ON A GRAPH PARAMETER RELATED TO VERTEX LABELINGS AND ITS APPLICATION TO MINIMUM RANK PROBLEMS IN GRAPH THEORY

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ABSTRACT

This thesis regards the minimum rank and minimum positive semidefinite rank of a simple graph. A graph parameter, called the minimum labeling degree (*mld*), is defined in terms of the concept of a vertex labeling of a graph, and its value is calculated for a few graph classes. It is proved here that there is a conception of *mld* that is independent of the notion of vertex labeling. Then, for a few other graph parameters β, including the zero-forcing number, a general inequality between *mld* and β is shown to hold. Further, it is demonstrated here that a certain upper bound for minimum rank in terms of minimum labeling degree holds for several classes of graphs for which minimum rank is known. Later, graphs whose complements both are $K_{3,2}$-free and have minimum labeling degree 2 are proved to have minimum positive semidefinite rank at most 4. Finally, two more labeling-independent conceptions of *mld* are given.

DEDICATION

This thesis is dedicated to my parents, Dennis and Shabnam Plaisted, and my siblings, Luke and Hannah. I would also like to dedicate this work to our family's pet mudkip...oh wait, I already mentioned Hannah.

ACKNOWLEDGEMENTS

Thanks a lot to Dr. Barioli for the introduction to the problem and the minimum labeling degree parameter, and the good suggestions regarding the content of the document. Thanks also to Drs. Nichols, van der Merwe, and Walters for serving on the committee.

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CHAPTER 1

INTRODUCTION

A *graph* G is a pair (V,E), where V is a set and E is a collection of sets $\{u,v\}$, where $u,v \in V$. If $G=(V,E)$, then V and E are referred to as the *vertex set* and the *edge set* of G, respectively. If G is not initially defined as a given vertex set-edge set pair, then the labels $V(G)$, $E(G)$ denote the vertex and edge sets of G, respectively. The cardinality of V is called the *order* of G and will be denoted $|G|$. The cardinality of E is called the *size* of G, and will be denoted $size(G)$. A graph $G=(V,E)$ is said to be *simple* if and only if each pair of elements of V has at most one edge between them, $\{v,v\}$ is not an element of E for any $v \in V$, and the elements of E are not assumed to have any inherent direction. Since we will only be concerned with simple graphs, the label "graph" may be understood to mean "simple graph" in this thesis.

An $n \times n$ matrix A is said to be *symmetric* if and only if for each i,j such that $1 \le i,j \le n$, $A_{ij} = A_{ji}$. If $G=(V,E)$ is a graph, then the set $S(G)$ is defined to contain exactly the $n \times n$ symmetric real matrices A such that the entry A_{ij}, with $i \neq j$, is nonzero if and only if $\{v_i, v_j\} \in E$. Hence, for any $A \in S(G)$, the diagonal entries A_{ii}, $1 \le i \le n$, can be any real number. We say that G is the *graph of a matrix* A, and write $\Gamma(A) = G$, if and only if $A \in S(G)$.

The *row space* (respectively, *column space*) of an $n \times n$ matrix A is the subspace of $\mathbb{R}^n$ spanned by the row vectors (respectively, column vectors) of A. It is known that, for any matrix A with real entries, the dimension of the row space of A is equal to the dimension of the column space of A. This number is referred to as the rank of A, and denoted $rank(A)$.

In general, the goal of a *minimum rank problem* is to determine, for a given set S of matrices, the value of $min(\{rank(A) : A \in S\})$. The goal of the *minimum rank problem for a graph* G is to determine $min(\{rank(A) : A \in S(G)\})$. The minimum rank of G is denoted $mr(G)$.

An $n \times n$ square matrix A is said to be *positive definite* if and only if A is such that $x^T A x > 0$ for each nonzero vector $x \in \mathbb{R}^n$. If $x^T A x \ge 0$ for every vector $x \in R^n$, then A is referred to as *positive semidefinite*.

The *minimum positive semidefinite rank problem* for a graph G is that of determining the minimum rank among matrices in the set $S_+(G) = \{A \in S(G) : A$ is positive semidefinite$\}$. The minimum positive semidefinite rank of a graph G is denoted $mr_+(G)$. Since the set of numbers for which $mr_+(G)$ is a minimum is a subset of that for which $mr(G)$ is a minimum, it is clear that $mr(G) \leq mr_+(G)$ holds for every graph G.

The outline for the remainder of the thesis is as follows: In Chapter 2, we define some basic terms related to graphs, as well as define graph classes that will be relevant to later sections. Also in Chapter 2, we state several results, a few of which characterize the class of graphs having certain minimum rank, and most of which give the minimum rank or minimum positive semidefinite rank for some class of graphs. Chapter 3 surveys several attempts to bound or partially characterize minimum rank or minimum positive semidefinite rank using another graph parameter. In Chapter 4, we define the notion of a vertex labeling of a graph and a graph parameter in terms of this notion, the minimum labeling degree (mld), and prove that there is a definition of mld that is independent of the notion of vertex labelings. It is also shown that a general relationship exists between mld and a couple of parameters that occur in known and conjectured bounds of minimum rank. It is proved in Chapter 5 that a certain upper bound of $mr(G)$ that is in terms of $mld(G)$ holds for several classes of graphs. It is shown in Chapter 6 that graphs whose complements are $K_{3,2}$-free and have mld 2 have $mr_+ \leq 4$, and this result is applied to show that $mr_+ \leq 4$ for a couple of particular graph classes, including the complements of what are called beehives. In Chapter 7, two other vertex labeling-independent definitions of mld, one in terms of a certain vertex-removal scheme, and the other based on a notion similar to that of a k-tree, are given.

CHAPTER 2

ELEMENTARY GRAPH THEORY TERMS, GRAPH CLASSES, AND RELEVANT THEOREMS AND CONJECTURES

2.1 Terms Related to Graphs

We first define some elementary graph theory terms. Definitions of all of the terms defined in this section can be found in [7]. In a graph $G = (V,E)$, two vertices $v_1, v_2 \in V$ are said to be *adjacent* provided that $\{v_1, v_2\} \in E$. Elements $v \in V$, $e \in E$ are called *incident* exactly when $e = \{v,u\}$, for some $u \in V$. For any element $v \in V$, the *neighborhood* of v in G, denoted $N_G(v)$, is defined to contain exactly the vertices that are adjacent to v in G. That is, $N_G(v) = \{u \in V : \{u,v\} \in E\}$. The *degree* of a vertex v in G, denoted $deg_G(v)$, is defined to be $|N_G(v)|$. The neighborhood (respectively, degree) of v in G may be denoted $N(v)$ (respectively, $deg(v)$) when it is clear that the neighborhood (respectively degree) of v in G is being referred to. An element $v \in V$ such that $deg(v) = 0$ is called an *isolated vertex*. The *minimum degree* (respectively, *maximum degree*) among all vertices of a graph G is denoted $\delta(G)$(respectively, $\Delta(G)$). A graph $G = (V,E)$ is called *k-regular* if and only if, for each $v \in V$, $deg(v) = k$.

The *complement* G_c of a graph $G = (V,E)$ is defined to be the graph with vertex set V and edge set $\{\{u,v\} : u,v \in V \text{ and } \{u,v\} \notin E\}$; that is, the edge set of G_c contains exactly the edges between the vertices of G that are not edges of G. A graph and its complement are shown in Figures 1 and 2.

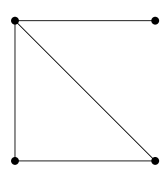

Figure 1 A graph

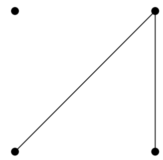

Figure 2 The complement of the graph in Figure 1

A graph $H=(V_H,E_H)$ is called a *subgraph* of $G=(V_G,E_G)$ if and only if $V_H \subseteq V_G$ and $E_H \subseteq \{\{u,v\} \in E_G : u,v \in V_H\}$. H is further said to be a *vertex-induced subgraph* of G exactly when $E_H = \{\{u,v\} : u,v \in V_H \textit{ and } \{u,v\} \in E_G\}$ (that is, if E_H contains every edge of G that is between each pair of elements of V_H).

