are labeled $\overline{(i,j)}$ and they correspond to the vertices of $grid(n,m)$. In $\overline{grid(n,m)}$ each vertex $\overline{(i,j)}$ is adjacent to every other vertex except the vertices $\{\overline{(i-1,j)}, \overline{(i+1,j)}, \overline{(i,j-1)}, \overline{(i,j+1)}\}$, if they exist. For any given vertex $\overline{(i,j)}$ we call these vertices non-adjacent-vertices of $\overline{(i,j)}$, and taken together for any vertex $\overline{(i,j)}$ we call it the non-adjacent-vertex-set of $\overline{(i,j)}$, $NAVS(\overline{(i,j)})$. If while finding $NAVS(\overline{(i,j)})$ there is a $\overline{(k,l)}$ where $k=0$ or $k>n,m$ or $l=0$ or $l>n,m$, then that vertex does not exist and so (k,l) has a smaller $NAVS(\overline{(i,j)})$.

We call any vertex $\overline{(i,j)}$ with exactly four vertices in $NAVS(\overline{(i,j)})$ an *interior vertex*, any vertex with exactly two vertices in $NAVS(\overline{(i,j)})$ a *corner vertex*, and any vertex with exactly three vertices in $NAVS(\overline{(i,j)})$ an *edge vertex*.

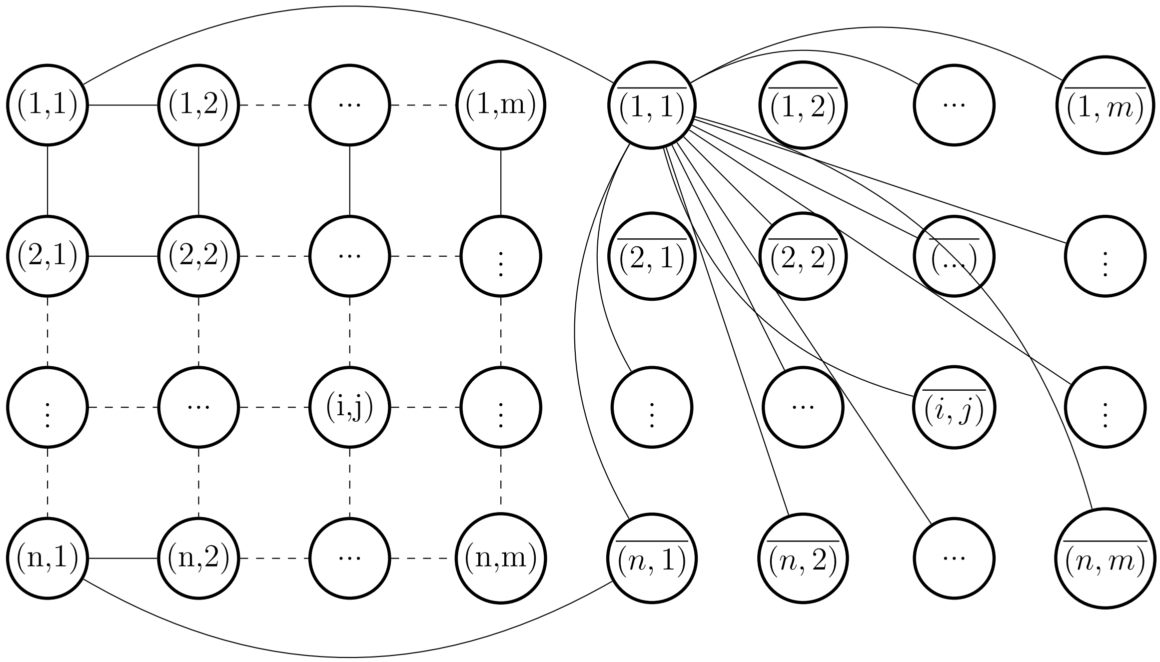

FIGURE 2.1

Some Edges and Vertices for $CP(grid(n,m))$
Not all edges are drawn

2.2 Induced Paths in Complementary Prisms

First we will provide two proofs for results on complementary prisms in general and then give characterizations of the induced paths in $CP(grid(n,m))$.

THEOREM 2.1. *Induced paths in complementary prisms have at most two prism edges.*

Proof: (refer to Figure 2.2) Let G and $\overline{G}$ be the two complementary graphs of the prism. Suppose to the contrary that there exists an induced path P in the prism that contains at least three or more prism edges. Then there exists $u_a, u_b, u_c \in G$ and $\overline{u_a}, \overline{u_b}, \overline{u_c} \in \overline{G}$ such that P has the edges $u_a\overline{u_a}, u_b\overline{u_b}, u_c\overline{u_c}$. Since G and $\overline{G}$ are complements, the following is true: either u_a is adjacent to u_b or $\overline{u_a}$ is adjacent to $\overline{u_b}$, and either u_a is adjacent to u_c or $\overline{u_a}$ is adjacent to $\overline{u_c}$, and either u_b is adjacent to u_c or $\overline{u_b}$ is adjacent to $\overline{u_c}$. Since P is an induced path, no vertex on the path is adjacent to three other vertices. Without loss of generality assume u_a is adjacent to u_b, then u_a is not adjacent to u_c and so $\overline{u_a}$ is adjacent to $\overline{u_c}$. Now u_b is not adjacent to u_c since u_b would be adjacent to three other vertices, and so $\overline{u_b}$ is adjacent to $\overline{u_c}$, this is a contradiction because then $\overline{u_c}$ is adjacent to three different vertices on the path. □

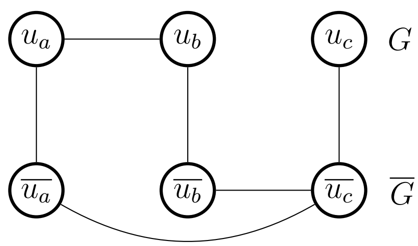

FIGURE 2.2

Induced Paths can not have three prism edges

THEOREM 2.2. *Suppose an induced path in a complementary prism contains the two prism edges $a\overline{a}$ and $b\overline{b}$, then either G or $\overline{G}$ has exactly two adjacent vertices of the induced path. Furthermore, the induced path has the form $\overline{u_1}...\overline{u_n}\overline{a}ab\overline{b}\overline{w_1}...\overline{w_k}$ or $v_1...v_na\overline{a}\overline{b}bx_1...x_k$ for $k, n \geq 0$.*

Proof: (refer to Figure 2.3) Suppose an induced path P in a complementary prism contains prism edges $a\overline{a}$ and $b\overline{b}$ with $a, b \in G$ and $\overline{a}, \overline{b} \in \overline{G}$. By **Theorem 2.1** there are only two prism edges. Clearly either a is adjacent to b or $\overline{a}$ is adjacent to $\overline{b}$. Suppose that a is adjacent to b in G. Then the edge ab is on the induced path, and $\overline{a}ab\overline{b}$ is a subpath of P. The whole path is of the form $a_la_{l-1}...a_1\overline{a}ab\overline{b}b_1...b_k$. Since there are no additional prism edges then all $a_la_{l-1}...a_1$ and $b_1...b_k$ are in $\overline{G}$ and the only edge in G is ab. A similar argument holds assuming $\overline{a}$ is adjacent to $\overline{b}$ for $\overline{G}$ and the other subpath $a\overline{a}\overline{b}b$. □

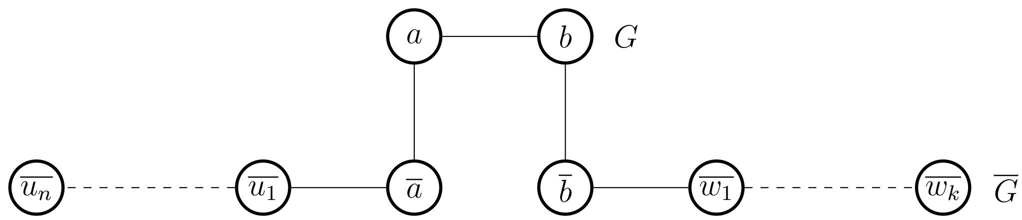

FIGURE 2.3

An induced path in a complementary prism with two prism edges

We now show that in $\overline{grid(n,m)}$, induced paths have at most four vertices. We will also describe how these four vertices can be arranged. The following lemmas provide us with information about the (NAVS) of vertices that will help in providing a proof for the nature of induced paths in $\overline{grid(n,m)}$.

LEMMA 2.3. *For any $\overline{(a,b)}, \overline{(c,d)} \in \overline{grid(n,m)}$; if $\overline{(a,b)} \in NAVS(\overline{(c,d)})$, then $NAVS(\overline{(a,b)}) \cap NAVS(\overline{(c,d)}) = \emptyset$ and $\overline{(c,d)} \in NAVS(\overline{(a,b)})$.*

Proof: (refer to Figure 2.4) Let $\overline{(a,b)} \in NAVS(\overline{(c,d)})$. The $NAVS(\overline{(c,d)})$ is $\{\overline{(c-1,d)}, \overline{(c+1,d)}, \overline{(c,d-1)}, \overline{(c,d+1)}\}$, so $\overline{(a,b)}$ is one of those vertices. Suppose that $\overline{(a,b)} = \overline{(c-1,d)}$, then the $NAVS(\overline{(a,b)})$ is $\{\overline{(c-2,d)}, \overline{(c,d)}, \overline{(c-1,d-1)}, \overline{(c-1,d+1)}\}$ which has no common members of the $NAVS(\overline{(c,d)})$. Repeat for the other three vertices in the $NAVS(\overline{(c,d)})$ and it is clear that $NAVS(\overline{(a,b)}) \cap NAVS(\overline{(c,d)}) = \emptyset$. $\square$

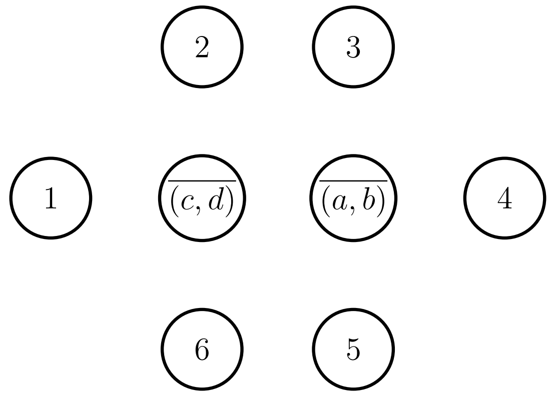

FIGURE 2.4

$$\overline{(a,b)} \in NAVS(\overline{(c,d)}) \text{ and } \overline{(c,d)} \in NAVS(\overline{(a,b)}) \;)$$
$$NAVS(\overline{(a,b)}) = \{ \; \overline{(c,d)}, 3, 4, 5\} \text{ and } NAVS(\overline{(c,d)}) = \{ \; \overline{(a,b)}, 1, 2, 6\}$$

LEMMA 2.4. *If there are two different vertices $\overline{a}, \overline{b} \in \overline{grid(n,m)}$ such that $NAVS(\overline{a}) \cap NAVS(\overline{b}) \neq \emptyset$, then the two vertices are adjacent in $\overline{grid(n,m)}$.*

Proof: (refer to Figure 2.5) Assume to the contrary that two vertices $\overline{a}$ and $\overline{b}$ share at least one member of their (NAVS) in common and $\overline{a}$ is not adjacent to $\overline{b}$. Then $\overline{a} \in NAVS(\overline{b})$ and $\overline{b} \in NAVS(\overline{a})$, and by **Lemma 2.2** $NAVS(\overline{a}) \cap NAVS(\overline{b}) = \emptyset$ contradicting our assumption. Therefore $\overline{a}$ is adjacent to $\overline{b}$. $\square$

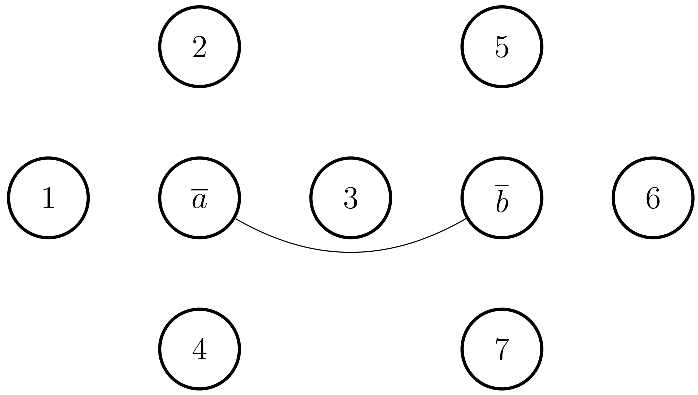

FIGURE 2.5

$NAVS(\overline{a}) \cap NAVS(\overline{b}) = \{3\}$
$\overline{a}$ is adjacent to $\overline{b}$ on the complement side

LEMMA 2.5. *If there are three vertices $\overline{a}, \overline{b}, \overline{c} \in \overline{grid(n,m)}$ such that $NAVS(\overline{a}) \cap NAVS(\overline{b}) \cap NAVS(\overline{c}) \neq \emptyset$, then these vertices induce a triangle.*

Proof: (refer to Figure 2.6) Apply the previous lemma three times and we see that $\overline{a}$ is adjacent to $\overline{b}$ and $\overline{c}$, and $\overline{b}$ is adjacent to $\overline{c}$. So they induce a triangle.$\square$

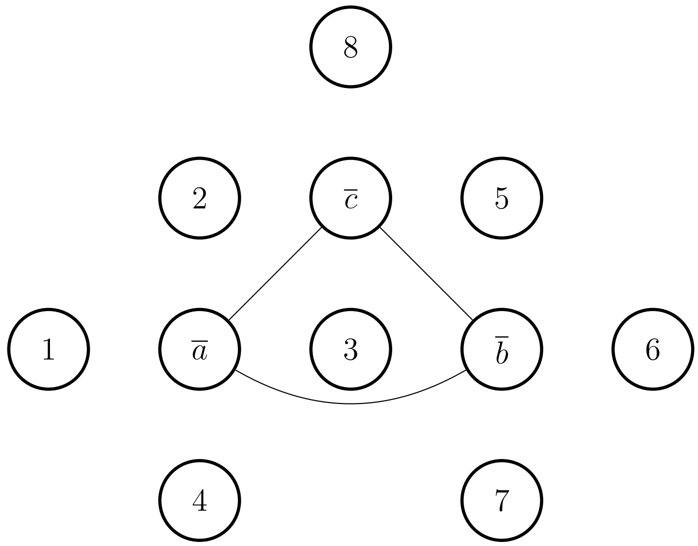

FIGURE 2.6

$NAVS(\overline{a}) \cap NAVS(\overline{b}) \cap NAVS(\overline{c}) = \{3\}$
$\overline{a}$, $\overline{b}$, and $\overline{c}$ induce a triangle

THEOREM 2.6. *Induced paths in $CP(grid(n,m))$ have at most four vertices in the $\overline{grid(n,m)}$ side.*

Proof: (refer to Figure 2.7) Using the notation established in **Section 2.1** the following vertices in $\overline{grid(4,4)}$ form an induced path; $\overline{(1,3)}, \overline{(1,1)}, \overline{(1,4)}, \overline{(1,2)}$. Therefore it is possible to have four vertices in an induced path in $\overline{grid(n,m)}$. Now we show that every induced path in $CP(grid(n,m))$ has at most four vertices in the complement side. Assume to the contrary that an induced path in $CP(grid(n,m))$ has the five vertices $\overline{a}$, $\overline{b}$, $\overline{c}$, $\overline{d}$, $\overline{e}$ in $\overline{grid(n,m)}$. Without loss of generality, assume that starting from one end of the induced path the first of these we encounter on the path is $\overline{a}$. Since this is an induced path, $\overline{a}$ is adjacent to at most one of $\overline{b}$, $\overline{c}$, $\overline{d}$, or $\overline{e}$. Without loss of generality, assume that $\overline{a}$ is not adjacent to $\overline{c}$, $\overline{d}$, nor $\overline{e}$. We have then that $\overline{a} \in NAVS(\overline{c})$, $\overline{a} \in NAVS(\overline{d})$, and $\overline{a} \in NAVS(\overline{e})$. Therefore, it follows that $NAVS(\overline{c}) \cap NAVS(\overline{d}) \cap NAVS(\overline{e}) \neq \emptyset$ and by **Lemma 2.4**, $\overline{c}$, $\overline{d}$, and $\overline{e}$ induce a triangle and it follows that five vertices do not induce a path, contradicting our assumption. Therefore any induced path in $CP(grid(n,m))$ has at most four vertices in $\overline{grid(n,m)}$. □

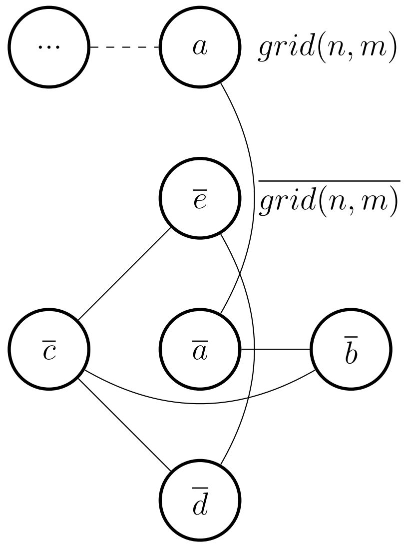

FIGURE 2.7

Five vertices in $\overline{grid(n,m)}$
$\overline{c}$, $\overline{d}$, and $\overline{e}$ induce a triangle.