2.2 Classes of Graphs

Next, we define several classes of graphs that will be referred to in later sections. The definitions of these classes may also be found in [7].

A *path* P_n is a graph with n vertices v_i, $1 \le i \le n$ whose edge set has the form $\{\{v_1,v_2\},\{v_2,v_3\}...,\{v_{n-2},v_{n-1}\},\{v_{n-1},v_n\}\}$. The graph P_5 is shown in Figure 3.

Figure 3 The path on 5 vertices, P_5

A *cycle* C_n is a graph with n vertices v_i, $1 \le i \le n$ with edge set $\{\{v_1,v_2\},\{v_2,v_3\}...,\{v_{n-2},v_{n-1}\},\{v_{n-1},v_n\},\{v_n,v_1\}\}$. The graph C_n may also be defined as a 2-regular graph on n vertices. The graph C_6 is shown in Figure 4.

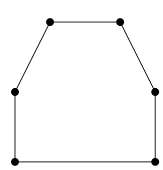

Figure 4 The cycle on 6 vertices, C_6

A graph is called *cyclic* if and only if it contains a cycle as a subgraph and *acyclic* if and only if it is not cyclic. A graph is said to be *unicyclic* if and only if it contains exactly one cycle as a subgraph.

A graph G is said to be a *cactus* if and only if, for each pair of distinct cycle subgraphs H_1, H_2 of G, the set $V(H_1) \cap V(H_2)$ contains at most one element.

$G = (V,E)$ is said to be *connected* if and only if its vertex set satisfies the following property: For any $v', v'' \in V(G)$, there exists a path between v' and v'' that is a subgraph of G. A *connected component*, or simply a *component*, of a graph, is a maximal connected subgraph of that graph.

An acyclic graph is often called a *forest*. A forest is shown in Figure 5. An acyclic graph that is also connected is called a *tree*. The labels "forest" and "tree" are appropriate since, under the given definitions, a forest is a graph in which each connected component is a tree. A graph $G = (V,E)$ is referred to as *complete* if and only if for each $u, v \in V$, $\{u,v\} \in E$. The complete graph on n vertices is denoted K_n. A complete graph is also referred to as a *clique*, and the graph K_n can be referred to as the n-clique. The graph K_5 is shown in Figure 6.

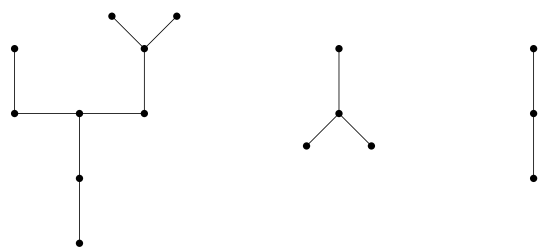

Figure 5 A forest

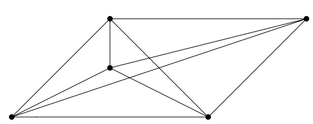

Figure 6 The complete graph on 5 vertices, K_5

A graph $G = (V,E)$ that can be constructed from K_{k+1} by adding one vertex at a time such that each added vertex v is adjacent to exactly k vertices when it is added, where these k vertices induce K_k in G, is called a *k-tree*. Note that, under this definition, unless a graph G consists of a single vertex, G

is a tree if and only if it is a 1-tree. A 2-tree is shown in Figure 7.

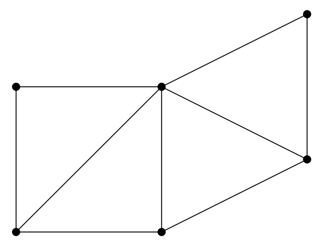

Figure 7 A 2-tree

G is called a *linear singly edge-articulated cycle graph* (or an *LSEAC graph*) if and only it is G has k induced cycles $C^1, C^2 ... C^k$ such that $\bigcup_{i=1}^{k} E(C^i) = E(G)$, of which there is an ordering $\{C^i : 1 \leq i \leq k\}$ such that, for $i \geq 2$, $j \leq i-2$, C^i shares exactly one edge with C^{i-1}, and does not share any vertices with C^j. An LSEAC graph is shown in Figure 8.

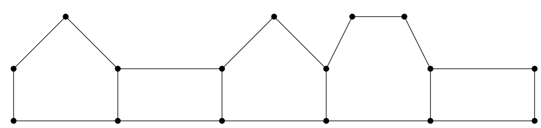

Figure 8 An LSEAC graph

A *ciclo* $C_k(H)$ is a graph that can be constructed from C_k by adding k copies of H, each of which shares a unique edge with C_k. The graph $C_4(K_3)$ is shown in Figure 9.

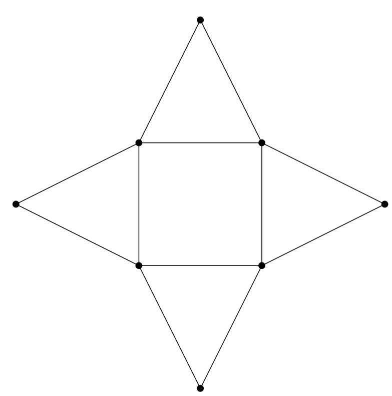

Figure 9 The ciclo $C_4(C_3)$

A graph is called *m-partite* exactly when there exist m subsets $V_1 \ldots, V_m$ which partition $V(G)$ such that if $u, v \in V_i$, for any $1 \leq i \leq m$, $\{u, v\} \notin E(G)$(in the case that $m = 2$, an m-partite graph may be referred to as *bipartite*). A *complete m-partite graph G* is an m-partite graph with partitioning sets $V_1 \ldots, V_m$ such that, if $u \in V_i, v \in V_j$, for $i \neq j$, then $\{u, v\} \in E(G)$. A complete m-partite graph with partitioning sets $V_1, V_2 \ldots, V_m$ is denoted $K_{|V_1|,|V_2| \ldots, |V_m|}$. The graph $K_{1,k}$ is referred to as a *k-star*. Note that, in a k-star with $k \geq 3$, exactly one vertex has degree greater than 2. A *generalized k-star*, for $k \geq 3$, is defined to be a tree T in which exactly one vertex v has degree greater than 2. The graphs $K_{1,5}$ and $K_{3,4}$ are shown in Figure 10 and Figure 11, respectively.

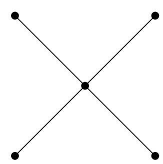

Figure 10 The 5-star

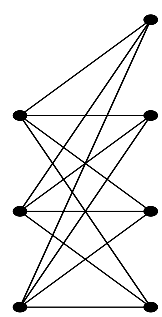

Figure 11 The complete bipartite graph $K_{3,4}$

The *m,k-pineapple* is the graph obtained from K_m by adding k vertices, each of which has degree one in the resulting graph and is adjacent to the same vertex $v \in V(K_m)$. The 5,3-pineapple is shown in

Figure 12.

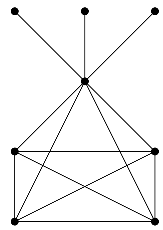

Figure 12 The 5,3-pineapple

The *halfgraph* H_s on $2s$ vertices is defined as that such that $V(H_s) = U \cup W$, where $U = \{u_i : 1 \leq i \leq s\}$ and $W = \{w_j : 1 \leq j \leq s\}$, and $E(H_s) = \{\{u_i, w_j\} : i \leq j\}$. The graph H_5 is shown in Figure 13.