THEOREM 2.7. *If an induced path in $CP(grid(n,m))$ has three or more consecutive vertices in $\overline{grid(n,m)}$, then that induced path has at most one prism edge.*

Proof: (refer to Figure 2.8) Suppose the we have an induced path P in $CP(grid(n,m))$ that contains two prism edges $a\overline{a}$ and $b\overline{b}$ and three consecutive vertices in $\overline{grid(n,m)}$. By **Theorem 2.2** P is of this form $\overline{u_1}...\overline{u_n}\overline{a}ab\overline{b}\overline{w_1}...\overline{w_k}$. By **Theorem 2.6** we can have at most four vertices in $\overline{grid(n,m)}$, and P is of the form $\overline{u_1 u_2 a}ab\overline{b}$ or $\overline{a}ab\overline{b}\overline{w_1 w_2}$. Suppose P is $\overline{u_1 u_2 a}ab\overline{b}$ then $NAVS(\overline{u_1}), NAVS(\overline{u_2})$, and $NAVS(\overline{a})$ all contain $\overline{b}$, and by **Lemma 2.4** they induce a triangle and we have contradiction. If P is $\overline{a}ab\overline{b}\overline{w_1 w_2}$ a similar contradiction follows. □

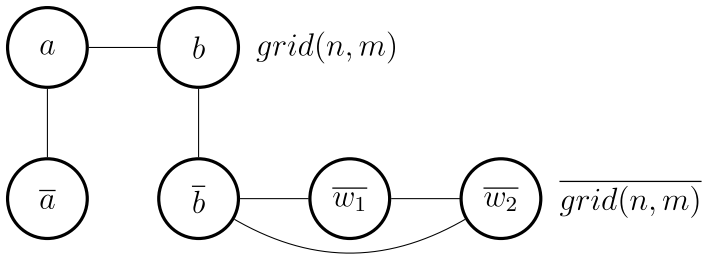

FIGURE 2.8

Two prism edges and three consecutive vertices in $\overline{grid(n,m)}$
$\overline{b}$, $\overline{w_1}$, and $\overline{w_2}$ induce a triangle.

THEOREM 2.8. *If ab is an edge in $grid(n,m)$ of $CP(grid(n,m))$, then any induced path in $CP(grid(n,m))$ containing the vertices $a, b, \overline{a}, \overline{b}$, has at most two more vertices, which can only be in $\overline{grid(n,m)}$, with one adjacent to $\overline{a}$ and the other adjacent to $\overline{b}$.*

Proof: (refer to Figure 2.9) Let P be an induced path containing the edge ab and the vertices $\overline{a}, \overline{b}$. Since P is induced, the prism edges $a\overline{a}$ and $b\overline{b}$ are in P. By **Theorem 2.1**, P does not have another prism edge, and any other vertices on P are in $\overline{grid(n,m)}$. By **Theorem 2.6** we only have two more vertices on P. So, P contains the subpath $\overline{a}ab\overline{b}$ and possibly two more vertices only in $\overline{grid(n,m)}$. Now suppose, without loss of generality, that the induced path is $\overline{a}, a, b, \overline{b}, \overline{c}, \overline{d}$. This contradicts **Theorem 2.7**, and similarly we can not have $\overline{c}, \overline{d}, \overline{a}, a, b, \overline{b}$. Thus, the induced path is of the form $\overline{c}, \overline{a}, a, b, \overline{b}, \overline{d}$. $\square$

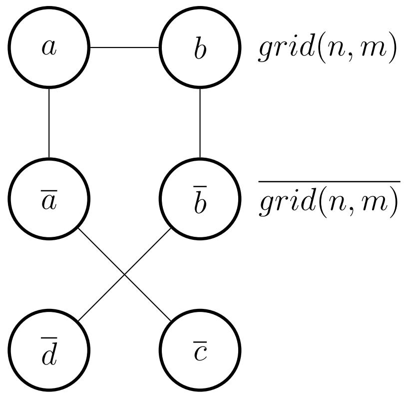

FIGURE 2.9

Induced path of the form $\overline{c}, \overline{a}, a, b, \overline{b}, \overline{d}$

2.3 Four Types of Induced Paths in $CP(grid(n,m))$ With Four Vertices In $\overline{grid(n,m)}$

Induced paths in $CP(grid(n,m))$ with four vertices in $\overline{grid(n,m)}$ follow four particular patterns that we will call **B-paths, L-paths, Z-paths, and U-paths**.

A **B-path** is an induced path with the following vertices; either $\overline{(a+1,b)}, \overline{(a+3,b)}, \overline{(a,b)}, \overline{(a+2,b)}$ (a vertical B-path) **or** $\overline{(a,b+1)}, \overline{(a,b+3)}, \overline{(a,b)}, \overline{(a,b+2)}$ (a horizontal B-path).

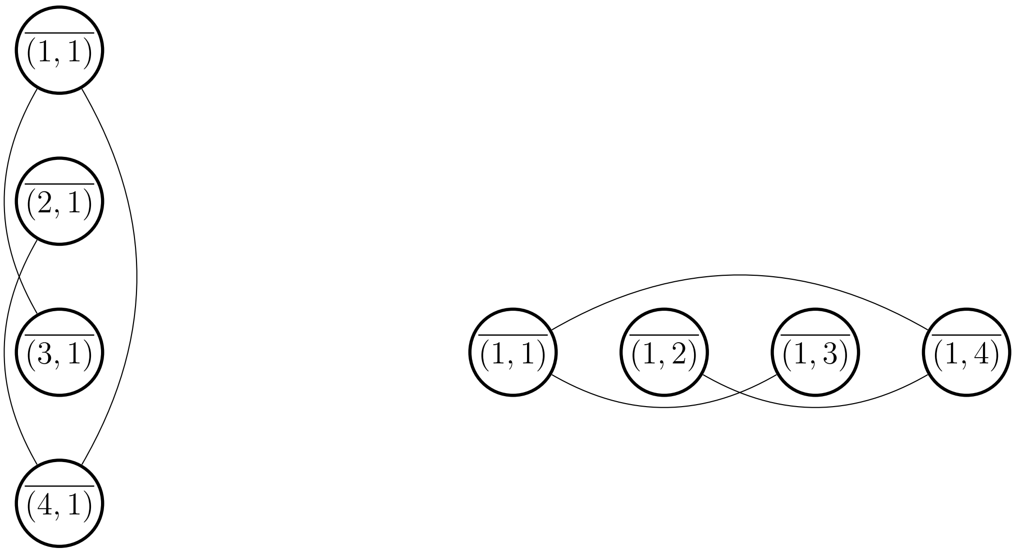

FIGURE 2.10

B-paths

An **L-path** is an induced path with the following vertices; either $\overline{(a+2,b)}, \overline{(a,b)}, \overline{(a+2,b\pm 1)}, \overline{(a+1,b)}$ **or** $\overline{(a,b)}, \overline{(a+2,b)}, \overline{(a,b\pm 1)}, \overline{(a+1,b)}$ **or** $\overline{(a,b+2)}, \overline{(a,b)}, \overline{(a\pm 1,b+2)}, \overline{(a,b+1)}$ **or** $\overline{(a,b)}, \overline{(a,b+2)}, \overline{(a\pm 1,b)}, \overline{(a,b+1)}$.

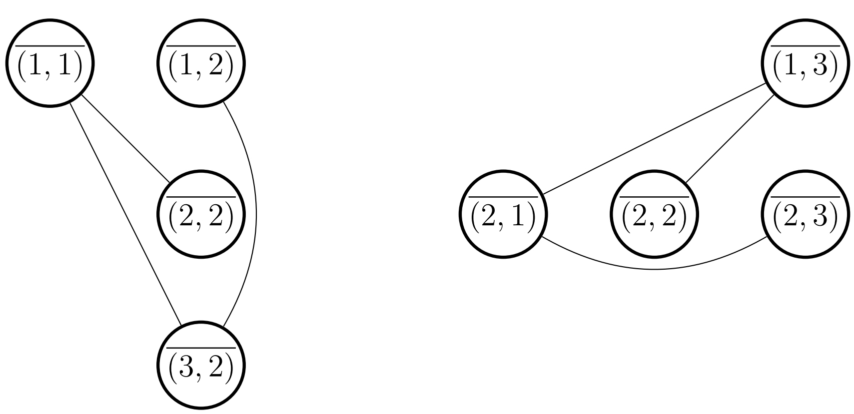

FIGURE 2.11

L-paths

A **Z-path** is an induced path with the following vertices; either $\overline{(a+1,b)}, \overline{(a+2,b+1)}, \overline{(a,b)}, \overline{(a+1,b+1)}$ **or** $\overline{(a+1,b-1)}, \overline{(a,b)}, \overline{(a+2,b-1)}, \overline{(a+1,b)}$ **or** $\overline{(a+1,b+1)}, \overline{(a,b)}, \overline{(a+1,b+2)}, \overline{(a,b+1)}$ **or** $\overline{(a-1,b+1)}, \overline{(a,b)}, \overline{(a-1,b+2)}, \overline{(a,b+1)}$.

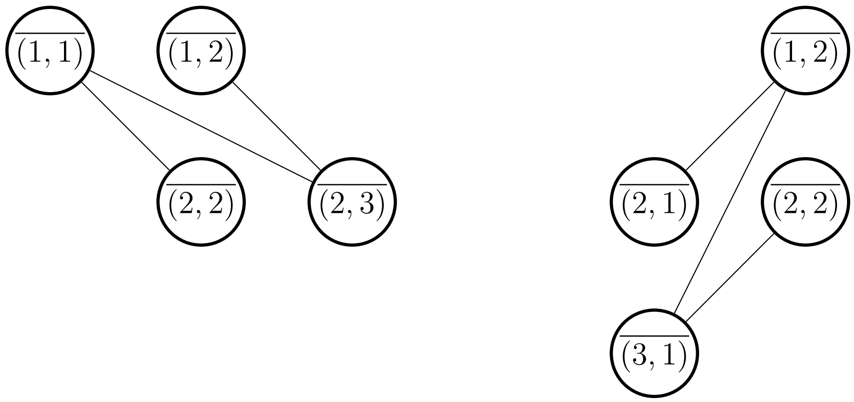

FIGURE 2.12

Z-paths

A **U-path** is of the form described in **Theorem 2.8** with the following vertices $\overline{(a+1,b+1)}, \overline{(a,b)}, (a,b), (a+1,b), \overline{(a+1,b)}, \overline{(a,b+1)}$ **or** $\overline{(a+1,b+1)}, \overline{(a,b)}, (a,b), (a,b+1), \overline{(a,b+1)}, \overline{(a+1,b)}$.

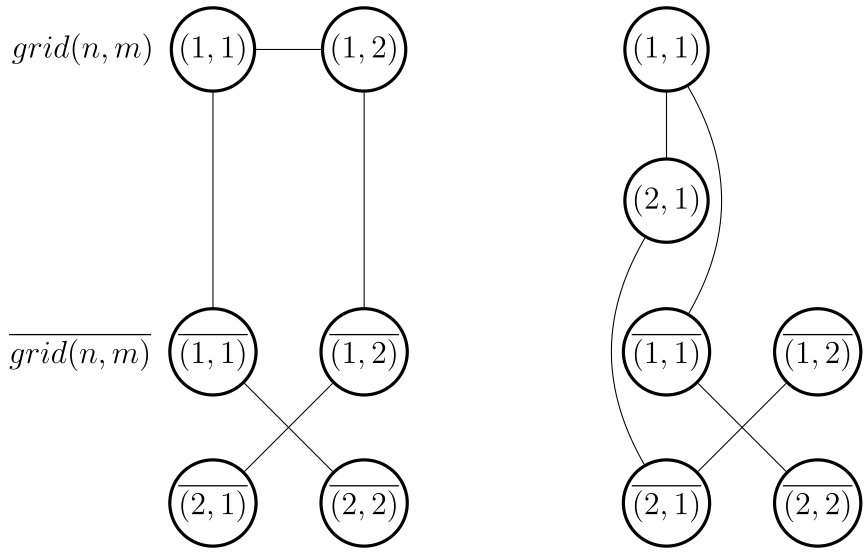

FIGURE 2.13

U-paths

THEOREM 2.9. *If an induced path in $CP(grid(n,m))$ has two prism edges and four vertices in $\overline{grid(n,m)}$, then the induced path is a U-path having exactly six vertices.*

Proof: Let P be an induced path in $CP(grid(n,m))$ with two prism edges and four vertices in $\overline{grid(n,m)}$. By **Theorem 2.8** the path must be of the form $\overline{u_1}\overline{u}uv\overline{v}\overline{v_1}$ since we have exactly four vertices in $\overline{grid(n,m)}$. Since P is an induced path, $\overline{u_1}$ is not adjacent to $\overline{v}$ nor $\overline{v_1}$ in $\overline{grid(n,m)}$, and $\overline{u}$ is not adjacent to $\overline{v}$ nor $\overline{v_1}$ in $\overline{grid(n,m)}$. Consider the specific case $u = (a,b)$ and $v = (a,b+1)$. Then $\overline{u} = \overline{(a,b)}$, $\overline{v} = \overline{(a,b+1)}$, $\overline{u_1} \in NAVS(\overline{v})$, and $\overline{v_1} \in NAVS(\overline{u})$. The vertices in $NAVS(\overline{v})$ are $\overline{(a-1,b+1)}$, $\overline{(a,b+2)}$ and $\overline{(a+1,b+1)}$, so $\overline{u_1}$ must be one of these. Similarly, the vertices in $NAVS(\overline{u})$ are $\overline{(a-1,b)}$, $\overline{(a,b-1)}$ and $\overline{(a+1,b)}$, so $\overline{v_1}$ must be one of these. Suppose that $\overline{u_1}$ is $\overline{(a,b+2)}$, then any possible $\overline{v_1}$ will be adjacent to $\overline{u_1}$ and we do not have a path. Similarly $\overline{v_1}$ can not be $\overline{(a,b-1)}$. We can not have $\overline{u_1} = \overline{(a-1,b+1)}$ and $\overline{v_1} = \overline{(a+1,b)}$ since these vertices are adjacent in $\overline{grid(n,m)}$. Similarly we can not have $\overline{u_1} = \overline{(a+1,b+1)}$ and $\overline{v_1} = \overline{(a-1,b)}$. Therefore we get the **U-path** $\overline{(a-1,b+1)(a,b)}(a,b)(a,b+1)\overline{(a,b+1)(a-1,b)}$ or the **U-path** $\overline{(a+1,b+1)(a,b)}(a,b)(a,b+1)\overline{(a+1,b)(a-1,b1)}$.

Theorem 2.10. *If an induced path in $CP(grid(n,m))$ has one prism edge and four vertices in $\overline{grid(n,m)}$, then the graph induced in $\overline{grid(n,m)}$ by these four vertices is a* **B-path, L-path, or Z-path**.

Proof: (refer to Figure 2.14) Using **Theorem 2.6** and without loss of generality the induced path has the form $\overline{a}\overline{b}\overline{c}\overline{d}dv_1...v_s$. Since this is a path we know the following $\overline{c} \in NAVS(\overline{a})$, $\overline{d} \in NAVS(\overline{a})$, $\overline{b} \in NAVS(\overline{d})$, and $\overline{a} \in NAVS(\overline{d})$. We also know that $\overline{b} \notin NAVS(\overline{a})$, $\overline{b} \notin NAVS(\overline{c})$, $\overline{c} \notin NAVS(\overline{b})$, and $\overline{c} \notin NAVS(\overline{d})$. Since the path is induced the subpath $\overline{a}\overline{b}\overline{c}\overline{d}$ is a path in $\overline{grid(n,m)}$. Starting with end vertex $\overline{a}$ the remaining vertices can only be in the following locations acording to their $NAVS$. If we are close to the edge of the $\overline{grid(n,m)}$ then some of these may not exist.

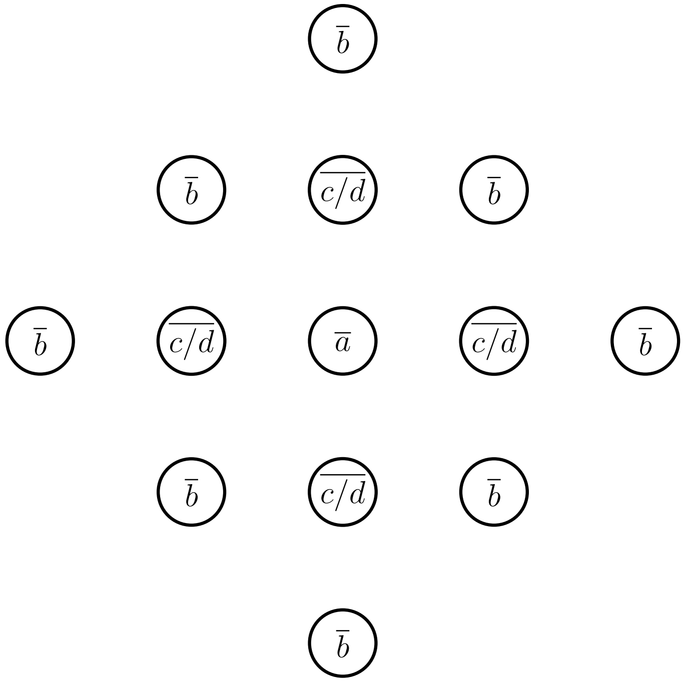

FIGURE 2.14

Possible locations of $\overline{b}, \overline{c}, \overline{d}$ relative to $\overline{a}$

Starting at $\overline{a}$ we go to any $\overline{b}$ in the image above. Once we choose a $\overline{b}$ we go to $\overline{c}$ such that $\overline{b} \notin NAVS(\overline{c})$. From here we go to any $\overline{d} \notin NAVS(\overline{c})$ and $\overline{d} \in NAVS(\overline{b})$. Following this algorithm we only produce **L-paths** (Figure 2.14), **B-paths** (Figure 2.15), or **Z-paths** (Figure 2.16). $\square$

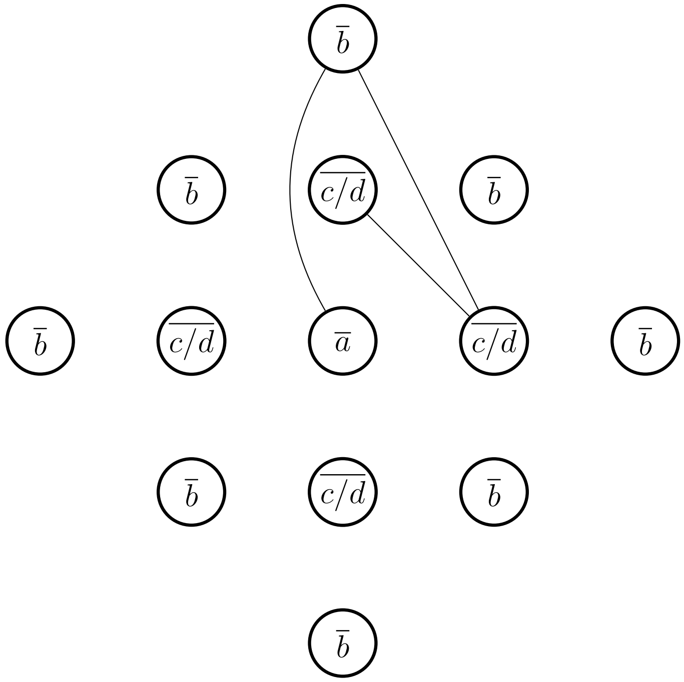

FIGURE 2.15

An **L-path** produced by the algorithm

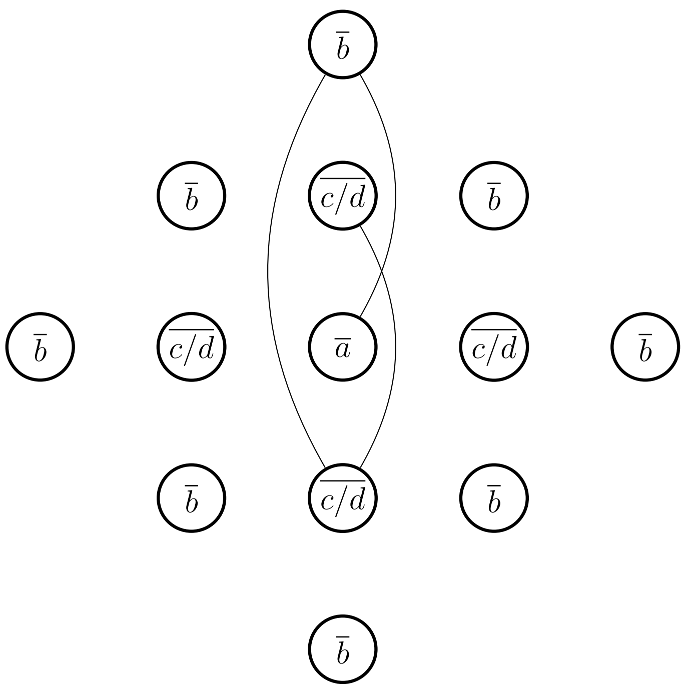

FIGURE 2.16

A **B-path** produced by the algorithm

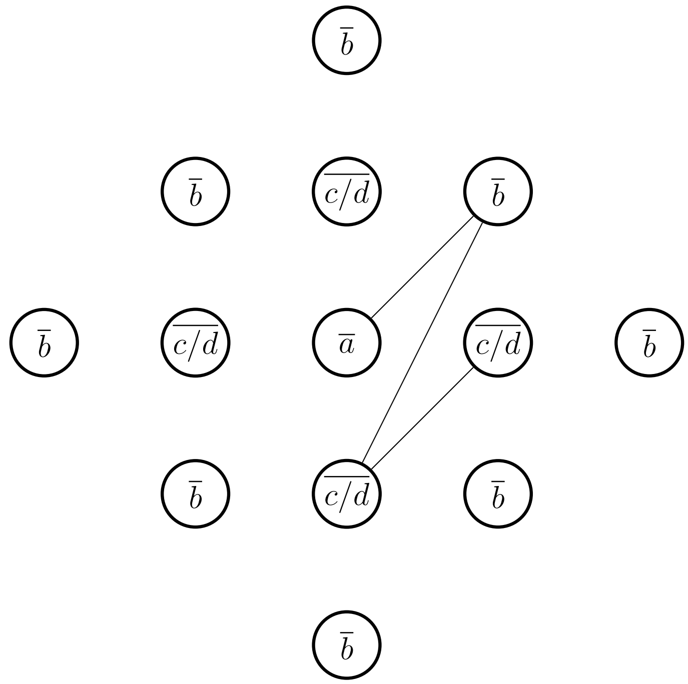

FIGURE 2.17

A **Z-path** produced by the algorithm

COROLLARY 2.11. *Any induced path in* $CP(grid(n,m))$ *that has four vertices in* $\overline{grid(n,m)}$ *is a* **U-path** *or has a* **B-path, L-path, or Z-path** *as a subpath.*

The proof of **Corollary 2.11** follows from **Theorem 2.9** and **Theorem 2.11** □

CHAPTER 3

INDUCED PATH NUMBER OF SPECIFIC $CP(grid(n,m))$

3.1 Specific Examples of $\rho(CP(grid(n,n)))$

Now, since there are nm vertices in $\overline{grid(n,m)}$ and each induced path in a cover for $CP(grid(n,m))$ contains at most four vertices from $\overline{grid(n,m)}$, we have $\rho(\overline{grid(n,m)}) \geq \lceil \frac{nm}{4} \rceil$. Using the above notation system we provide $\rho(CP(grid(n,n)))$ for $n = 1,2,3,4,5,6,7$.

For $n = 1$ it is trivial and $\rho(CP(grid(1,1))) = 1 = \lceil \frac{1}{4} \rceil = \lceil \frac{1^2}{4} \rceil$

For $n = 2$ $\rho(CP(grid(2,2))) \neq \lceil \frac{2^2}{4} \rceil$ because all the vertices would need to be in the induced path and the four vertices in $grid(2,2)$ clearly induce a cycle. It is easy to show however, that $\rho(CP(grid(2,2)) = 2 = \lceil \frac{2^2}{4} \rceil + 1$ with the following induced paths; $P_1 = \overline{(1,1)}, \overline{(2,2)}, (2,2), (1,2)$; $P_2 = \overline{(1,2)}, \overline{(2,1)}, (2,1), (1,1)$

For $n = 3$ the induced paths are the following: $P_1 = (2,2), (2,3), \overline{(2,3)}, \overline{(1,2)}, \overline{(3,3)}$; $P_2 = (3,3), (3,2), (3,1), (2,1), (1,1), (1,2), (1,3), \overline{(1,3)}, \overline{(2,2)}$; $P_3 = \overline{(3,1)}, \overline{(1,1)}, \overline{(3,2)}, \overline{(2,1)}$, and we have $\rho(CP(grid(3,3))) = 3 = \lceil \frac{3^2}{4} \rceil$

For $n = 4$ (refer to Figure 3.1) the induced paths are the following: $P_1 = \overline{(4,1)}, \overline{(2,1)}, \overline{(4,2)}, \overline{(3,1)}, (3,1), (3,2), (3,3), (3,4), (4,4)$; $P_2 = \overline{(3,3)}, \overline{(4,4)}, \overline{(2,3)}, \overline{(4,3)}, (4,3), (4,2), (4,1)$; $P_3 = \overline{(1,4)}$,

$\overline{(3,4)}, \overline{(1,3)}, \overline{(2,4)}, (2,4), (2,3), (2,2), (2,1), (1,1)$; $P_4 = \overline{(2,2)}, \overline{(1,1)}, \overline{(3,2)}, \overline{(1,2)}, (1,2), (1,3), (1,4)$, and we have $\rho(CP(grid(4,4))) = 4 = \lceil \frac{4^2}{4} \rceil$.

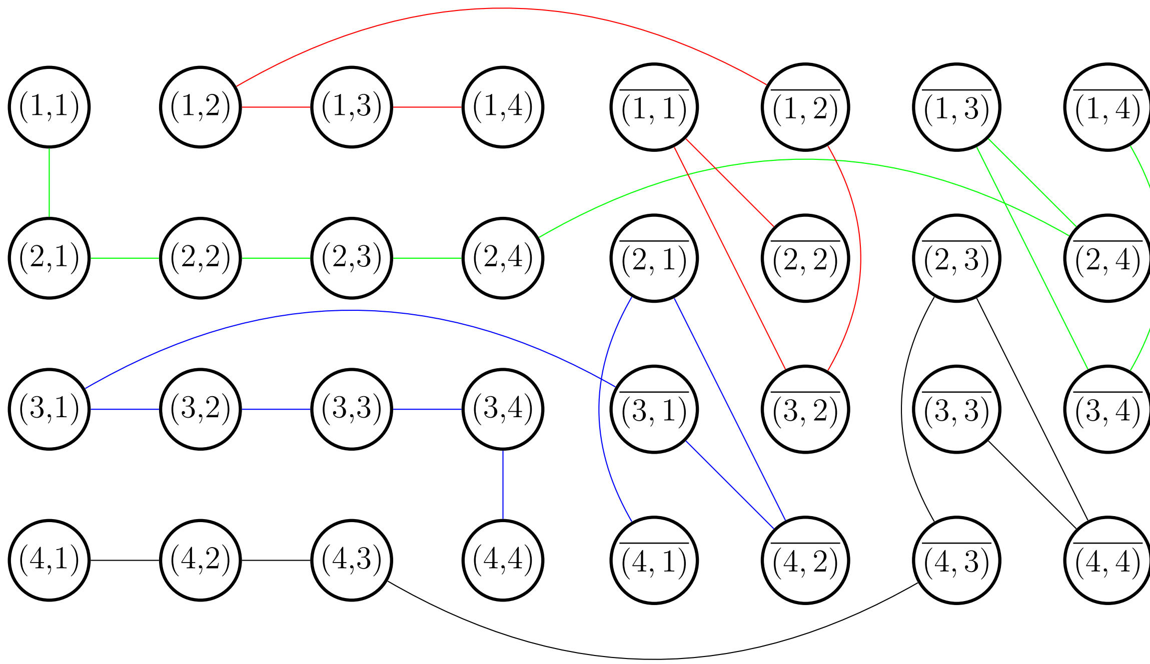

FIGURE 3.1

Induced paths for $CP(grid(4,4))$

For $n = 5$ the induced paths are the following: $P_1 = \overline{(3,3)}, (3,3), (2,3), (1,3), (1,4), (1,5), (2,5), (3,5), (4,5), (5,5), (5,4), (5,3), (5,2), (5,1), (4,1), (3,1), (2,1), (1,1)$; $P_2 = \overline{(2,5)}, \overline{(2,3)}, \overline{(1,5)}, \overline{(2,4)}, (2,4), (3,4), (4,4), (4,3), (4,2), (3,2), (2,2), (1,2)$; $P_3 = \overline{(3,5)}, \overline{(5,5)}, \overline{(3,4)}, \overline{(4,5)}$; $P_4 = \overline{(4,4)}, \overline{(4,2)}, \overline{(5,4)}, \overline{(4,3)}$; $P_5 = \overline{(5,1)}, \overline{(5,3)}, \overline{(4,1)}, \overline{(5,2)}$; $P_6 = \overline{(3,1)}, \overline{(1,1)}, \overline{(3,2)}, \overline{(2,1)}$; $P_7 = \overline{(1,2)}, \overline{(1,4)}, \overline{(2,2)}, \overline{(1,3)}$, and we have $\rho(CP(grid(5,5))) = 7 = \lceil \frac{5^2}{4} \rceil$

For $n = 6$ the induced paths are the following: $P_1 = \overline{(2,2)}, \overline{(1,1)}, \overline{(3,2)}, \overline{(1,2)}, (1,2), (1,3), (1,4), (1,5), (1,6)$; $P_2 = \overline{(1,6)}, \overline{(3,6)}, \overline{(1,5)}, \overline{(2,6)}, (2,6), (2,5), (2,4), (2,3), (2,2), (2,1), (1,1)$; $P_3 = \overline{(4,1)}, \overline{(2,1)}, \overline{(5,1)}, \overline{(3,1)}, (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,6), (5,6), (5,5), (5,4), (5,3), (5,2)$; $P_4 = \overline{(3,5)}, \overline{(5,5)}, \overline{(2,5)}, \overline{(4,5)}, (4,5), (4,4), (4,3), (4,2), (4,1), (5,1), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)$; $P_5 = \overline{(1,4)}, \overline{(3,4)}, \overline{(1,3)}, \overline{(2,4)}$; $P_6 = \overline{(3,3)}, \overline{(5,3)}, \overline{(2,3)}, \overline{(4,3)}$;

$P_7 = \overline{(6,2)}, \overline{(4,2)}, \overline{(6,1)}, \overline{(5,2)}$; $P_8 = \overline{(6,4)}, \overline{(4,4)}, \overline{(6,3)}, \overline{(5,4)}$; $P_9 = \overline{(6,6)}, \overline{(4,6)}, \overline{(6,5)}, \overline{(5,6)}$, and we have $\rho(CP(grid(6,6))) = 9 = \lceil \frac{6^2}{4} \rceil$

For $n = 7$ the induced paths are the following: $P_1 = \overline{(5,2)}, \overline{(7,2)}, \overline{(4,2)}, \overline{(6,2)}, (6,2), (6,1), (7,1)$; $P_2 = \overline{(6,1)}, \overline{(4,1)}, \overline{(7,1)}, \overline{(5,1)}, (5,1), (5,2), (5,3), (6,3), (7,3), (7,2)$; $P_3 = \overline{(5,6)}, \overline{(7,6)}, \overline{(4,6)}, \overline{(6,6)}, (6,6), (6,7), (7,7)$; $P_4 = \overline{(6,7)}, \overline{(4,7)}, \overline{(7,7)}, \overline{(5,7)}, (5,7), (5,6), (5,5), (6,5), (7,5), (7,6)$; $P_5 = \overline{(3,6)}, \overline{(2,7)}, \overline{(3,5)}, \overline{(3,7)}, (3,7), (4,7), (4,6), (4,5), (4,4), (5,4), (6,4), (7,4)$; $P_6 = \overline{(3,2)}, \overline{(3,4)}, \overline{(2,2)}, \overline{(3,3)}, (3,3), (4,3), (4,2), (4,1), (3,1), (2,1), (1,1), (1,2), (1,3)$; $P_7 = \overline{(1,7)}, (1,7), (2,7), (2,6), (3,6), (3,5), (3,4)$; $P_8 = \overline{(2,4)}, \overline{(2,6)}, \overline{(2,3)}, \overline{(2,5)}, (2,5), (1,5), (1,6)$; $P_9 = \overline{(1,5)}, \overline{(1,3)}, \overline{(1,6)}, \overline{(1,4)}, (1,4), (2,4), (2,3), (2,2), (3,2)$; $P_{10} = \overline{(1,1)}, \overline{(3,1)}, \overline{(1,2)}, \overline{(2,1)})$; $P_{11} = \overline{(5,3)}, \overline{(7,3)}, \overline{(4,3)}, \overline{(6,3)})$; $P_{12} = \overline{(5,5)}, \overline{(7,5)}, \overline{(4,5)}, \overline{(6,5)})$; $P_{13} = \overline{(5,4)}, \overline{(7,4)}, \overline{(4,4)}, \overline{(6,4)})$, and we have $\rho(CP(grid(7,7))) = 13 = \lceil \frac{7^2}{4} \rceil$

3.2 Proof of $\rho(CP(grid(n,m)))$ for $n, m \geq 4$

THEOREM 3.1. *For $n, m \geq 4$, $\rho(CP(grid(n,m))) = \lceil \frac{nm}{4} \rceil$*

Proof: We proceed by examining the 16 possible modulo cases.