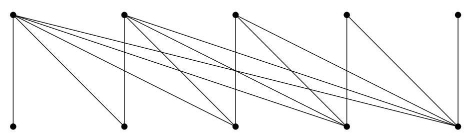

Figure 13 The halfgraph on 10 vertices, H_5

Let $G = (V,E)$ be a graph, and label the edges of G $e_1, e_2 ..., e_{|E|}$. Then the *line graph of G*, denoted $L(G)$, is the graph such that $V(L(G)) = \{v_i : 1 \leq i \leq |E|\}$, and $E(L(G)) = \{\{v_i, v_j\} :$ e_i and e_j share exactly one vertex in G$\}$. A tree and its line graph are shown in Figure 14.

Figure 14 A tree and its line graph

A few operations between two graphs will be used. If G^* is a subgraph of G, then the *difference* G-G^* is the subgraph of G induced by $V(G) \setminus V(G^*)$. The graph K_4-K_1, K_1 being by definition of complete graph the graph on a single vertex, is shown in Figure 15.

Figure 15 K_4 and the difference $K_4 - K_1$

If $G_1 = (V_1, E_1)$, and $G_2 = (V_2, E_2)$ are graphs, then the *join* of G_1 and G_2, $G_1 \vee G_2$, is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup \{\{v', v''\} : v' \in V_1, v'' \in V_2\}$. The graph $P_4 \vee C_3$ is shown in Figure 16.

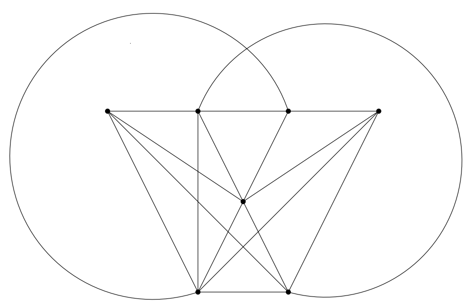

Figure 16 The join $P_4 \vee C_3$

Let $G' = (V, E')$, $G'' = (U, E'')$ be graphs, and let $V = \{v_i : 1 \leq i \leq k'\}$, $U = \{u_j : 1 \leq j \leq k''\}$. Then the *Cartesian product* of G_1 and G_2, denoted $G_1 \diamond G_2$, is the graph with vertex set $\{w_{ij} : 1 \leq i \leq k', 1 \leq j \leq k''\}$ and edge set $\{\{w_{i_1,j_1}, w_{i_2,j_2}\}$: either $\{u_{i_1}, u_{i_2}\} \in E'$ and $j_1 = j_2$ or $\{v_{j_1}, v_{j_2}\} \in E''$ and $i_1 = i_2\}$. The graph $P_3 \diamond C_3$ is shown in Figure 17.

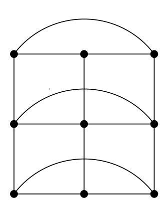

Figure 17 The Cartesian product of P_3 and C_3, $P_3 \diamond C_3$

If G_1, G_2 are graphs, then the *corona* of G_1 with G_2, $G_1 \circ G_2$, is the graph that is obtained from

G_1 by adding $|G_1|$ copies of G_2, each of which shares a unique vertex with G_1. The graph $C_5 \circ K_3$ is shown in Figure 18.

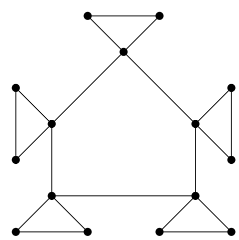

Figure 18 The corona of C_5 with K_3, $C_5 \circ K_3$

The final graph class necessary to define depends on the notion of the Cartesian product operation. The *hypercube graph on 2 vertices*, denoted Q_1, is defined to be K_2, and the *hypercube graph on* 2^n *vertices*, denoted Q_n, is defined to be $Q_{n-1} \diamond K_2$. The graphs Q_2 and Q_3 are shown in Figure 19.

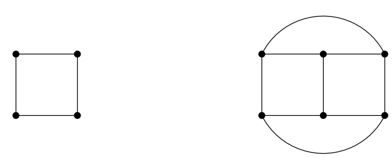

Figure 19 The hypercubes Q_2 and Q_3

2.3 Graphs for which Minimum Rank Is Known

We next survey a few results regarding the minimum ranks of graphs in certain classes. The following theorems each give the minimum rank for graphs in one of the graph classes defined in the last section. Several of the graph classes for which minimum rank is given below is one of the graphs for which minimum labeling degree is calculated in Chapter 5.

Theorem 2.1: If G is connected and $|G| \geq 2$, then $mr(G) = 1$ if and only if $G = K_n$, for some $n \geq 2$. [11]

Theorem 2.2: For any graph G, $mr(G) = n - 1$ if and only if $G = P_n$, for some n. [11]

Theorem 2.3: If G is a cycle on n vertices, where $n \geq 3$, then $mr(G) = n - 2$. [16]

Theorem 2.4: If $G = K_{m,n}$ with $n \geq m, 2$, and $= 2$ otherwise. [10]

Theorem 2.5: If G is the complement of a cycle, then $mr(G) = 3$. [1]

Theorem 2.6: If G is the complement of a tree on at least 4 vertices and G_c is not a star, then $mr(G) = 3$. [1]

Theorem 2.7: If $G = P_s \diamond C_t$, then $mr(G) = st - min(s, 2t)$. [1]

Theorem 2.8: If $G = H_s$, the halfgraph on $2s$ vertices, then $mr(G) = s$. [8]

Theorem 2.9: If $G = L(T)$, where T is a tree, then $mr(G) = |T| - k$, where k is the number of vertices of degree one in T. [1]

2.4 Characterizations of Graphs with Certain Minimum Rank

For some numbers k, the set of graphs G for which $mr(G) = k$ or $mr(G) \leq k$, under certain conditions, has been characterized. Each of the following theorems provides such a characterization. Note that Theorems 2.1 and 2.2 give classifications of the graphs G for which $mr(G) = 1$ and $mr(G) = 2$, respectively.

Theorem 2.10: If G is connected, then $mr(G) = 2$ if and only if $G = \bigvee_{i=1}^{r} G_i$, for some $r > 1$, where either:

. (a) $G_i = K_{m_i} \cup K_{n_i}$, for some $m_i \geq 1$, $n_i \geq 0$

(b) $G_i = K_{m_i}$, for some $m_i \geq 3$,

and option (b) occurs at most twice. [4]

Theorem 2.11: If G is 2-connected, then $mr(G) = |G| - 2$ if and only if G is a linear 2-tree. [14]

2.5 Relevant Results on Minimum Positive Semidefinite Rank

The following are a couple of results related to minimum positive semidefinite rank that are relevant to Chapter 6. The second of these has to do with the notion of an orthogonal representation of a graph. If $G = (V, E)$ is a graph on n vertices, then a *k-dimensional orthogonal representation* of G is a bijective function r from D to V, where D is a set of n nonzero vectors in $\mathbb{R}^k$, such that $w_1, w_2 \in D$ are orthogonal if and only if $\{r(w_1), r(w_2)\} \notin E$.

Theorem 2.12: If $G_c = C_n$ for some $n \geq 5$, then $mr_+(G) = 3$.

Theorem 2.13: Let $Y = (V_Y, E_Y)$ be a graph such that the order of Y is at least two, Y does not contain a set of four independent vertices, and there is an orthogonal representation r of Y in $\mathbb{R}^4$ satisfying, for all $u_i, u_j, u_k \in \mathbb{R}^4$ mapped respectively by r to distinct $v_i, v_j, v_k \in V_Y$,

(1) $u_i \notin span(u_j)$

(2) $dim(span(u_i, u_j, u_k)) = 3$ if $\{v_i, v_j, v_k\}$ does not induce a cycle in Y.