Case 1: $n \equiv 0 \bmod 4$ and $m \equiv 0 \bmod 4$

It has already been shown that to cover all the vertices for $CP(grid(4,4))$ we need 4 induced paths. Now suppose $n = 4k$ and $m = 4l$ for $k, l \geq 1$. There are $32kl$ vertices in the prism. The rows and columns will increase by multiples of 4 in both $grid(n,m)$ and $\overline{grid(n,m)}$. We cover all the vertices with copies of the induced paths provided for $CP(grid(4,4))$, each copy has 4 induced paths and we will need l columns and k rows of the copies. Therefore we will need $4kl = \lceil \frac{16kl}{4} \rceil = \lceil \frac{nm}{4} \rceil$ induced paths to cover all the vertices.

Case 2: (efer to Figures 3.2 and 3.3) $n \equiv 0 \bmod 4$ and $m \equiv 1 \bmod 4$

Suppose $n = 4k$ and $m = 4l + 1$ for $k, l \geq 1$, and there are a total of $32kl + 8k$ vertices to cover. We cover $32kl$ vertices with copies of the induced paths used to cover $CP(grid(4,4))$ starting in the upper lefthand corner for a total of $4kl$ paths. What remains is the farthest right column of vertices in both $grid(n,m)$ and $\overline{grid(n,m)}$. To cover the remaining vertices in $grid(n,m)$ we extend the path corresponding to P_4 from the $CP(grid(4,4))$ case. The vertex $(1, m-1)$ will be the end of a path, continue it to $(1,m), (2,m), ...(n,m)$ and all the vertices in $grid(n,m)$ will be covered without a new path. Now we cover the remaining $4k$ vertices in $\overline{grid(n,m)}$ with k vertical **B-paths** for a total of $4kl + k = \lceil \frac{16kl+4k}{4} \rceil = \lceil \frac{4k(4l+1)}{4} \rceil = \lceil \frac{nm}{4} \rceil$ induced paths.

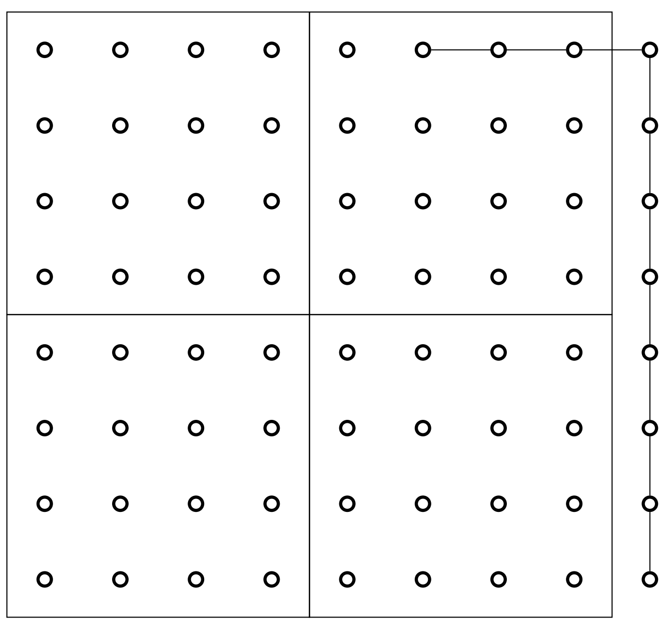

FIGURE 3.2

Example for $grid(8,9)$
4 copies of induced paths for $grid(4,4)$, with extended path

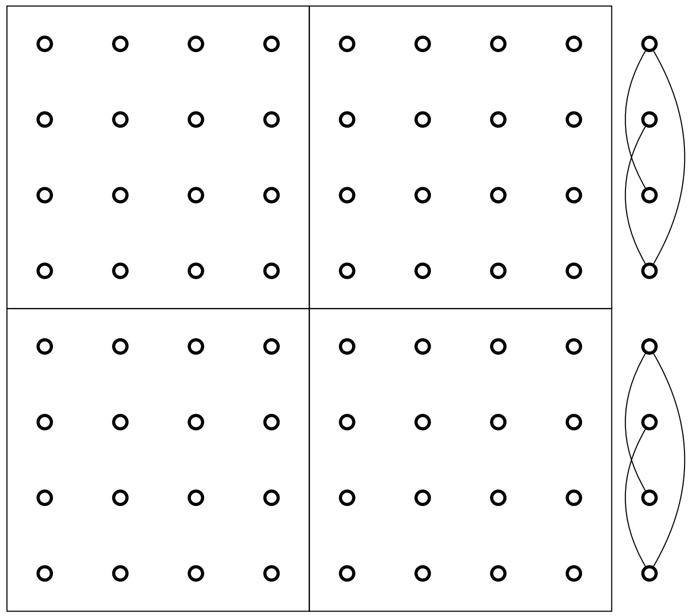

FIGURE 3.3

Example for $\overline{grid(8,9)}$
4 copies of induced paths for $\overline{grid(4,4)}$ and 2 **B-paths**

Case 3: (refer to Figures 3.4 and 3.5) $n \equiv 0 \bmod 4$ and $m \equiv 2 \bmod 4$

Suppose $n = 4k$ and $m = 4l + 2$ for $k, l \geq 1$, then the total number of vertices to cover is $32kl + 16k$. As before, we cover $32kl$ vertices with copies of the induced paths used to cover $CP(grid(4,4))$ starting in the upper lefthand corner for a total of $4kl$ induced paths. What remains are the farthest right 2 columns in both $grid(n,m)$ and $\overline{grid(n,m)}$. For the remaining vertices in $grid(n,m)$ we extend two paths corresponding to P_4 and P_1 in the original cover of $CP(grid(4,4))$. From the vertex $(1, m-2)$ continue to $(1, m-1), (2, m-1), ..., (n-1, m-1)$ and from the vertex $(n, m-2)$ continue to $(n, m-1), (n, m), (n-1, m), ...(1, m)$. Now we cover the $8k$ vertices in $\overline{grid(n,m)}$ with $2k$ vertical **B-paths** for a total of $4kl + 2k = \lceil \frac{16kl+8k}{4} \rceil = \lceil \frac{4k(4l+2)}{4} \rceil = \lceil \frac{nm}{4} \rceil$ induced paths.

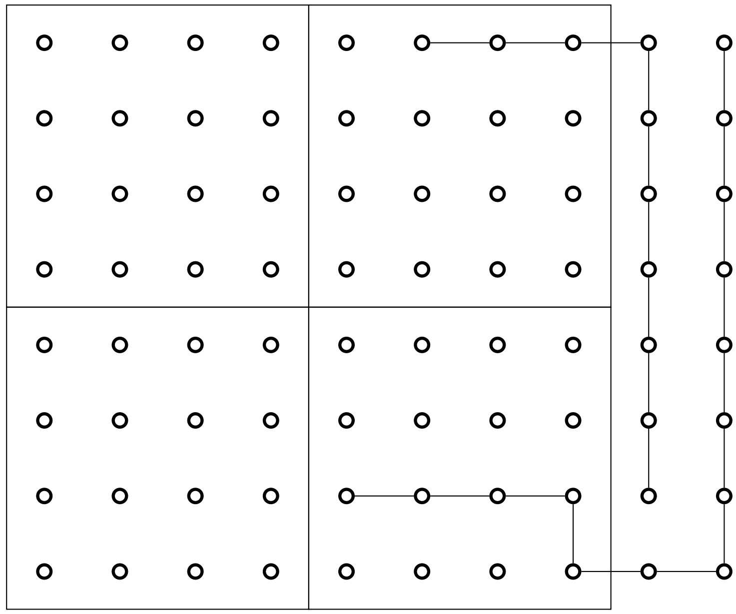

Figure 3.4

Example for $grid(8,10)$
4 copies of induced paths for $grid(4,4)$, with extended paths

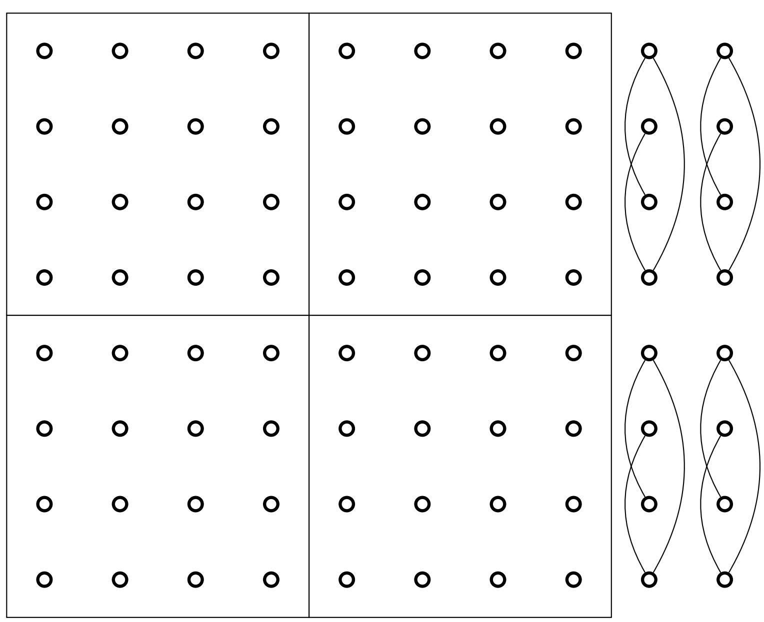

Figure 3.5

Example for $\overline{grid(8,10)}$
4 copies of induced paths for $\overline{grid(4,4)}$ and 4 **B-paths**

Case 4: (refer to Figures 3.6 and 3.7) $n \equiv 0 \bmod 4$ and $m \equiv 3 \bmod 4$

Suppose $n = 4k$ and $m = 4l + 3$ for $k, l \geq 1$, then the total number of vertices to cover is $32kl + 24k$. As before we cover $32kl$ vertices with copies of the induced paths used to cover $CP(grid(4,4))$ but this time starting 1 column to the right in the upper lefthand corner for a total of $4kl$ induced paths. What remains are the farthest right 2 columns and farthest left column in both $grid(n,m)$ and $\overline{grid(n,m)}$. To cover the remaining vertices in $grid(n,m)$ we continue the paths corresponding to P_4, P_1, and P_2 in the original cover of $CP(grid(4,4))$. From vertex $(1, m-2)$ continue to $(1, m-1), (2, m-1), ..., (n-1, m-1)$. From vertex $(n, m-2)$ continue to $(n, m-1), (n, m), (n-1, m), ..., (1, m)$. From vertex $(n, 2)$ continue to $(n, 1), (n-1, 1), ..., (1, 1)$. Now we cover the $12k$ vertices in $\overline{grid(n,m)}$ with $3k$ vertical **B-paths** for a total of $4kl + 3k = \lceil \frac{16kl+12k}{4} \rceil = \lceil \frac{4k(4l+3)}{4} \rceil = \lceil \frac{nm}{4} \rceil$ induced paths.

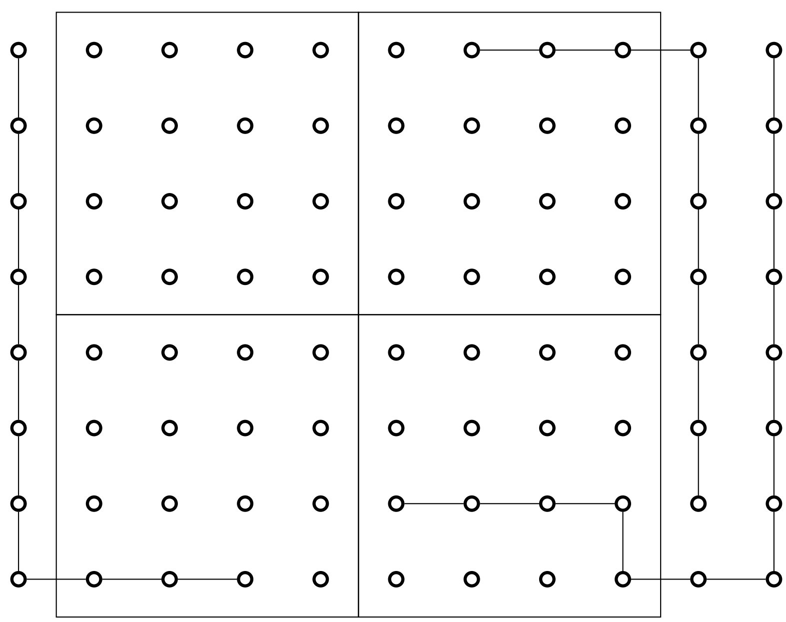

Figure 3.6

Example for $grid(8,11)$
4 copies of induced paths for $grid(4,4)$, with extended paths

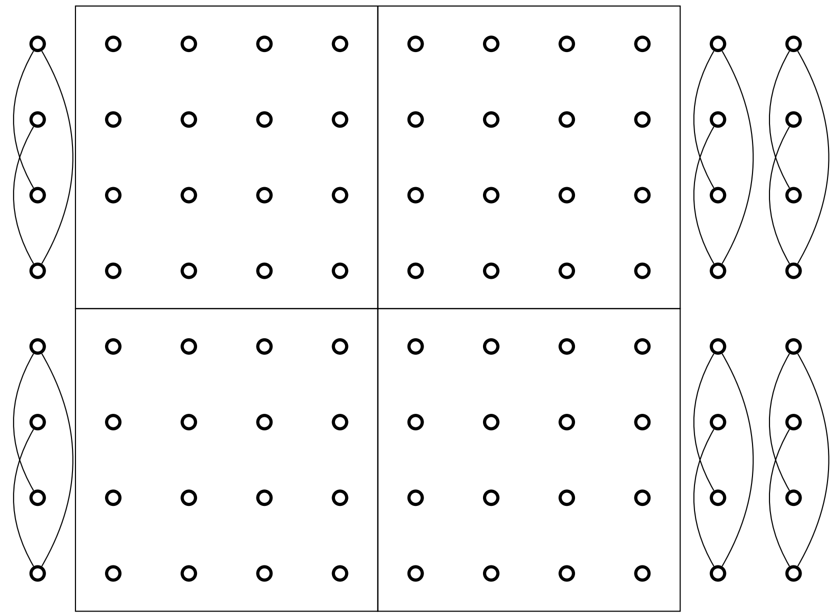

FIGURE 3.7

Example for $\overline{grid(8,11)}$
4 copies of induced paths for $\overline{grid(4,4)}$ and 6 **B-paths**

Case 5: $n \equiv 1 \bmod 4$ and $m \equiv 0 \bmod 4$

This is isomorphic to **Case 2**

Case 6: (refer to Figures 3.8 and 3.9) $n \equiv 1 \bmod 4$ and $m \equiv 1 \bmod 4$

Suppose $n = 4k + 1$ and $m = 4l + 1$ for $k, l \geq 1$, so the total number of vertices to cover is $32kl + 8k + 8l + 2$. As before we cover $32kl$ vertices with copies of the induced paths used to cover $CP(grid(4,4))$ starting in the upper lefthand corner for a total of $4kl$ induced paths. What remains are the farthest right column and bottom row in both $grid(n,m)$ and $\overline{grid(n,m)}$. To cover the remaining vertices in $grid(n,m)$ we extend a path corresponding to P_4 in the original cover of $CP(grid(4,4))$. From the vertex $(1, m-1)$ we continue to $(1,m), (2,m), ..., (n,m), (n,m-1), (n,m-2)..., (n,1)$. For the remaining vertices in $\overline{grid(n,m)})$ we cover the farthest right column with k vertical **B-paths** starting at the top, with 1 vertex left at the bottom right corner. We cover the bottom row starting on the left with l horizontal **B-paths** and the same vertex leftover that will be a singleton vertex path. This brings the total number of induced paths needed to $4kl + l + k + 1 = \lceil \frac{16kl+4l+4k+1}{4} \rceil = \lceil \frac{(4k+1)(4l+1)}{4} \rceil = \lceil \frac{nm}{4} \rceil$.

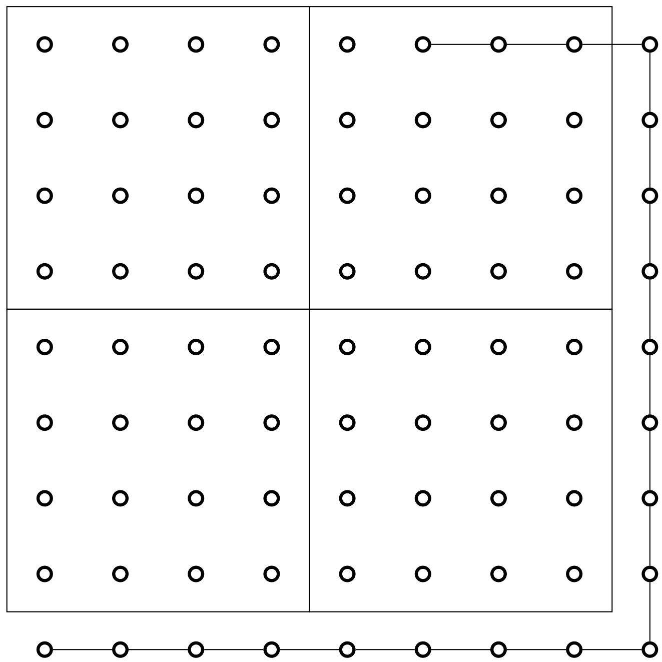

FIGURE 3.8

Example for $grid(9, 9)$
4 copies of induced paths for $grid(4, 4)$, with extended paths

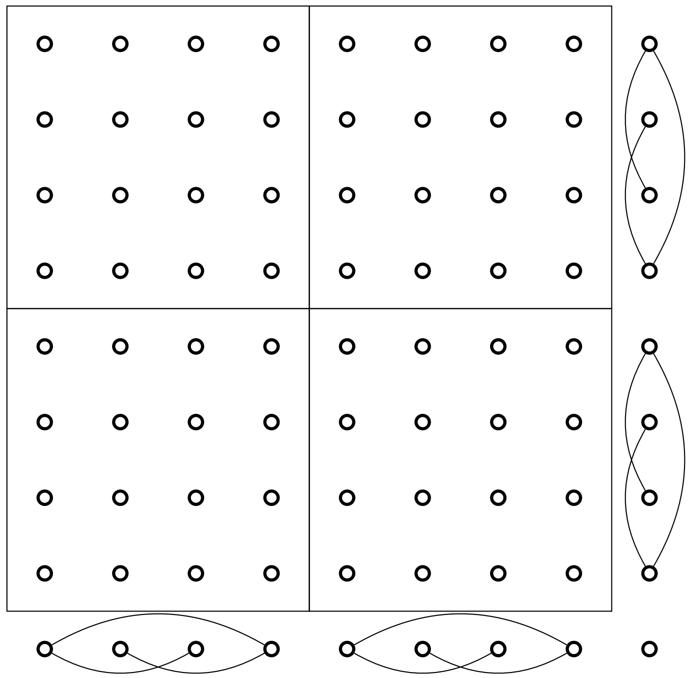

FIGURE 3.9

Example for $\overline{grid(9,9)}$
4 copies of induced paths for $\overline{grid(4,4)}$, 4 **B-paths**, and one vertex

Case 7: (refer to Figures 3.10 and 3.11) $n \equiv 1 \bmod 4$ and $m \equiv 2 \bmod 4$

Suppose $n = 4k+1$ and $m = 4l+2$ for $k, l \geq 1$, then the total number of vertices to cover is $32kl + 16k + 8l + 4$. As before we cover $32kl$ vertices with copies of the induced paths used to cover $CP(grid(4,4))$ starting at the top and 1 column to the right for a total of $4kl$ induced paths. What remains are the farthest right column, farthest left column, and bottom row in both $grid(n,m)$ and $\overline{grid(n,m)}$. To cover the remaining vertices in $grid(n,m)$ we extend a path corresponding to P_4 and P_2 in the original cover of $CP(grid(4,4))$. From vertex $(1, m-1)$ we continue to $(1,m), (2,m), ..., (n,m), (n,m-1), ..., (n,1)$ and from the vertex $(n-1,2)$ we continue to $(n-1,1), (n-2,1), ...(1,1)$. For the remaining vertices in $\overline{grid(n,m))}$ we start on the upper left corner for the left column and use k vertical **B-paths**, with 1 vertex leftover in the bottom left. Starting at the bottom left we use l horizontal **B-paths** to cover the bottom row except for 2 vertices. Now we use $k-1$ vertical bars starting at the upper right corner to cover that column except for the 5 bottom vertices in the right column and the 2 right vertices in the bottom row. We cover these with two induced paths $P_a = \overline{(n,m-1)}, \overline{(n-2,m)}, \overline{(n,m)}$ and $P_b = \overline{(n-4,m)}, \overline{(n-1,m)}, \overline{(n-3,m)}$ This comes out to a total of $4kl+k+l+(k-1)+2 = 4kl+2k+l+1 = \lceil \frac{16kl+8k+4l+2}{4} \rceil = \lceil \frac{(4k+1)(4l+2)}{4} \rceil = \lceil \frac{nm}{4} \rceil$ induced paths.

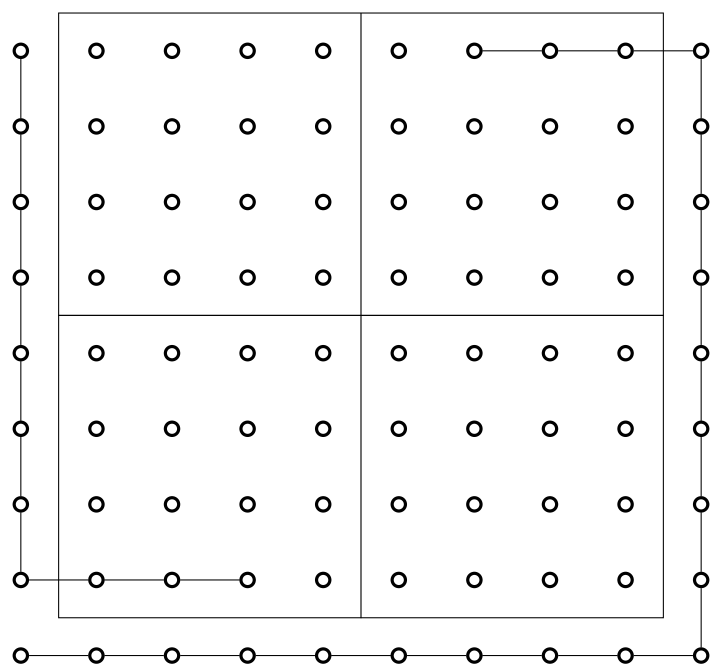

FIGURE 3.10

Example for $grid(9,10)$
4 copies of induced paths for $grid(4,4)$, with extended paths

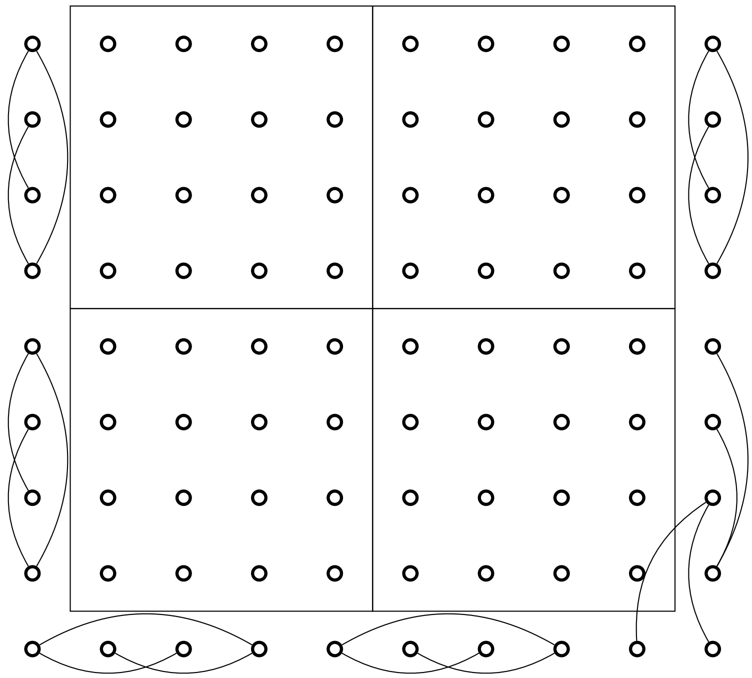

FIGURE 3.11

Example for $\overline{grid(9,10)}$
4 copies of induced paths for $\overline{grid(4,4)}$, 5 **B-paths**, and two paths of three vertices

Case 8: (refer to Figures 3.12 and 3.13) $n \equiv 1 \bmod 4$ and $m \equiv 3 \bmod 4$

Suppose $n = 4k+1$ and $m = 4l+3$ for $k, l \geq 1$, then the total number of vertices to cover is $32kl+24k+8l+6$. As before we cover $32kl$ vertices with copies of the induced paths used to cover $CP(grid(4,4))$ leaving the leftmost column, the 2 rightmost columns and the top row for a total of $4kl$ induced paths. To cover the remaining vertices in $grid(n,m)$ we extend the paths corresponding to P_4 and P_2 in the original cover of $CP(grid(4,4))$. From vertex $(n,2)$ continue to $(n,1), (n-1,1), ..., (1,1), (1,2), ...(1,m), (2,m), ...(n,m)$. From vertex $(2,m-2)$ continue to $(2,m-1), (3,m-1), ..., (n,m-1)$. For the remaining vertices in $\overline{grid(n,m))}$ we start on the lower left corner for the left column and use k vertical **B-paths**, with 1 leftover in the upper left corner. Starting at that vertex use l horizontal **B-paths** leaving the last 3 vertices on the right side of the top row. Now use 1 **L-path** covering those 3 vertices and 2 vertices of the farthest right column. Now use 1 more **L-path** starting in the second from last column to cover 3 of those vertices and the vertex $\overline{(4,m)}$. Now we use the induced path $P_a = \overline{(n,m)}, \overline{(3,m)}, \overline{(n,m-1)}$ and the remaining vertices are covered with $2(k-1)$ vertical **B-path**. This comes out to a total of $4kl + k + l + 1 + 1 + 1 + 2(k-1) = 4kl + 3k + l + 1 = \lceil \frac{16kl+12k+4l+3}{4} \rceil = \lceil \frac{(4k+1)(4l+3)}{4} \rceil = \lceil \frac{nm}{4} \rceil$ induced paths.

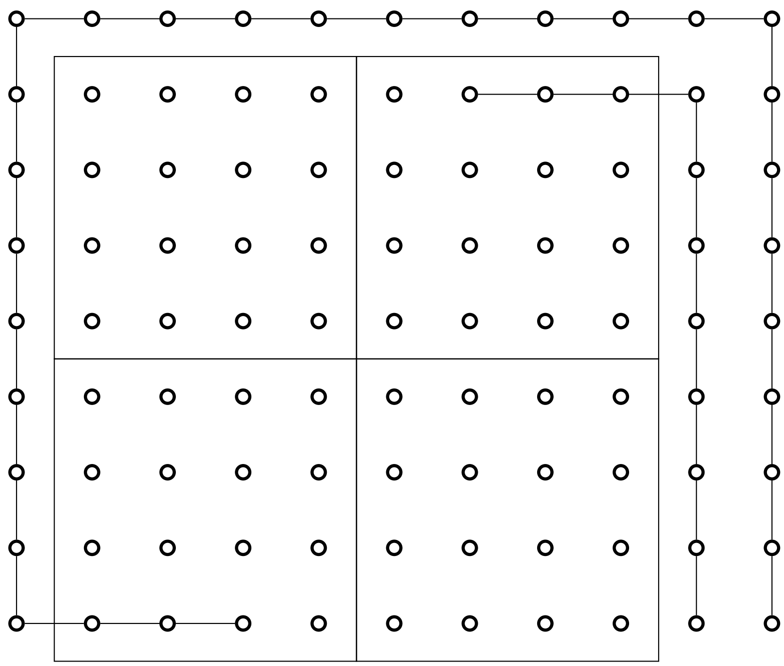

FIGURE 3.12

Example for $grid(9,11)$
4 copies of induced paths for $grid(4,4)$, with extended paths

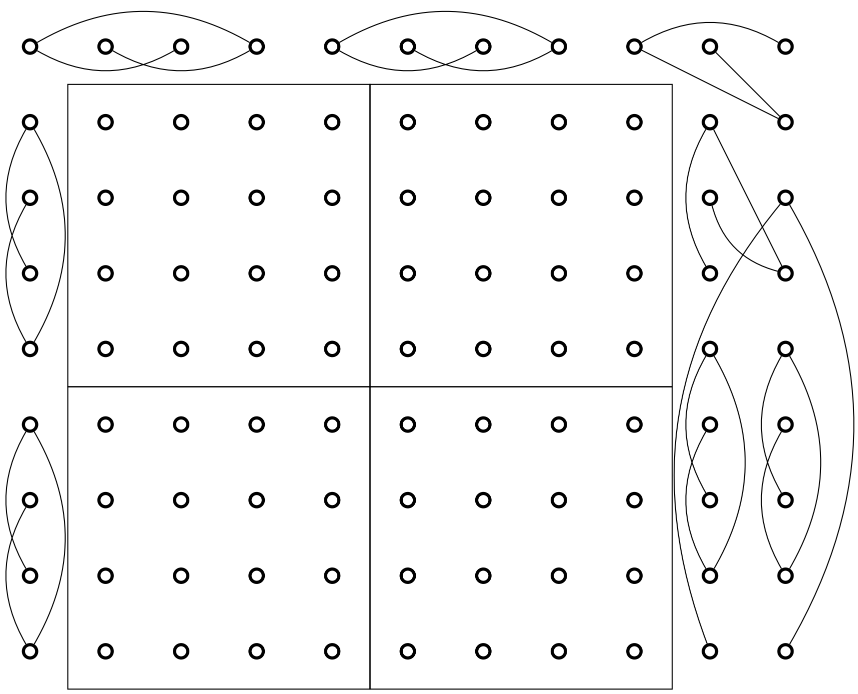

FIGURE 3.13

Example for $\overline{grid(9,11)}$
4 copies of induced paths for $\overline{grid(4,4)}$, 6 **B-paths**, 2 **L-paths**, and 1 path with 3 vertices

Case 9: $n \equiv 2 \bmod 4$ and $m \equiv 0 \bmod 4$

This is isomorphic to **Case 3**

Case 10: $n \equiv 2 \bmod 4$ and $m \equiv 1 \bmod 4$

This is isomorphic to **Case 7**

Case 11: (refer to Figures 3.14 and 3.15) $n \equiv 2 \bmod 4$ and $m \equiv 2 \bmod 4$

Suppose $n = 4k+2$ and $m = 4l+2$ for $k, l \geq 1$, then the total number of vertices to cover is $32kl + 16k + 16l + 8$. As before we cover $32kl$ vertices with copies of the induced paths used to cover $CP(grid(4,4))$ leaving the leftmost column, the rightmost column, the top row, and the bottom row for a total of $4kl$ induced paths.To cover the remaining vertices in $grid(n,m)$ we extend the paths corresponding to P_2 and P_4 in the original cover of $CP(grid(4,4))$. From vertex $(n-1,2)$ continue to $(n-1,1), (n-2,1), ..., (1,1), (1,2), ...(1,m)$. From vertex

$(2, m-1)$ to $(2, m), (3, m), ..., (n, m)$. For the remaining vertices in $\overline{grid(n, m))}$ we start on the upper left corner for the left column and use k vertical **B-paths**, with 2 leftover in the bottom left corner. We cover those 2 vertices with 1 **L-path** that also covers 3 vertices of the bottom row. The remaining vertices in the bottom row are covered with $l-1$ horizontal **B-paths** with 3 vertices left. We cover those 3 with 1 more **L-path** that also covers 2 of the vertices in the rightmost column. We cover the remainder of the column with k vertical **B-paths**. We cover the top row with l horizontal **B-paths**. This is a total of $4kl + k + 1 + (l-1) + 1 + k + l = 4kl + 2k + 2l + 1 = \lceil \frac{16kl+8k+8l+4}{4} \rceil = \lceil \frac{(4k+2)(4l+2)}{4} \rceil = \lceil \frac{nm}{4} \rceil$ induced paths.