Let X be a graph that can be constructed starting with Y and adding one vertex at a time, such that the newly added vertex is adjacent to all prior vertices except at most two nonadjacent vertices. Then there is an orthogonal representation of X in $\mathbb{R}^4$ satisfying (1) and (2); in particular, $mr(X) \leq mr_+(X) \leq 4$ [13]

In the following result, Theorem 2.14, a k-tree is understood to be defined slightly differently that it was above. In the paper in which the following theorem is proved, a k-tree given to be a graph constructed from K_k (rather than K_{k+1}) by adding one vertex at a time, where each added vertex is adjacent to exactly k vertices at the point that it is added, such that these k vertices form K_k. Thus, the only difference between the set of k-trees as defined in section 2.2 as the set of k-trees as defined in the paper in which Theorem 2.14 occurs is that the latter contains K_k and the former does not. After Theorem 2.14, the term k-tree will refer to the notion thereof defined in section 2.2 for the remainder of the thesis.

Theorem 2.14: If G is the complement of a 2-tree, then $mr_+(G) \leq 4$. [13]

Theorem 2.15: If G is the complement of a unicyclic graph, then $mr_+(G) \leq 4$. [13]

CHAPTER 3

KNOWN AND CONJECTURED BOUNDS AND PARTIAL CHARACTERIZATIONS OF $mr(G)$ AND $mr_+(G)$ USING OTHER GRAPH PARAMETERS

We now examine known and conjectured relationships between the two minimum rank parameters of interest and other graph parameters.

3.1 Known Bounds and Partial Characterizations of $mr(G)$

Two parameters for which there exists an expression in terms of the parameter *mr* for at least some graph classes are the path cover number $P(G)$ and maximum multiplicity $M(G)$. A *path cover* of G is a partition of $V(G)$ such that each element of the partition induces a path in G. The *path cover number* of G is the minimum cardinality among all path covers of G. The *maximum multiplicity* of a graph G, $M(G)$, is defined as the maximum multiplicity among all eigenvalues of matrices in $S(G)$.

Theorem 3.1: If G is a tree, then $|G| - mr(G) = P(G) = M(G)$.[9]

In fact, one equality from this theorem holds for graphs in general.

Theorem 3.2: If G is a graph, then $|G| - mr(G) = M(G)$. [11]

Another parameter that has been used to bound the value of $mr(G)$ has to do with associating each of the vertices of G with one of two colors. Suppose that the vertices of G are partitioned into two subsets V' and V'' such that the vertices in V' are black, and those in V'' are white. Suppose that the following *color-change rule* (CCR) is applied to $V(G)$: Change the color of a vertex v in V'' to black if and only if v is the unique white neighbor of a vertex in V'. The set V' is called a *zero-forcing set* of G if and only if repeatedly applying CCR until it can no longer be applied results in all vertices of G being black. The minimum cardinality among all zero-forcing sets of G is called the *zero-forcing number* of G, and written $Z(G)$. [2] The following theorem relates this parameter to minimum rank.

Theorem 3.3: For any graph G, $Z(G) \geq M(G)$.[1]

From this it follows that a lower bound for $mr(G)$ is the difference $|G| - Z(G)$.

Another parameter that has been used to obtain an upper bound on the value of $mr(G)$ is the clique cover number. A *clique cover* of G is a collection of complete subgraphs of G, such that each edge of G is contained in at least one of the elements of the collection. The *clique cover number* of G, denoted $cc(G)$, is the minimum number of elements in a clique cover of G.

Theorem 3.4: If G is a graph, then $mr(G) \leq cc(G)$.[11]

3.2 A Conjectured Upper Bound of $mr(G)$

We now discuss a *proposed* upper bound on the parameter $mr(G)$, given in what is known as the δ-conjecture.

Conjecture 3.5 (Form 1 of δ-conjecture): If G is a graph, then $mr(G) \leq |G| - \delta(G)$.[5]

Since the minimum degree of a graph is equal to one more than the maximum degree of its complement, the δ-conjecture can also be stated as follows:

Conjecture 3.6 (Form 2 of δ-conjecture): If G is a graph, then $mr(G) \leq \Delta(G_c) + 1$.

Equality between $mr(G)$ and $\Delta(G_c) + 1$ is attained when G is, for example, the complement of a cycle. For any such graph G, $\Delta(G_c) + 1 = 2 + 1 = 3 = mr(G)$. However, $\Delta(G_c)$ can be much larger, and in fact arbitrarily larger, than $mr(G)$. For instance, consider the graph G composed of a complete graph on k vertices and an isolated vertex, denoted $K_k \cup \{v\}$, whose complement is the k-star (the 5-star is shown in Figure 10). In this case, clearly $\Delta(G_c) = k$. In any matrix in $S(K_k \cup \{v\})$, a non-diagonal entry corresponding to a pair of vertices (u_1, u_2) is nonzero if and only if $u_1, u_2 \in V(K_k)$. So there is an $A \in S(K_k \cup \{v\})$ such that the columns corresponding to the vertices of K_k are equal, and the column corresponding to v is the zero vector. This matrix A has rank 1, and since there is no element of $S(K_k \cup \{v\})$ having rank 0, $mr(K_k \cup \{v\}) = 1$. Thus $\Delta(G_c)$ is $k - 1$ greater than $mr(G)$ in this case. Hence, the difference between $\Delta(G_c)$ and $mr(G)$ can be arbitrarily large if G_c is a k-star.

As will be discussed in section A.1, the arbitrarily large discrepancy between $mr(G)$ and $\Delta(G_c)$ for the case that G_c is a k-star is due to the fact that the neighborhood of the vertex of maximal degree in a k-star is a set of what are known as duplicate vertices.

3.3 Known Bounds and Partial Characterizations of $mr_+(G)$

We now survey some bounds and partial characterizations of minimum positive semidefinite rank of a graph.

Recall that a *k-dimensional orthogonal representation* for G is a bijective function r from D

to V, where D is a set of n nonzero vectors in R^k, such that $w_1, w_2 \in D$ are orthogonal if and only if $\{r(w_1), r(w_2)\} \notin E$. The parameter $d(G)$ denotes the minimum dimension among all orthogonal representations of G. [13]

Fact: If A is positive semidefinite and symmetric, then A can be expressed as $W^T W$, where W is an $k \times n$ matrix, with $k = rank(A)$.

In what follows, it is proved using this Fact that, if G has no isolated vertices, then $mr_+(G) = d(G)$. The Fact will be denoted (*).

Let G be a graph with no isolated vertices and let $A \in S_+(G)$. By (*), $A = W^T W$, where W is a $k \times n$ matrix. So, an entry $A_{ij} = 0$ if and only if the inner product $W_i * W_j = 0$, where W_i,W_j are the ith and jth columns of W, respectively. In other words, $A_{ij} = 0$ if and only if W_i,W_j are orthogonal. By definition of $S_+(G)$, an entry $A_{ij} = 0$ exactly when the vertices $v_i, v_j \in V(G)$ are not adjacent, and so it follows that $v_i, v_j \in V(G)$ are not adjacent if and only if the column vectors W_i,W_j are orthogonal. Because, by (*), the number of rows of W is the rank of A, it follows that, if each element of a set of $|G|$ k-dimensional vectors can be associated with a vertex of G such that two of the vectors are orthogonal exactly when the two associated vertices are not adjacent, then $mr_+(G) \leq k$. Since G has no isolated vertices, each element of any such set of $|G|$ k-dimensional vectors must be nonzero. Thus, such a set is the domain of a k-dimensional orthogonal representation of G. So, $mr_+(G) \leq k$ where k is the dimension of some orthogonal representation of G. There cannot be an orthogonal representation r of G of dimension k' smaller than k, since then the vectors in the domain of r would form the rows of a matrix W' such that $A' = {W'}^T W'$ for some $A' \in S_+(G)$, where k' is the rank of A', which contradicts the assumption that A has minimal rank among matrices in $S_+(G)$. Hence, $mr_+(G) \leq d(G)$.