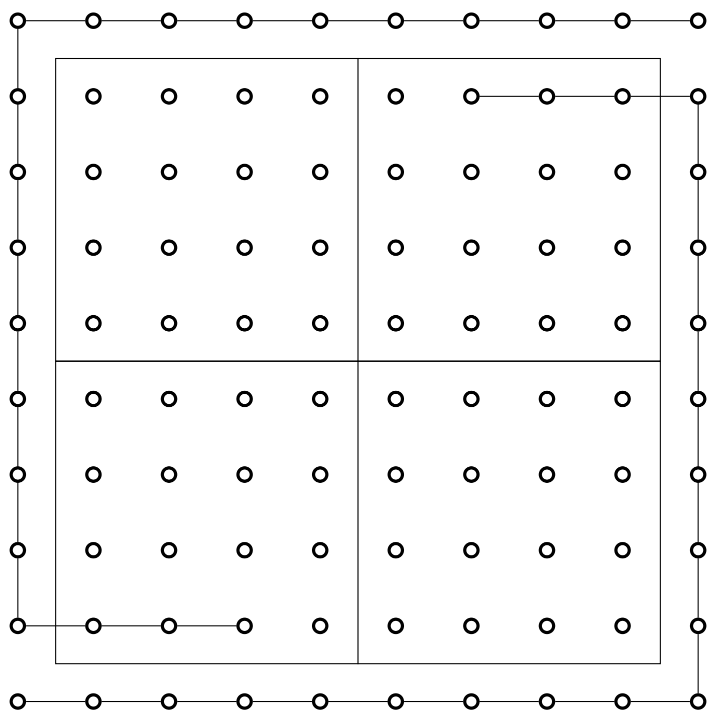

FIGURE 3.14

Example for $grid(10, 10)$
4 copies of induced paths for $grid(4, 4)$, with extended paths

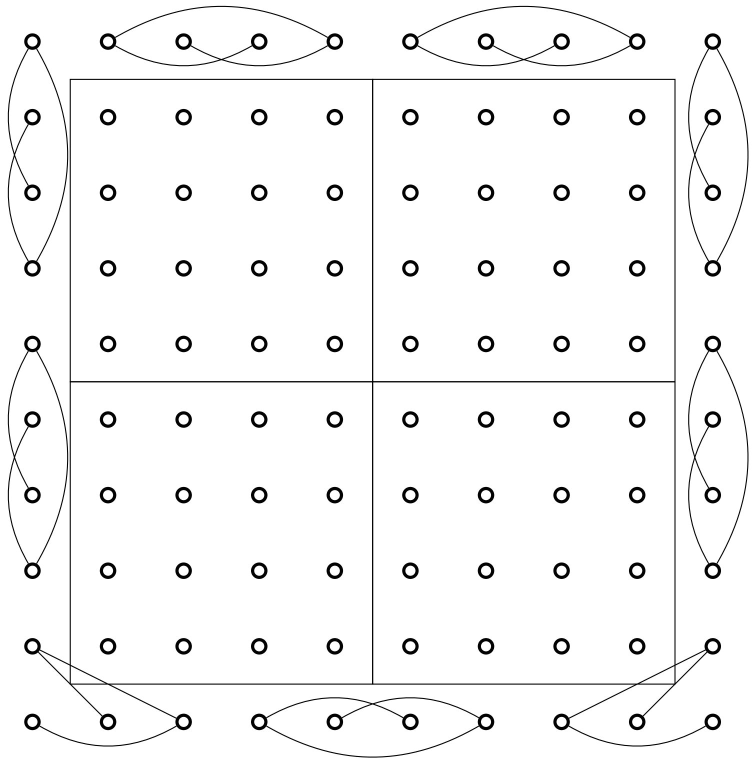

FIGURE 3.15

Example for $\overline{grid(10,10)}$
4 copies of induced paths for $\overline{grid(4,4)}$, 7 **B-paths**, and 2 **L-paths**

Case 12: (refer to Figures 3.16 and 3.17) $n \equiv 2 \bmod 4$ and $m \equiv 3 \bmod 4$

Suppose $n = 4k+2$ and $m = 4l+3$ for $k, l \geq 1$, then the total number of vertices to cover is $32kl+24k+16l+12$. As before we cover $32kl$ vertices with copies of the induced paths used to cover $CP(grid(4,4))$ leaving the leftmost column, the 2 rightmost columns, the top row, and the bottom row for a total of $4kl$ induced paths. To cover the remaining vertices in $grid(n,m)$ we extend the paths corresponding to P_2 and P_4 in the original cover of $CP(grid(4,4))$. From vertex $(n-1,2)$ continue to $(n-1,1),(n-2,1),...,(1,1),(1,2),...,(1,m),(2,m),...,(n,m)$. From $(2,m-2)$ continue to $(2,m-1),(3,m-1),...,(n,m-1),(n,m-2),...(n,1)$. For the remaining vertices in $\overline{grid(n,m))}$ we start on the upper left corner for the left column and use k vertical **B-paths**, with 2 leftover in the bottom left corner. We cover those 2 vertices with 1 **L-path** that also covers 3 vertices of the bottom row. We cover the remaining vertices in the bottom row with l horizontal **B-paths**. We cover the top row with l horizontal **B-paths** leaving 2 vertices left in the top row. For the second from the right column we start at the top and use k vertical **B-paths** with 1 uncovered vertex. For the rightmost column we start second from the top and use k vertical **B-paths** leaving 1 vertex. Those remaining two vertices form 1 more induced path. This comes to a total of $4kl+3k+2l+2 = \lceil \frac{16kl+12k+8l+6}{4} \rceil = \lceil \frac{(4k+2)(4l+3)}{4} \rceil = \lceil \frac{nm}{4} \rceil$ induced paths.

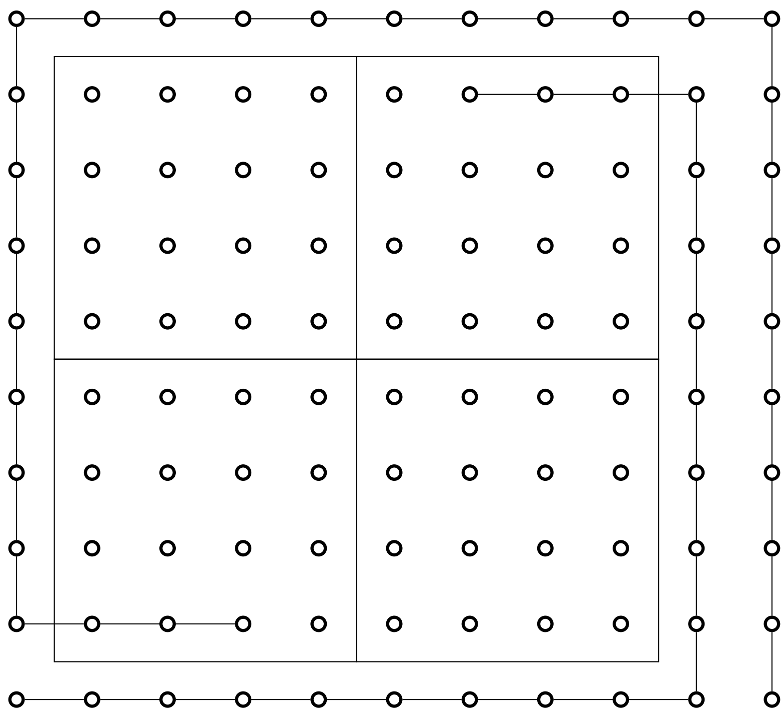

FIGURE 3.16

Example for $grid(10,11)$
4 copies of induced paths for $grid(4,4)$, with extended paths

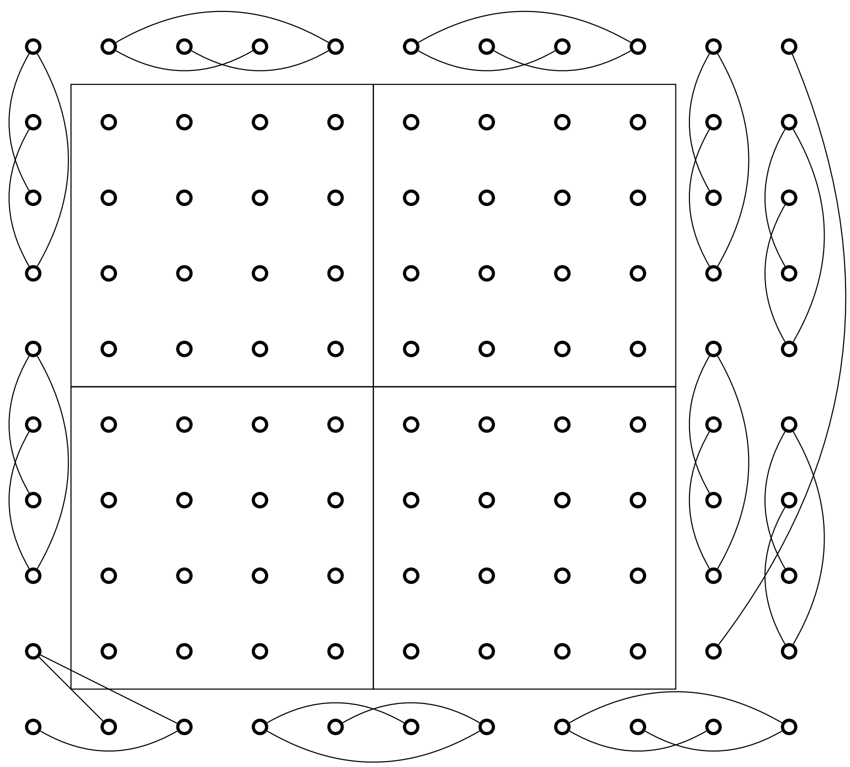

FIGURE 3.17

Example for $\overline{grid(10, 11)}$
4 copies of induced paths for $\overline{grid(4, 4)}$, 10 **B-paths**, 1 **L-path**, and 1 two vertex path

Case 13: $n \equiv 3 \bmod 4$ and $m \equiv 0 \bmod 4$

This is isomorphic to **Case 4**

Case 14: $n \equiv 3 \bmod 4$ and $m \equiv 1 \bmod 4$

This is isomorphic to **Case 8**

Case 15: $n \equiv 3 \bmod 4$ and $m \equiv 2 \bmod 4$

This is isomorphic to **Case 12**

Case 16: (refer to Figures 3.18 and 3.19) $n \equiv 3 \bmod 4$ and $m \equiv 3 \bmod 4$

Suppose $n = 4k + 3$ and $m = 4l + 3$ for $k, l \geq 1$, then the total number of vertices to cover is $32kl + 24k + 24l + 18$. As before we cover $32kl$ vertices with copies of the induced paths used to cover $CP(grid(4, 4))$ leaving the leftmost column, the 2 rightmost columns, the top row, and the 2 bottom rows for a total of $4kl$ induced paths. To cover the remaining vertices in $grid(n,m)$ we extend the paths corresponding to P_4 and P_2 in the original cover of $CP(grid(4,4))$. From vertex $(n-2,2)$ continue to $(n-2,1),(n-3,1),...,(1,1),(1,2),...,(1,m),(2,m),...,(n,m),(n,m-1),..,(n,1)$. From vertex $(2,m-2)$ continue to $(2,m-1),(3,m-1),...,(n-1,m-1),(n-1,m-2),...,(n-1,1)$. For the remaining vertices in $\overline{grid(n,m))}$ we start on the upper left corner for the left column and use k vertical **B-paths**, with 3 leftover in the bottom left corner. We cover those 3 vertices with 1 **L-path** that also covers 2 vertices of the bottom row. We cover the second from the bottom row and the bottom row each with l horizontal **B-paths**, leaving 2 vertices and 1 vetex leftover respectivley. We use an **L-path** to cover the 3 bottom vertices in the rightmost column and 1 vertex at $\overline{(n-2,m-1)}$. Now we cover the rest of the 2 right most columns using k vertical **B-paths** each. We cover the rest of the top row with l horizontal **B-paths**. The vertex $\overline{(n-1,m-1)}$ will be 1 induced path in the cover. This comes out to a total of $4kl+3k+3l+3 = \lceil\frac{16kl+12k+12l+9}{4}\rceil = \lceil\frac{(4k+3)(4l+3)}{4}\rceil = \lceil\frac{nm}{4}\rceil$ induced paths. $\square$

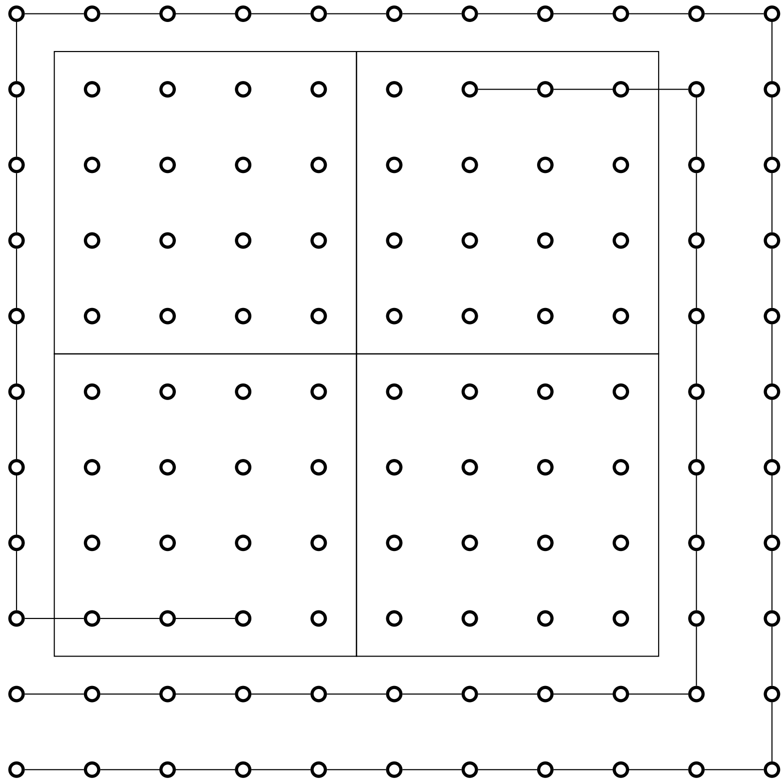

FIGURE 3.18

Example for $grid(11, 11)$
4 copies of induced paths for $grid(4, 4)$, with extended paths

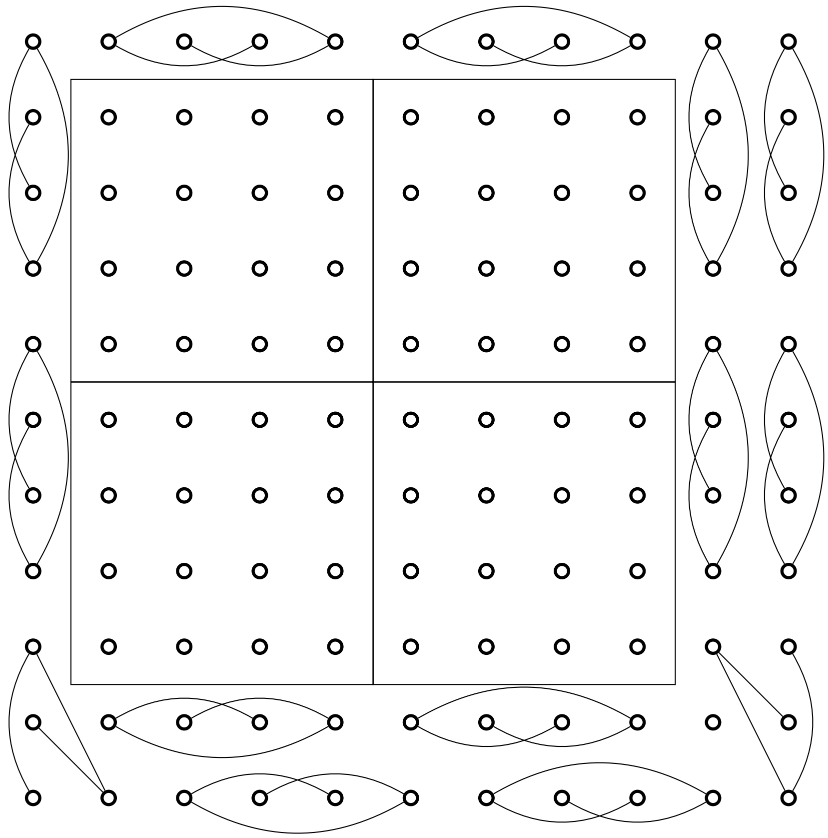

FIGURE 3.19

Example for $\overline{grid(11,11)}$
4 copies of induced paths for $\overline{grid(4,4)}$, 12 **B-paths**, 2 **L-paths**, and 1 vertex

COROLLARY 3.2.

$$\rho(CP(grid(n,n))) = \begin{cases} \lceil \frac{n^2}{4} \rceil + 1 & n = 2 \\ \lceil \frac{n^2}{4} \rceil & m \neq 2; m \geq 1 \end{cases}$$

Proof follows from the above theorem and the examples already provided.□

3.3 Special Cases where $n, m < 4$

Now to address the special cases where $n = 2$; first we will show that $\rho(CP(grid(2,m))) > \lceil \frac{2m}{4} \rceil$ for $m = 3, 4, 6$, then we will provide some examples of induced path covers for $n = 2$. We will use these to compute $\rho(CP(grid(2,m)))$ in general.

Lemma 3.3. $\rho(CP(grid(2,3))) > 2$

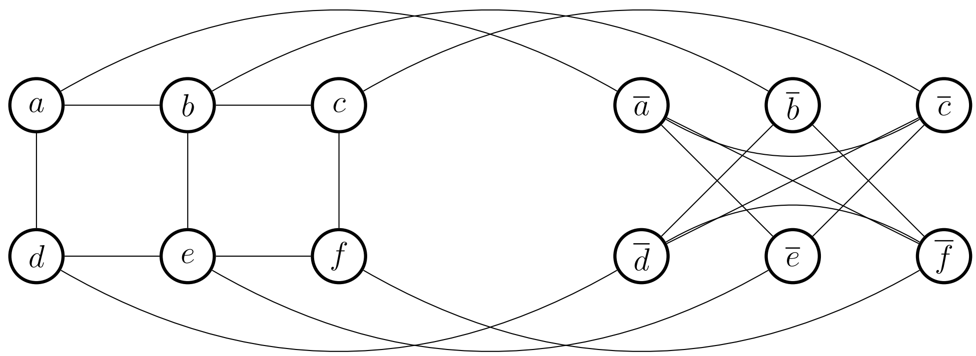

Figure 3.20

$CP(grid(2,3))$
All edges are shown

Proof: (refer to Figure 3.20) Suppose we have two induced paths P_1 and P_2 to cover the twelve vertices of $CP(grid(2,3))$. By **Theorem 2.6**, P_1 and P_2 have at most four vertices in $\overline{grid(2,3)}$. So both P_1 and P_2 have three vertices in $\overline{grid(2,3)}$, or there are four vertices in one path and two vertices in the other path in $\overline{grid(2,3)}$. A path cannot have six vertices in $grid(2,3)$, because we would induce multiple cycles. Suppose one of the paths, P_1, has five vertices in $grid(2,3)$. In order to not induce a cycle, and without loss of generality, the five vertices are d, a, b, c, f and they induce a subpath $dabcf$. P_1 must have at least two vertices in $\overline{grid(2,3)}$, but if we continue the subpath to either $\overline{d}$ or $\overline{f}$, there are no more vertices in $\overline{grid(2,3)}$ that can be in P_1, since otherwise we induce a cycle. So the paths do not have five vertices in $grid(2,3)$ and do not have less than two vertices in $grid(2,3)$.

Now we show that the paths cannot have four vertices in $\overline{grid(2,3)}$. Suppose P_1 has four vertices in $\overline{grid(2,3)}$, then by **Corollary 2.11** the four vertices are in a **U-path** in $CP(grid(2,3))$ or they form a **B-path, Z-path, or L-path** in $\overline{grid(2,3)}$. A **B-path** is not possible in $\overline{grid(2,3)}$. Suppose P_1 has a **Z-path**. Without loss of generality let it be $\overline{e}\,\overline{a}\,\overline{f}\,\overline{b}$.

P_1 has at least two vertices in $grid(2,3)$, and does not have two prism edges by **Theorem 2.7**. So we can not continue to both b and e. Using symmetry, we continue to e. From here we can only go to d and no farther. Now P_2 must have the vertices $a, b, c, f, \overline{d}, \overline{c}$. However, c is adjacent to $\overline{c}$, b, and f, and so P_2 is not a path, which is a contradiction. Now suppose P_1 is an **L-path** and using symmetry let it be $\overline{df}\,\overline{ae}$. P_2 has non-adjacent vertices $\overline{b}$ and $\overline{c}$. In order for those vertices to be on the same path, P_2 has two prism edges. Using **Theorem 2.8**, P_2 is $\overline{b}bc\overline{c}$ with no additional vertices. Therefore P_1 has vertices $a, d, e, f, \overline{d}, \overline{f}, \overline{a}, \overline{e}$, but P_2 is an induced path so it has four prism edges which contradicts **Theorem 2.1**. Finally suppose P_1 is a **U-path**. Using symmetry, P_1 can be $\overline{f}\,\overline{b}bc\overline{c}\,\overline{e}$, $\overline{f}\,\overline{b}be\overline{e}\,\overline{c}$, or $\overline{b}\,\overline{f}fc\overline{c}\,\overline{e}$. If P_1 is $\overline{f}\,\overline{b}bc\overline{c}\,\overline{e}$, then vertex d in P_2 is adjacent to three vertices in P_2, which is a contradiction. If P_1 is $\overline{f}\,\overline{b}be\overline{e}\,\overline{c}$, then in P_2 the vertices $c, f, \overline{a}, \overline{d}$ all have degree one, but a path can only have two vertices with degree one. If P_1 is $\overline{b}\,\overline{f}fc\overline{c}\,\overline{e}$, then P_2 contains the cycle $abed$, which is a contradiction. So each path must contain exactly three vertices in $\overline{grid(2,3)}$.

We now show that neither P_1 nor P_2 can have four vertices in $grid(2,3)$. Suppose P_1 has four vertices in $grid(2,3)$, we know it must also have exactly three vertices in $\overline{grid(2,3)}$. P_1 can not have two prism edges because this would contradict **Theorem 2.2**. The four vertices in $grid(2,3)$ in P_1 without loss of generality can be $\{a, d, e, f\}$, $\{a, b, e, f\}$ $\{a, d, c, f\}$, or $\{a, d, e, c\}$. Suppose P_1 has the vertices $\{a, d, e, f\}$ which induce the subpath $adef$. If we continue to either $\overline{a}$ or $\overline{f}$ then we can only continue to one more vertex in each case, but we need three vertices in $\overline{grid(2,3)}$, and therefore have a contradiction. The case is similar for $\{a, b, e, f\}$. Suppose P_1 has the vertices $\{a, d, c, f\}$. These vertices do not induce a subpath of $grid(2,3)$ and so in order for them to be in the same path P_1 must have two prism edges, but this is a contradiction to **Theorem 2.2**. The case is similar for P_1 containing the vertices $\{a, d, e, c\}$. So P_1 and P_2 both contain exactly three vertices in both $grid(2,3)$ and $\overline{grid(2,3)}$.

Suppose P_1 and P_2 both contain exactly three vertices in both $grid(2,3)$ and $\overline{grid(2,3)}$. Using **Theorem 2.2** each path can only have one prism edge so the three vertices in $grid(2,3)$ and $\overline{grid(2,3)}$ each induce subpaths. Without loss of generality, we can assume that the vertex a is on P_1 and the subpath in P_1 is abc, ade, or abe in $grid(2,3)$. Suppose P_1 has abc and P_1 needs three more vertices in $\overline{grid(2,3)}$. Using symmetry, continue P_1 to $\overline{c}$. From here we can go to $\overline{e}$, or $\overline{d}$. If we continue to $\overline{e}$, we can not get a third vertex. If we continue to $\overline{d}$, the only vertex to go to is $\overline{f}$. Now P_2 has vertices $\{d, e, f, \overline{a}, \overline{b}, \overline{e}\}$, but $\overline{b}$ is not adjacent to any other vertex in P_2 which is a contradiction. Suppose P_1 has subpath abe, then P_2 has vertices $\{d, c, f\}$ in $grid(2,3)$ and d is not adjacent to c or f. So P_2 must have two prism edges, but this is a contradiction. Finally suppose P_1 has ade. If we continue to $\overline{e}$, we can only go to $\overline{c}$, and therefore we can not get the third vertex of $\overline{grid(2,3)}$. If we continue to $\overline{a}$, we can go to to $\overline{f}$ or $\overline{c}$. If we continue to $\overline{c}$, we can not get our third vertex. Instead, continue to $\overline{f}$, then to $\overline{b}$. Then P_2 has the vertices $\{b, c, f, \overline{d}, \overline{c}, \overline{e}\}$, but c is adjacent to three of these vertices, which is a contradiction. So there are not two disjoint induced paths that cover every vertex in $CP(grid(2,3))$. $\square$

LEMMA 3.4. $\rho(CP(grid(2,4))) > 2$

Proof: Suppose that $\rho(CP(grid(2,4))) = 2$, then there are two disjoint induced paths, P_1 and P_2, that cover all the vertices and they each have four vertices in $\overline{grid(2,4)}$. To cover $\overline{grid(2,4)}$ with two induced paths, they are two **B-paths**, two **L-paths**, or 2 **U-paths**. If we use two **U-paths** we will not cover all the vertices in $grid(2,4)$, since they each have two vertices in $grid(2,4)$. If we use two **B-paths** we will have the subpaths $\overline{(1,3)}, \overline{(1,1)}, \overline{(1,4)}, \overline{(1,2)}$ in P_1 and $\overline{(2,3)}, \overline{(2,1)}, \overline{(2,4)}, \overline{(2,2)}$ in P_2. By **Theorem 2.7** we have at most one prism edge to extend P_1. If for instance we extend P_1 from $\overline{(1,2)}$ to $(1,2)$ and stop at $(1,2)$, then neither P_1 nor P_2 can cover $(1,1)$. Instead we continue to $(2,2)$, and agiain neither P_1 nor P_2 can cover $(1,1)$. Extending P_1 to $(1,3)$ or P_2 to $(2,2)$ or $(2,3)$ lead to a similar contradiction. If we use two **L-paths**, we have at most one prism edge on each path by **Theorem 2.7** and we have the subpath $\overline{(2,1)}, \overline{(2,3)}, \overline{(1,1)}, \overline{(2,2)}$ in P_1, and $\overline{(1,4)}, \overline{(1,2)}, \overline{(2,4)}, \overline{(1,3)}$ in P_2. If we extend P_1 from $\overline{(2,1)}$ to $(2,1)$, then there is nowhere else to go without inducing a cycle. So P_2 has the remaining vertices and P_2 will contain a cycle. By extending P_1 from $\overline{(2,2)}$ to $(2,2)$, then we can not cover $(2,1)$ with either P_1 or P_2. □

Lemma 3.5. $\rho(CP(grid(2,6))) > 3$

Proof: Suppose that $\rho(CP(grid(2,6))) = 3$, then there are three induced paths and they each have four vertices in $\overline{grid(2,6)}$. The only way to do this is with two **L-paths** and one **B-path** or two **B-paths** and one **U-path** or three **U-paths**. Applying reasoning similar to the above claim, we can not cover all the vertices without the addition of another induced path. □

Now we provide specific $\rho(CP(grid(2,m)))$ for $m = 1,2,3,4,5,6,8,10,12,14$. For $m = 1$ it is trivial and $\rho(CP(grid(2,1))) = 1 = \lceil \frac{2*1}{4} \rceil$.

For $m = 2$ we have already shown $\rho(CP(grid(2,2)) = 2 = \lceil \frac{2*2}{4} \rceil + 1$.

For $m = 3$ (refer to Figure 3.21) we know $\rho(CP(grid(2,3)) > 2$ but we cover all vertices with the following three induced paths. $P_1 = (1,1),(1,2),(1,3),\overline{(1,3)}$; $P_2 = (2,3),(2,2),(2,1),\overline{(2,1)}$; $P_3 = \overline{(2,2)}, \overline{(1,1)}, \overline{(2,3)}, \overline{(1,2)}$, and $\rho(CP(grid(2,3)) = 3 = \lceil \frac{2*3}{4} \rceil + 1$.

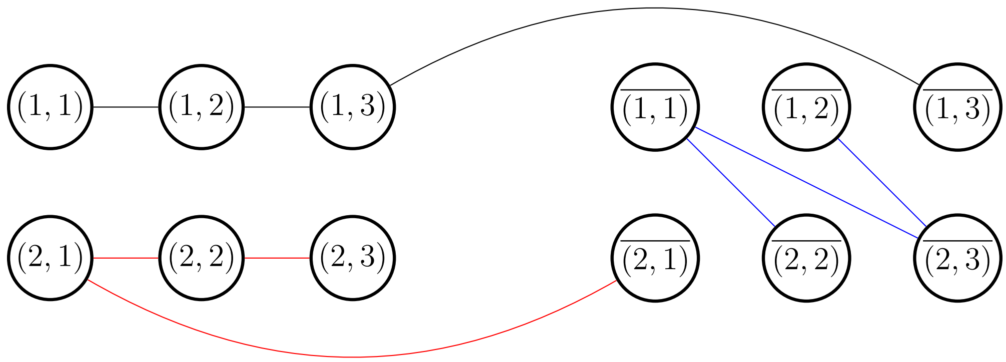

FIGURE 3.21

Paths for $\rho(CP(grid(2,3)))$

For $m = 4$ we know $\rho(CP(grid(2,4)) > 2$, but we cover all vertices with the following three induced paths. $P_1 = (1,1),(1,2),(1,3),(1,4),\overline{(1,4)},\overline{(2,1)}$; $P_2 = (2,1),(2,2),(2,3),(2,4),\overline{(2,4)},\overline{(1,3)}$; $P_3 = \overline{(2,2)},\overline{(1,1)},\overline{(2,3)},\overline{(1,2)}$, and $\rho(CP(grid(2,4)) = 3 = \lceil \frac{2*4}{4} \rceil + 1$

For $m = 5$ (refer to Figure 3.22) we cover all the vertices with the following three induced paths. $P_1 = (1,1),(2,1),(2,2),(2,3),(2,4),\overline{(2,4)},\overline{(1,3)},\overline{(2,5)},\overline{(1,4)}$; $P_2 = (2,5),(1,5),(1,4),(1,3),(1,2),\overline{(1,2)},\overline{(2,3)},\overline{(1,1)},\overline{(2,2)}$; $P_3 = \overline{(2,1)},\overline{(1,5)}$, and $\rho(CP(grid(2,5)) = 3 = \lceil \frac{2*5}{4} \rceil$

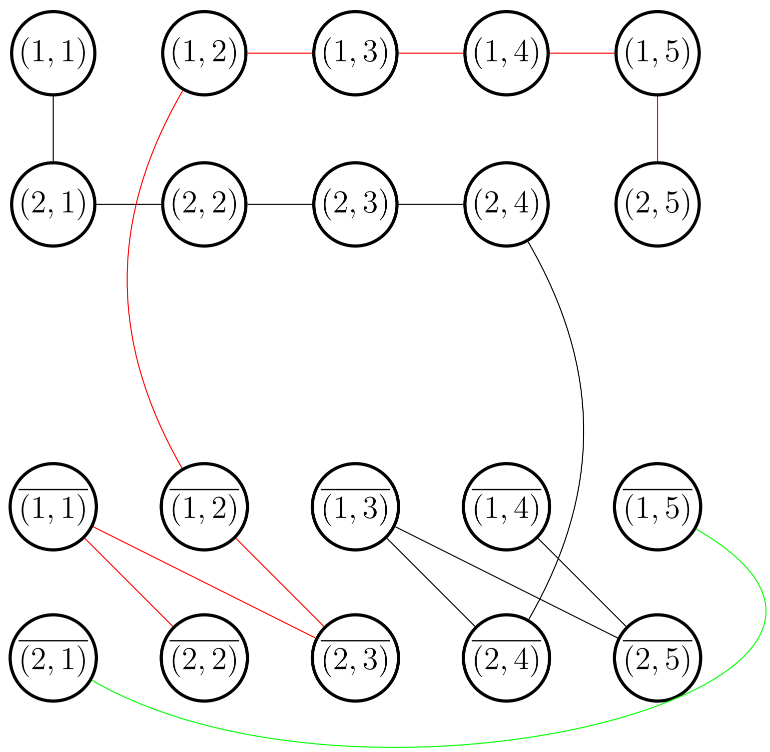

FIGURE 3.22

Paths for $\rho(CP(grid(2,5)))$

For $m = 6$ we know $\rho(CP(grid(2,6)) > 3$ but we cover all vertices with the following four induced paths. $P_1 = (1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\overline{(1,6)},\overline{(2,5)}$; $P_2 = (2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\overline{(2,6)},\overline{(1,5)}$; $P_3 = \overline{(2,1)},\overline{(2,3)},\overline{(1,1)},\overline{(2,2)}$; $P_4 = \overline{(1,4)},\overline{(1,2)},\overline{(2,4)},\overline{(1,3)}$, and $\rho(CP(grid(2,6)) = 4 = \lceil\frac{2*6}{4}\rceil + 1$

For $m = 8$ (refer to Figure 3.23) we cover all the vertices with the following four induced paths. $P_1 = \overline{(2,1)},\overline{(2,3)},\overline{(1,1)},\overline{(2,2)},(2,2)$; $P_2 = \overline{(1,3)},\overline{(2,4)},\overline{(1,2)},\overline{(1,4)},(1,4),(1,5),(1,6),(2,6),(2,7),(2,8),(1,8)$; $P_3 = \overline{(2,6)},\overline{(1,5)},\overline{(2,7)},\overline{(2,5)},(2,5),(2,4),(2,3),(1,3),(1,2),(1,1),(2,1)$; $P_4 = \overline{(1,8)},\overline{(1,6)},\overline{(2,8)},\overline{(1,7)},(1,7)$, and $\rho(CP(grid(2,8)) = 4 = \lceil\frac{2*8}{4}\rceil$.