If we suppose that $mr_+(G) < d(G)$, then there must exist $A'' \in S_+(G)$ that satisfies $rank(A'') = x < d(G)$. Since A'' is positive semidefinite, A'' can be expressed as ${W''}^T W''$, where W'' is an $x \times n$ matrix. Then the n x-dimensional vectors that are the columns of W'' form an orthogonal representation of degree x for G, contradicting the assumption that $d(G) > x$. So, by contradiction, there is no $A \in S_+(G)$ such that $rank(A) < d(G)$. Therefore, if G has no isolated vertices, then $mr_+(G) = d(G)$.

The *minimum semidefinite rank* of a graph G on n vertices $v_i, 1 \leq i \leq n$, denoted $msr(G)$, is the minimum rank among matrices in the set $S_H(G)$, defined as that containing exactly the matrices with *complex* entries such that, for $i \neq j$, the ijth entry is nonzero if and only if the ith and jth vertices of G are adjacent.

Since the set of matrices of which $msr(G)$ is the minimum rank contains that of which $mr_+(G)$ is the minimum rank, it follows that $msr(G) \leq mr_+(G)$.[15]

In [6], three lower bounds for $msr(G)$ (and thus, for $mr_+(G)$), where G is connected, are proved. One of the bounds has to do with a parameter called the forest measure of G, which is defined in terms of the tree size of G. The *tree size* of G is the maximum order among all induced trees of G, and is denoted $ts(G)$. The *forest measure* of G, denoted $fm(G)$, is the maximum value, among all induced forests F of G, of the quantity $ts(T_1) + ts(T_2) + ts(T_k) - c = |T_1| + |T_2| + |T_k| - c$, where the $T_i, 1 \leq i \leq k$, are the components of F, and c is the number of the T_i that are not isolated vertices.

Theorem 3.7: If G is a connected graph, then $mr_+(G) \geq fm(G)$.[6]

Another bound to which mr_+ is shown to be related is a parameter known as the independence number. If G is a graph, then an *independent set of vertices* $V' \subseteq V(G)$ is one such that there is no edge in G between any pair of elements of V'. The maximum cardinality among all independent sets of vertices in G is called the *independence number* of G. A general relationship between $mr_+(G)$ and $i(G)$ can be derived from Theorem 3.7 as follows.

Note that, if F is an induced forest of G, then any components of F that are not isolated vertices have at least two vertices. Since the expression for $fm(G)$ only subtracts one for each component that is not an isolated vertex, each component of an induced forest that achieves $fm(G)$ contributes at least one to the value of $fm(G)$. Finally, note that the graph induced by a maximal independent set of vertices of G is an induced forest of G. These considerations together imply that $fm(G) \geq i(G)$. Hence, the following is implied by Theorem 3.7.

Theorem 3.8: If G is a connected graph, then $mr_+(G) \geq i(G)$.[6]

Because each component of an induced forest F of G that achieves $fm(G)$ contributes at least one to the value of $fm(G)$, it follows that $fm(G) \geq ts(G) - 1$. Thus, the following is also true.

Theorem 3.9: If G is a connected graph, then $mr_+(G) \geq ts(G) - 1$.[6]

Another lower bound for $mr_+(G)$ is what is known as the OS-number of G, denoted $OS(G)$, that has to do with certain ordered subsets of $V(G)$ called OS-sets. Let G be a graph on n vertices. An *OS-set* of a graph G is an ordered set $O = \{v_i \in V(G) : 1 \leq i \leq m\}$ such that, for each $v_j \in O$, there exists $w \in V(G)$ such that $\{w, v_j\} \in E(G)$ and $\{w, v_i\} \notin E(G)$ for any $i < j$ for which v_i, v_j are in the same component of the subgraph of G induced by $\{v_i \in O : i \leq j\}$. The *OS-number* of G is defined to be the maximum cardinality among all OS-sets of G.[12] In the following example, a maximal OS-set is constructed for a certain class of graphs.

Example 3.10: Let G be a cycle such that $|G| = n$. Let $O = v_1, v_2 ..., v_{n-2}$, where v_i is adjacent to v_{i-1} for $i \geq 2$. Let u' be the vertex that is adjacent to v_{n-2} besides v_{n-3}. If $i < n-2$, then in any subset O^* of O containing exactly v_i and all vertices that occur before v_i in O, v_{i+1} is a vertex outside of O^* that is adjacent to v_i but not adjacent to any other vertex in the same component of the subgraph induced by O^* (or in fact, to any vertex in this subgraph, period). If $i = n-2$, then u' is a vertex that is adjacent to v_i but not to any other vertex in O. So, O is an OS-set. Suppose for contradiction that O' is an OS-set of G having $n-1$ vertices. Then the one vertex w in $V(G) - O'$ must be adjacent to the last vertex u in O' but not be adjacent to any other vertices in the same component as u in the subgraph of G induced by O'. Since G is a cycle, there is a vertex u^* besides u in O' to which w is adjacent. By definition of OS-set, u^* must lie in a different component of the subgraph of G induced by O' than u. But this cannot be the case since O' contains $n-1$ vertices and G is 2-connected. So, O is a maximal OS set of G.

Theorem 3.11: For any graph G, $OS(G) \leq mr_+(G)$. [12]

CHAPTER 4

THE MINIMUM LABELING DEGREE: DEFINITION, CALCULATION EXAMPLES, AND A COUPLE OF ELEMENTARY RESULTS

The parameter that we define in this chapter, the minimum labeling degree, is similar to the OS-number in that it is defined in terms of ordered sets of vertices (although, as will be seen, the parameter is defined in terms of orderings of the whole vertex set, and we use the term “vertex labelings” instead of “orderings”).

4.1 Definition of the Parameter and Calculation for a Couple of Graph Classes

For a finite set P with k elements, a *labeling* f is a bijection between P and $\{j : 1 \leq j \leq n\}$. In the following, a labeling of the vertex set of a given graph G will be referred to as a *vertex labeling* of G.

Definition: The *degree of a vertex labeling* f of G is the maximum number of vertices to which any $v' \in V(G)$ is adjacent in the subgraph induced by the set $\{v \in V(G) : f(v) \leq f(v')\}$. The degree of a vertex labeling f of G will be denoted $ld_G(f)$. The minimum degree among all possible vertex labelings of G will be called the *minimum labeling degree* of G, and denoted $mld(G)$.

One class of graphs for which it is simple to calculate the minimum labeling degree is that of k-trees. Recall that a k-tree is a graph that can be built beginning with K_{k+1} and adding one vertex at a time, such that each added vertex v is adjacent to exactly k vertices when it is added, where these k vertices induce K_k in G. The calculation of $mld(G)$ if G is a k-tree follows easily from the definitions of $mld(G)$ and k-tree.

Example 4.1: If G is a k-tree, then $mld(G) = k$.

Proof. Let G be a k-tree on n vertices. Denote the $k+1$ vertices of the original K_{k+1} by u_i, for $1 \leq i \leq k+1$, and let v_1, v_2...,v_{n-k-1} be the vertices that are added to K_{k+1} to form G. Without loss of generality, assume that the v_i were added in order of increasing subscript. The labeling f for which $f(u_i) = i$, for

$1 \leq i \leq k+1$ and $f(v_j) = k+j+1$, for $1 \leq j \leq n-k$ clearly has degree k. So $mld(G) \leq k$. Since G has minimum degree k, the last vertex w that occurs in any vertex labeling f^* of G has degree at least k in the subgraph of G induced by $\{u \in V(G) : f^*(u) \leq f^*(w)\}$(i.e., G). Thus, $mld(G) = k$. □

The following example shows the calculation of mld for a cycle.

Example 4.2: If G is a cycle, then $mld(G) = 2$.