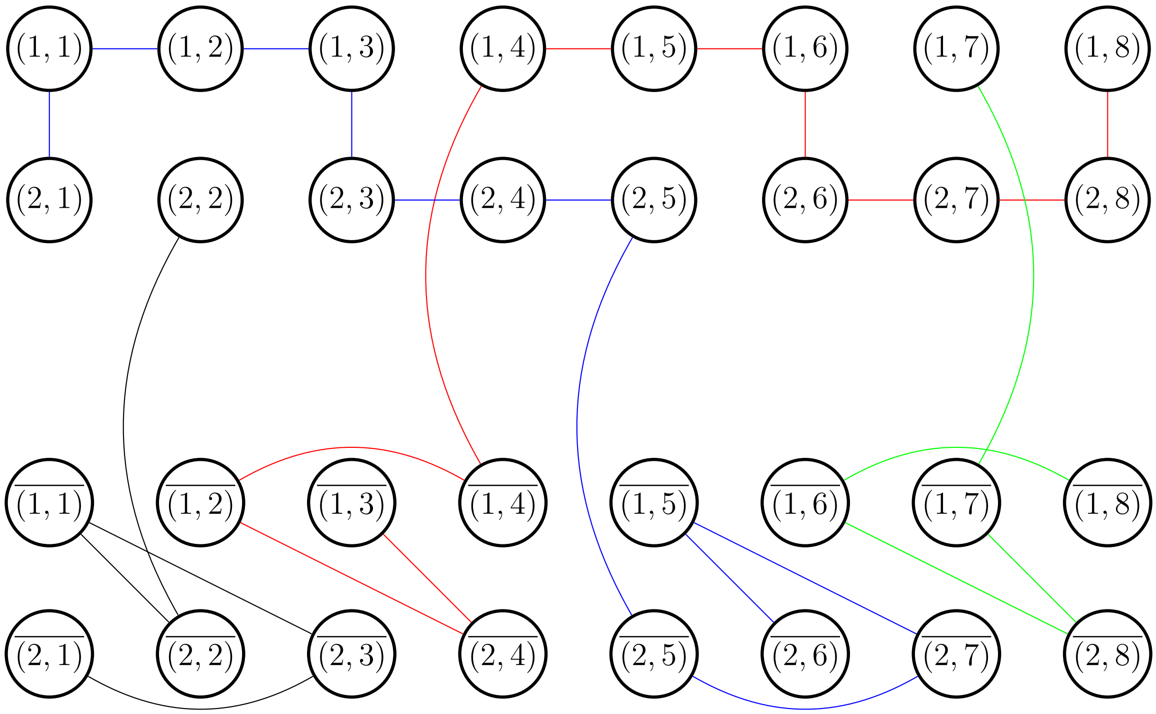

FIGURE 3.23

Paths for $\rho(CP(grid(2,8)))$

For $m = 10$ we cover all the vertices with the following five induced paths. $P_1 = \overline{(2,1)}, \overline{(2,3)}, \overline{(1,1)}, \overline{(2,2)}, (2,2)$; $P_2 = \overline{(1,3)}, \overline{(1,5)}, \overline{(1,2)}, \overline{(1,4)}, (1,4)$; $P_3 = \overline{(2,5)}, \overline{(1,6)}, \overline{(2,4)}, \overline{(2,6)}, (2,6), (2,7), (2,8), (1,8), (1,9), (1,10), (2,10)$; $P_4 = \overline{(1,8)}, \overline{(2,7)}, \overline{(1,9)}, \overline{(1,7)}, (1,7), (1,6), (1,5), (2,5), (2,4), (2,3), (1,3), (1,2), (1,1), (2,1)$; $P_5 = \overline{(2,10)}, \overline{(2,8)}, \overline{(1,10)}, \overline{(2,9)}, (2,9)$, and $\rho(CP(grid(2,10)) = 5 = \lceil \frac{2*10}{4} \rceil$

For $m = 12$ we cover all the vertices with the following six induced paths. $P_1 = \overline{(2,1)}, \overline{(2,3)}, \overline{(1,1)}, \overline{(2,2)}, (2,2)$; $P_2 = \overline{(1,3)}, \overline{(1,5)}, \overline{(1,2)}, \overline{(1,4)}, (1,4)$; $P_3 = \overline{(2,5)}, \overline{(1,6)}, \overline{(2,4)}, \overline{(2,6)}, (2,6), (2,7), (2,8), (1,8), (1,9), (1,10), (2,10), (2,11), (2,12), (1,12)$; $P_4 = \overline{(1,8)}, \overline{(2,7)}, \overline{(1,9)}, \overline{(1,7)}, (1,7), (1,6), (1,5), (2,5), (2,4), (2,3), (1,3), (1,2), (1,1), (2,1)$; $P_5 = \overline{(2,10)}, \overline{(2,8)}, \overline{(2,11)}, \overline{(2,9)}, (2,9)$; $P_6 = \overline{(1,12)}, \overline{(1,10)}, \overline{(2,12)}, \overline{(1,11)}, (1,11)$, and $\rho(CP(grid(2,12)) = 6 = \lceil \frac{2*12}{4} \rceil$

For $m = 14$ we cover all the vertices with the following seven induced paths. $P_1 = \overline{(2,1)}, \overline{(2,3)}, \overline{(1,1)}, \overline{(2,2)}, (2,2)$; $P_2 = \overline{(1,3)}, \overline{(1,5)}, \overline{(1,2)}, \overline{(1,4)}, (1,4)$; $P_3 = \overline{(2,5)}, \overline{(1,6)}, \overline{(2,4)}, \overline{(2,6)}, (2,6), (2,7), (2,8), (1,8), (1,9), (1,10), (2,10), (2,11), (2,12), (1,12), (1,13), (1,14), (2,14)$; $P_4 = \overline{(1,8)}, \overline{(2,7)}, \overline{(1,9)}, \overline{(1,7)}, (1,7), (1,6), (1,5), (2,5), (2,4), (2,3), (1,3), (1,2), (1,1), (2,1)$; $P_5 = \overline{(2,10)}, \overline{(2,8)}, \overline{(2,11)}, \overline{(2,9)}, (2,9)$; $P_6 = \overline{(1,12)}, \overline{(1,10)}, \overline{(1,13)}, \overline{(1,11)}, (1,11)$; $P_7 = \overline{(2,14)}, \overline{(2,12)}, \overline{(1,14)}, \overline{(2,13)}, (2,13)$, and $\rho(CP(grid(2,14)) = 7 = \lceil \frac{2*14}{4} \rceil$

THEOREM 3.6.

$$\rho(CP(grid(2,m))) = \begin{cases} \lceil \frac{2m}{4} \rceil + 1 & m = 2,3,4,6 \\ \lceil \frac{2m}{4} \rceil & m \neq 2,3,4,6; m \geq 1 \end{cases}$$

Proof: The cases for $m = 1,2,3,4,5,6,8,10,12,14$ have already been shown.

Case 1: Suppose that m is odd and $m \geq 5$.

An example has been provided for $m = 5$, for any other $m > 5$ and $m = 2k+1$ we have $2(2k+1) = 4k+2$ vertices in each $grid(2,m)$ and $\overline{grid(2,m)}$. Our first induced path is from $\overline{(2,1)}$ and $\overline{(1,m)}$. What remains in $\overline{grid(2,m)}$ are $2k$ vertices in the top row and $2k$ vertices in the bottom row shifted to one vertex to the right. We cover these with k **Z-paths**. Now to cover all the vertices in $grid(2,m)$ we extend two of the previous paths. We extend one path starting with $\overline{(1,2)}$ and continue to $(1,2), (1,3), ..., (1,m), (2,m)$, and another path starting at $\overline{(2,m-1)}$ and continuing to $(2,m-1), (2,m-2), ..., (2,1), (1,1)$ and we have a total of $k+1 = \lceil k + 1/2 \rceil = \lceil \frac{4k+2}{4} \rceil = \lceil \frac{2(2k+1)}{4} \rceil = \lceil \frac{2m}{4} \rceil$ induced paths to cover all the vertices

Case 2: Suppose that m is even and $m \geq 14$.

Since in the examples of $m = 8,10,12,14$ the induced paths all have four vertices in the $\overline{grid(2,m)}$ side, therefore any $m \geq 14$ and even is covered by making copies and combining the covers used for $m = 8,10,12,14$ for a total of $\lceil \frac{2m}{4} \rceil$ induced paths. Since $m = 2k$ for $k > 7$, $k = 4s + r$ for $s \geq 1$, and $0 \leq r1 \leq 3$. Therefore $m = 8(s-1) + (2r+8)$ with

$8 \leq 2r + 8 \leq 14$ so we have $s - 1$ copies of a cover for $CP(grid(2, 8))$ and one copy of $grid(2, 2r + 8)$. $\square$

3.4 Future Work

There are many more graphs for which we can try to calculate $\rho(G)$ and $\rho(CP(G))$. We also would like to find a proof for **Lemma 3.2** that does not require a "brute force" method and leave it to future research. The $NAVS$ for vertices was helpful for finding proofs in this thesis, and perhaps finding more theorems involving $NAVS$ for other types of graphs can help solve other open problems. For the case of $n = 3$ we were unable to provide a proof, but believe it is possible. From multiple examples we have the following **Conjecture 3.7**.

Conjecture 3.7.

$$\rho(CP(grid(3, m))) = \begin{cases} \lceil \frac{3m}{4} \rceil + 1 & m = 1, 2, 4 \\ \lceil \frac{3m}{4} \rceil & m \neq 1, 2, 4; m \geq 1 \end{cases}$$

We believe we can generalize $\rho(CP(grid(3, m)))$ by using constructions similar to the ones we provide in **Figures 3.24 and 3.25**. We believe however, we have to make a special argument for when $m = 4$ that might be similar to the arguments made in **Lemmas 3.3, 3.4, and 3.5**.

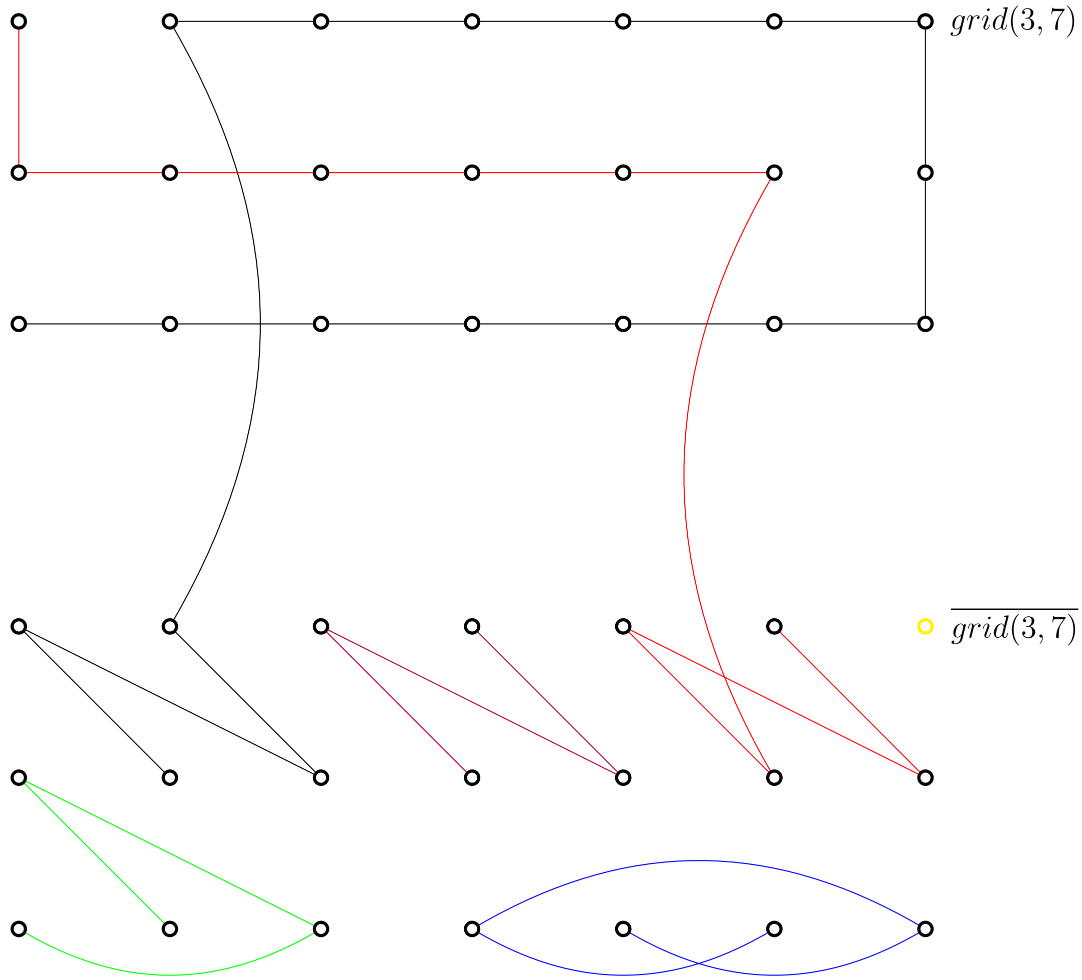

FIGURE 3.24

Six Induced Paths for $\rho(CP(grid(3,7)))$

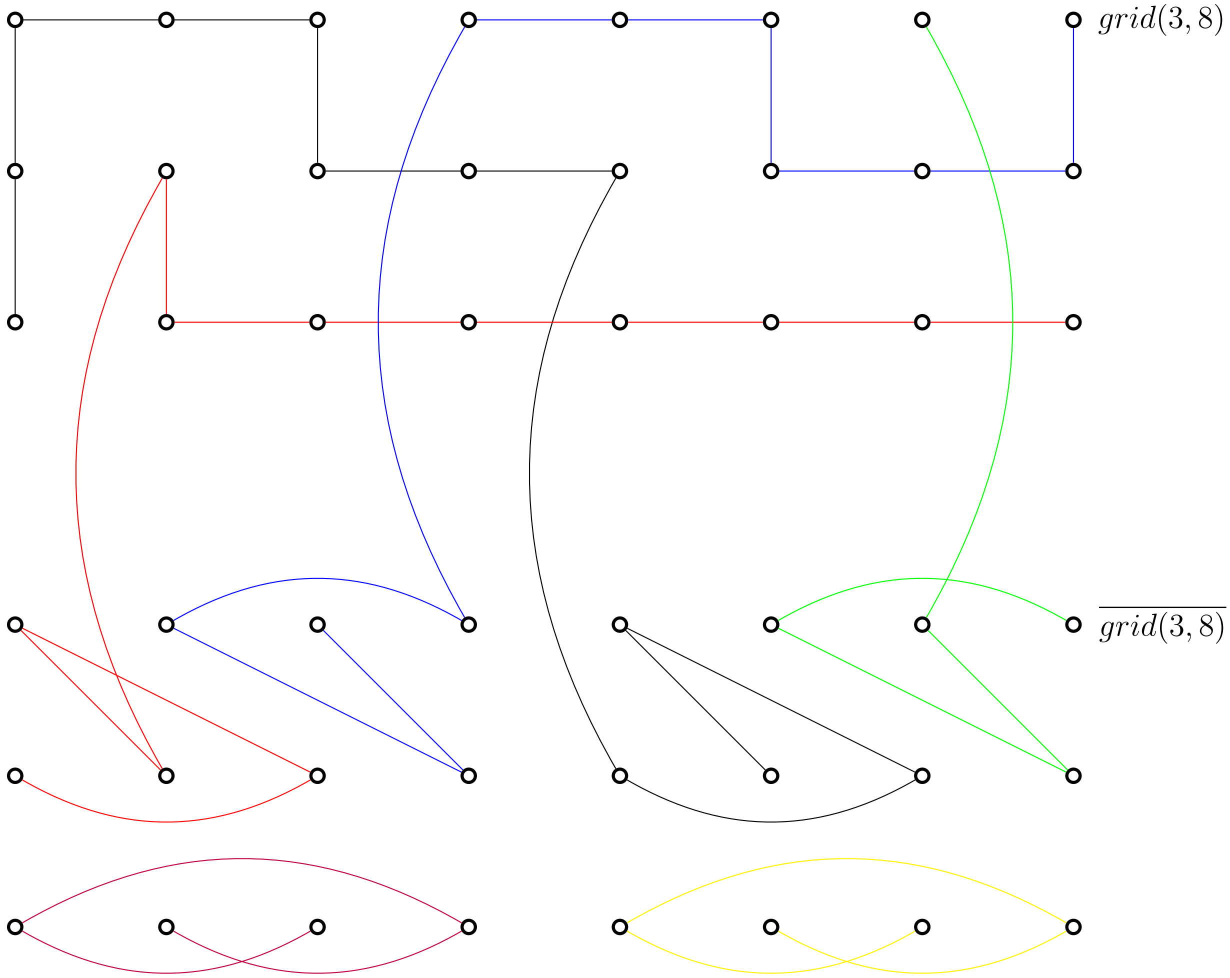

FIGURE 3.25

Six Induced Paths for $\rho(CP(grid(3,8)))$

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VITA

Jeffrey Christopher was born in Houston, Texas, to parents Tracy and Vance Christopher. He graduated from The University of the South in 2010 with a Bachelors in Science. Since 2010 he has taught all levels of high school mathematics. In 2016 he enrolled in the pre-professional mathematics program at the University of Tennessee at Chattanooga as a part time student. He graduated in May of 2019 with a Masters of Science.