Proof. Let G be a cycle on n vertices, and let f be an arbitrary vertex labeling of G. Since $\Delta(G) = 2$, it is clear that $mld(G) \leq 2$. Let the $n-1$ vertices u for which $f(u) \leq n-1$ of G be given. Because a cycle is 2-regular, the only remaining vertex v of G must have exactly two neighbors in the subgraph of G induced by $\{u \in V(G) : f(u) \leq f(v)\}$ (i.e., G). Since f was chosen to be an arbitrary vertex labeling of G, $mld(G) \geq 2$. Thus, $mld(G) = 2$. □

The next proposition expresses the mld of a disconnected graph in terms of the mld of each of its components.

Proposition 4.3: If G is disconnected with components G_1,G_2...,G_k, then $mld(G) = max(\{mld(G_i) : 1 \leq i \leq k\})$.

Proof. Let G be disconnected with connected components G_i, $1 \leq i \leq k$. For $1 \leq i \leq k$, let f_i be a vertex labeling of G_i with degree $mld(G_i)$. Consider the vertex labeling f of G such that, for all $v \in V(G)$, $f(v) = f_i(v) + |G_1|... + |G_{i-1}|$ when $v \in V(G_i)$. By definition of connected components, there are no edges $\{v', v''\} \in G$ such that v', v'' are in different components. Hence, $ld_G(f) \leq max(\{mld(G_i) : 1 \leq i \leq k\})$. So, $mld(G) \leq max(\{mld(G_i) : 1 \leq i \leq k\})$.

Now let f' be a vertex labeling of G with degree $mld(G)$. Assume without loss of generality that G_1 is the component of G that attains the maximum mld among all connected components of G. Let ${f_1}'$ be the labeling of all of the vertices of G_1 so that, for any $v^*, v^{**} \in V(G_1)$, ${f_1}'(v^*) \geq {f_1}'(v^{**})$ if and only if $f'(v^*) \geq f'(v^{**})$. Clearly, $ld_{G_1}({f_1}') \leq ld_G(f')$. Because f_1' is a vertex labeling of G_1, it follows that $mld(G_1) \leq ld_{G_1}({f_1}')$ (I) also. Since $ld_G(f') = mld(G)$ and $mld(G_1) = max(\{mld(G_i) : 1 \leq i \leq k\})$ by assumption, the inequality (I) can be written as $max(\{mld(G_i) : 1 \leq i \leq k\}) \leq mld(G)$.

Hence, the equality that constitutes the proposition has been shown. □

The next example shows that the mld of a cactus is ≤ 2 by showing that there is a vertex labeling of a cactus that has degree 2. The reasoning used in the proof is similar to that used to prove the necessary condition in the first proposition in the next section.

Example 4.4: If G is a cactus, then $mld(G) \leq 2$.

Proof. Clearly, since G is a cactus, G has minimum degree ≤ 2. Let V_1 be the set of vertices of G having degree ≤ 2, and let G_1 be the subgraph of G induced by $V(G) \setminus V_1$. Then G_1 is a cactus, and so G_1 has minimum degree ≤ 2. Let V_2 contain exactly the vertices of G_1 having degree ≤ 2 in G_1, and define G_2 to be the subgraph of G_1 induced by $V(G) \setminus (V_1 \cup V_2)$. Then G_2 is a cactus, and thus G_2 has minimum degree ≤ 2. By continuing deleting the set V_i of vertices of degree ≤ 2 in the cactus subgraph G_{i-1} of G_{i-2} to attain a cactus subgraph G_i of G_{i-1}, a subgraph, say G_{k-1}, will eventually be obtained such that $\Delta(G_{k-1}) \leq 2$. If we then define V_k to contain exactly the vertices of G_{k-1}, then a vertex labeling of G in which the first $|V_k|$ vertices are the elements of V_k (in any order), the next $|V_{k-1}|$ vertices are the elements of V_{k+1} (in any order)..., and the last $|V_1|$ vertices are the elements of V_1 (in any order) is a vertex labeling of G with degree 2. Hence, it follows that $mld(G) \leq 2$. □

4.2 A Labeling-Independent Conception of Minimum Labeling Degree

The following result leads to a characterization of graphs with *mld* m which is not in terms of the notion of vertex labeling.

Proposition 4.5: A graph G has a subgraph with minimum degree m if and only if $m \leq mld(G)$.

Proof. $(->)$ Let G be a graph that contains a subgraph with minimum degree m. Suppose that there are x vertices in this subgraph. Let v_x be the last of these x vertices that occurs in an arbitrary vertex labeling f of G. Then v_x is adjacent to at least m vertices whose images under f are smaller than that of v_x. Since f was assumed to be arbitrary, it can be concluded that every vertex labeling for G has degree $\geq m$. Thus, $mld(G) \geq m$.

$(<-)$ Assume that G is a graph which does not contain a subgraph with minimum degree m, and let $V(G) = \{v_i : 1 \leq i \leq n\}$, where $n = |G|$. Since G does not have a subgraph with minimum degree m, there is a nonempty set $V_1 \subseteq V(G)$ that contains exactly the vertices that are adjacent to fewer than m vertices in $V(G)$. Further, because G does not have a subgraph with minimum degree m, the subgraph G_1 of G induced by $V(G) \setminus V_1$ has minimum degree less than m. Hence, there is a nonempty set $V_2 \subseteq V(G) - V_1$ that contains exactly the vertices of $V(G) \setminus V_1$ with degree $< m$ in G_1. Since G does not have a subgraph of degree m, the subgraph G_2 of G induced by $V(G) \setminus (V_1 \cup V_2)$ has minimum degree less than m. By continuing deleting sets of vertices V_i to obtain induced subgraphs G_i of G, a subgraph, say G_{k-1}, will eventually be obtained in which every vertex has degree $< m$. If V_k is defined to contain exactly the vertices of G_{k-1}, then the vertex labeling of G in which the first $|V_k|$ vertices are the elements of V_k (in any order), the next $|V_{k-1}|$ vertices are the elements of V_{k-1}(in any order)..., and the last $|V_1|$ vertices are the elements of V_1(in any order) has degree $< m$. So, by contraposition, if $mld(G) \geq m$, then G has a subgraph with minimum degree m. □

Corollary 4.6: If G is a graph, then $mld(G) = max(\{\delta(H) : H \text{ is a subgraph of } G\})$.

Proof. Note that because G has only a finite number of subgraphs, the set $\{\delta(H) : H \text{ is a subgraph of } G\})$ has a maximum. Let $M = max(\{\delta(H) : H \text{ is a subgraph of } G\})$. So G has a subgraph with minimum degree M, and thus the $(->)$ direction of Proposition 4.5 implies that $mld(G) \geq M$.
Note that because $mld(G) \leq mld(G)$, the $(<-)$ direction of Proposition 4.5 implies that G has a subgraph with minimum degree $mld(G)$. So $mld(G)$ is an element of the set of which M is the maximum, and hence $mld(G) \leq M$. So $mld(G) = M$, which is the desired result. □

Note that any subgraph of G on some set of vertices has at most the number of edges as the subgraph of G induced by that set of vertices. Thus, if H is a subgraph of G on a set V' of vertices, and H' is the subgraph of G induced by V', then $\delta(H) \leq \delta(H')$. Hence, $max(\{m = \delta(H) : H \text{ is a subgraph of } G\}) = max(\{\delta(H) : H \text{ is an induced subgraph of } G\})$, and so the latter is equal to $mld(G)$ as well.
Example 4.7: If G is a cyclic cactus, then $mld(G) = 2$.

Proof. Since G is a cactus, the result of Example 4.4 implies that $mld(G) \leq 2$. Since G is cyclic, G has a subgraph with minimum degree 2, and thus $mld(G) \geq 2$ by Proposition 4.5. Hence, $mld(G) = 2$. □

4.3 A Couple of Results Regarding the Relationship between $mld(G)$, $\delta(G)$, and $\Delta(G)$

Corollary 4.6 makes it easy to see that the following is true:
Proposition 4.8: For any graph G, $\delta(G) \leq mld(G) \leq \Delta(G)$.

Proof. The left-hand inequality follows from Corollary 4.6 and the fact that G is a subgraph of itself.
To see the right-hand inequality, note that, for any subgraph H of G, $\delta(H) \leq \Delta(H) \leq \Delta(G)$, and hence $max(\{\delta(H) : H \text{ is a subgraph of } G\}) \leq \Delta(G)$. So Corollary 4.6 implies that $mld(G) \leq \Delta(G)$. □

The next result states that, in effect, if G is connected, equality on the right in Proposition 4.8 is attained if and only if equality is attained on the left.

Proposition 4.9: If G is a connected graph, then $mld(G) = \Delta(G)$ if and only if $\delta(G) = \Delta(G)$.

Proof. $(<-)$ Let $\delta(G) = \Delta(G)$. Then Proposition 4.8 implies that $mld(G) = \Delta(G)$.
$(->)$ Now let $\delta(G) \neq \Delta(G)$. It will be shown that every subgraph of G has minimum degree $< \Delta(G)$, and therefore that $mld(G) < \Delta(G)$. Let $V' \subset V(G)$ contain exactly the vertices of G attaining maximal degree in G. Because $\delta(G) \neq \Delta(G)$, V' is a proper subset of $V(G)$ and equivalently, $V(G) \setminus V'$ is nonempty. Then any subgraph of G that contains a vertex outside of V' must have minimum degree less that $\Delta(G)$. We are now left to consider subgraphs of G whose vertex sets are subsets of V'. Suppose for contradiction that some subset V^* of V' induces a subgraph H of G with minimum degree $\Delta(G)$. Then all of the vertices in V^* have degree $\Delta(G)$ in H. So $deg_H(v) = deg_G(v)$ for all $v \in V^*$. Thus, no vertices of H have any neighbors outside of H in G. But then H is disconnected from the subgraph of G induced by $V(G) \setminus V^*$, and since $V(G) \setminus V^*$ is nonempty (this follows since $V(G) \setminus V'$ is nonempty and $V^* \subseteq V'$), this contradicts the assumption that G is connected. Hence, by contradiction, there is no subset of V' that induces a subgraph with minimum degree $\Delta(G)$, and as noted before the assumption for contradiction, no subset of $V(G)$ that is not contained in V' can induce a subgraph of minimum degree $\Delta(G)$. Thus, G cannot have a subgraph with minimum degree $\Delta(G)$. So $mld(G) < \Delta(G)$. By contraposition, this direction follows. $\square$

4.4 Comparison with Zero-forcing Number, Path Cover Number, and Treewidth

We next compare the parameter *mld* to three other graph parameters: the zero-forcing number, path cover number, and treewidth. It should be noted that Proposition 4.15 below (and Proposition 4.10, which is a weaker result that Proposition 4.15) is a straightforward consequence of [BBF+, Corollary 2.14].

We first derive a general relationship between *mld* and the zero-forcing number.

Note that the color-change rule (CCR) for the zero-forcing number states that, given a graph G and a set B of initially-black vertices, a vertex v changes from white to black if and only if v is the *unique* white neighbor of a black vertex in G. It follows that, in applying CCR with a given initial set of black vertices in G, each vertex that is black at some point can force a color change in another vertex at most one time. Thus, once a black vertex has forced a color change in a vertex, it cannot force a color change in any other vertex. This observation is used in the proof of the relationship between *mld* and the zero-forcing number.

Proposition 4.10: For any graph G, $mld(G) \leq Z(G)$.

Proof. Let $mld(G) = m$. Then G has a subgraph H with minimum degree m. Let v be a vertex of H having degree m. Let $U \subseteq V(G)$ be a set of initially-black vertices, and let $|U| = k < m$. It will be shown that U is a not a zero-forcing set of G. Let k' be the number of elements of U that are vertices of H. Note that for a vertex $v \in H$ to force a color-change in one of its neighbors, both v and all of its other at least $m-1$ neighbors in G must be black after some number of applications of the color-change rule. Thus, no vertex in H will be able to force another vertex to change color unless there are at least m black vertices in H. Since $k' \leq k < m$, it is not possible that any vertex v of H will force a color change in one of its neighbors initially. Suppose for contradiction that there is some n such that, after n applications of CCR, there are m black vertices in H. By the above observation that each black vertex forces at most one vertex to change color, it follows that there were at least $m-k'$ initially black vertices outside of H. But this contradicts the assumption that there were only $k-k'$ elements of U outside of H, with $k < m$. By contradiction, there is no n such that, after n applications of CCR, there are m black vertices in H. Thus, since there are clearly more than m vertices in H (because $\delta(H) = m$), it follows that U is not a zero-forcing set for G. Hence, $m \leq Z(G)$, and so $mld(G) \leq Z(G)$. □

Recall that an m-tree is a graph that can be constructed from K_{m+1} by adding one vertex at a time, such that each added vertex is adjacent to exactly m vertices at the point that it is added, where these m vertices induce K_m in G. A graph is called a *partial m-tree* if and only if it is a subgraph of an m-tree. If G is a graph, then the smallest m such that G is a partial m-tree is called the *treewidth* of G, and is denoted $tw(G)$. Theorem 4.11 (respectively, 4.12) gives a general relationship between the path cover number (respectively, treewidth) and zero forcing number.

Theorem 4.11: For any graph G, $P(G) \leq Z(G)$.[2]

Theorem 4.12: For any graph G, $tw(G) \leq Z(G)$.[3]

As the following examples show, there is no general inequality relationship between $mld(G)$ and $P(G)$.

Example 4.13: If G is a complete graph on n vertices, then it is known that $P(G) = \lceil n/2 \rceil$. However, since G is an $(n-1)$-regular graph, Corollary 4.6 implies that $mld(G) = n-1$. Hence, for $n \geq 4$,

$mld(G) > P(G)$ for $G = K_n$.

Example 4.14: If G is the k-star, then clearly $P(G) = k-1$. But since no subgraph of G has minimum degree greater than 1, it follows that $mld(G) = 1$. So for $k \geq 3$, $mld(G) < P(G)$ if $G = K_{1,k}$.

Hence, no general inequality holds between the parameters $mld(G)$ and $P(G)$. However, the following shows that such a relationship does exist between $mld(G)$ and $tw(G)$.

Proposition 4.15: For any graph G, $mld(G) \leq tw(G)$.

Proof. Let $tw(G) = m$. Then G is a subgraph of an m-tree G^*. Suppose for contradiction that $mld(G) > m$. Then Corollary 4.6 implies that G has a subgraph H of minimum degree $> m$. Because G is a subgraph of G^*, it follows that H is a subgraph of G^*. By Corollary 4.6, $mld(G^*) > m$, which contradicts the result of Example 4.1 that mld of an m-tree is m. Therefore, $mld(G) \leq m$, and since G is an arbitrary subgraph of an m-tree, $mld(G) \leq tw(G)$ for any graph G. □

CHAPTER 5

AN EXPRESSION IN TERMS OF mld THAT IS AN UPPER BOUND OF $mr(G)$ FOR G IN ANY OF SEVERAL GRAPH CLASSES

Note that, combining two propositions proved above, we have the following.

Corollary 5.1: For any graph G, $\delta(G) \leq mld(G) \leq Z(G)$. Hence, $|G| - Z(G) \leq |G| - mld(G) \leq |G| - \delta(G)$.

Recall the following statements, two of which are theorems and one of which is a conjecture.

Theorem 3.2: If G is a graph, then $|G| - mr(G) = M(G)$. [11]

Theorem 3.3: For any graph G, $Z(G) \geq M(G)$.[1]

Conjecture 4.1(Form 1 of δ-conjecture): For any graph G, $mr(G) \leq |G| - \delta(G)$.[5]

Hence the value of $|G| - mld(G)$ is between the values of a known lower bound and a conjectured (and thought by many to be correct) upper bound of $mr(G)$. This fact motivates the comparison of the value of $|G| - mld(G)$ with that of $mr(G)$. In each of the following examples, it is proved that $mr(G) \leq |G| - mld(G)$ for G in a certain class of graphs.

Example 5.2: If $G = K_{m,n}$, with $2 \leq m \leq n$, then $mr(G) = 2$. Since each vertex not attaining minimal degree in G is adjacent only to vertices having minimal degree in G, there cannot be a subgraph of G having minimum degree greater than that of G. So $mld(G) = \delta(G) = m$, and thus $|G| - mld(G) = n \geq 2 = mr(G)$.

Example 5.3: If $G = H_s$, the half-graph on $2s$ vertices, then $mr(G) = s$. Let $V(H_s) = U \cup W$, where $U = \{u_i : 1 \leq i \leq s\}$ and $W = \{w_j : 1 \leq j \leq s\}$, and let $E(H_s) = \{\{u_i, w_j\} : i \leq j\}$. Then the subgraph of G induced by $\{u_i : i \leq \lceil s/2 \rceil\} \cup \{w_j : j \geq \lceil s/2 \rceil\}$, has minimum degree $\lceil s/2 \rceil$, and so $mld(G) \geq \lceil s/2 \rceil$. Let H be a subgraph of G. If H contains $\leq \lceil s/2 \rceil$ elements of either one of U or W, then because each edge of G consists of one element of U and one element of W, H must have minimum degree $\leq \lceil s/2 \rceil$. Assume that H contains exactly $\lceil s/2 \rceil + 1$ elements from each of U and W. Then $V(H)$ must contain some $u_{i'}$ for which $i' \geq \lceil s/2 \rceil$+1 and must contain some $w_{j'}$ for which $j' \leq s - \lceil s/2 \rceil$. By definition of $\lceil s/2 \rceil$, $s - \lceil s/2 \rceil < \lceil s/2 \rceil + 1$, and so $w_{j'}$ is not adjacent to $u_{i'}$. Hence, $w_{j'}$ has degree $\leq \lceil s/2 \rceil$ in H.

So H has minimum degree $\leq \lceil s/2 \rceil$. Although H was assumed to have exactly $\lceil s/2 \rceil + 1$ vertices from each of U and W, it is clear from the above reasoning that any subgraph of G that has more than $\lceil s/2 \rceil$ vertices from each of U and W has vertices in W that have degree $\leq \lceil s/2 \rceil$. Thus, no such subgraph of G has minimum degree greater than $\lceil s/2 \rceil$. It has thus been shown G has a subgraph of minimum degree $\lceil s/2 \rceil$, and that no subgraph of G has minimum degree $> \lceil s/2 \rceil$. By Corollary 4.6, $mld(G) = \lceil s/2 \rceil$. Because $s \leq 2s - \lceil s/2 \rceil$, the inequality holds.

The next proposition will be made use of in the example immediately following it.

Proposition 5.4: If G is a graph such that there exist $|G| - m$ vertices of degree smaller than m in G, then $mld(G) < m$.

Proof. Let H be a subgraph of G, and let V' be a subset of $V(G)$ containing exactly $|G| - m$ vertices of degree smaller than m. If $V(H)$ contains a vertex of V', then clearly $\delta(H) < m$. On the other hand, if all vertices of H are in $V(G) - V'$, then $V(H)$ has at most m vertices, and so $\delta(H) < m$. Hence, $\delta(H) < m$, and since H was an arbitrary subgraph of G, it follows that every subgraph of G has minimum degree less than m. By Proposition 4.5, $mld(G) < m$. □

Example 5.5: In any tree T such that $|T| \geq 4$ and T is not a star, there must be two vertices of degree at least 2. In the complement T_c of such a tree, these two vertices have degree at most $|T| - 3$. Hence, T_c has two vertices of degree less than $|T| - 2$. By the last proposition, $mld(T_c) < |T| - 2$, and hence $mld(T_c) \leq |T| - 3$. So, $mr(T_c) = 3 \leq |T| - mld(T_c)$.

Example 5.6: The minimum rank of the line graph of a tree T is known to be $|T| - k$, where k is the number of vertices of degree 1 in T. Note that $L(T)$ has a complete subgraph on $\Delta(T)$ vertices. So $mld(L(T)) \geq \Delta(T) - 1$. Suppose that $mld(L(T)) > \Delta(T) - 1$. Then $L(T)$ has a subgraph $L(H)$ with minimum degree $m > \Delta(T) - 1$. Because T is acyclic (by definition of tree), H contains an $(m+1)$-star. Then H has a vertex of degree $m+1$, and thus T has a vertex of degree $\geq m+1$, which contradicts the inference that $m > \Delta(T) - 1$. Thus, $mld(L(T)) = \Delta(T) - 1$. Since T is acyclic, $k \geq \Delta(T)$, and so $k - 1 \geq mld(L(T))$. This implies that $|L(T)| - (k-1) \leq |L(T)| - mld(L(T))$. It is known that $|L(T)| = |T| - 1$. So, the left side of this inequality is $|T| - 1 - k + 1 = |T| - k = mr(L(T))$. Hence we have $mr(L(T)) \leq |L(T)| - mld(L(T))$.

The inequality $mr(G) \leq |G| - mld(G)$ was verified for a number of additional graphs G, including:

Complete graphs

Complete multipartite graphs

Unicyclic graphs

Complete, cycle, house, and full house ciclos

Cartesian products between a path and a path, a path and a cycle, a complete graph and a complete graph

Line graphs of complete graphs

Complements of cycles, complete k-partite graphs for $k \geq 2$, linear 2-trees

Coronas of a complete graph with a complete graph, a cycle with a complete graph

Pineapples

Hypercubes

For some graphs G in certain graph classes, $|G| - mld(G)$ gave a sharper bound on $mr(G)$ than $|G| - \delta(G)$. Among these graph classes were:

Unicyclic graphs

Complements of complete k-partite graphs for $k \geq 2$, linear 2-trees

Halfgraphs

Pineapples

Line graphs of trees

CHAPTER 6

THE MINIMUM POSITIVE SEMIDEFINITE RANK OF CERTAIN GRAPHS WHOSE COMPLEMENTS HAVE *mld* 2

In this chapter we consider what can be concluded about the minimum positive semidefinite rank of certain graphs whose complements have *mld* 2.

6.1 Graphs Whose Complements Have *mld* 2 and No $K_{3,2}$ Subgraphs

The following observation will be used in the proof of Proposition 6.2.

Observation 6.1: If S is a k-dimensional vector space over $\mathbb{R}$ and $S_1 ..., S_n$ are subspaces of S of dimension $< k$, then $S - \bigcup_{i=1}^{n} S_i$ is nonempty.

The next proposition establishes an upper bound on mr_+ of certain graphs whose complements have *mld* 2.

Proposition 6.2: Let G be a graph such that $mld(G_c) = 2$ and G_c does not contain $K_{3,2}$ as a subgraph. Then $mr_+(G) \leq 4$.

Proof. Let $|G| = n$. Let f be a vertex labeling of G_c of minimal degree, and for $1 \leq i \leq n$, denote the vertex v for which $f(v) = i$ as v_i. Let $j \leq n$ and let r_j be an orthogonal representation of the subgraph of G induced by $\{v_i : i < j\}$ such that any set of three domain vectors of r_j is linearly independent. Denote the domain of r_j by $U_j = \{u_i : i < j\}$. Since the degree of f is 2, the degree of v_j in the subgraph of G_c induced by $\{v_i : i < j+1\}$ is ≤ 2. It will be shown that there exists u_j such that $r_{j+1} = r_j \cup \{(u_j, v_j)\}$ is an orthogonal representation of the subgraph of G induced by $\{v_i : i < j+1\}$ in which any set of three domain vectors is linearly independent. Consider three cases:

Case 1: Assume that the degree of v_j in the subgraph of G_c induced by $\{v_i : i < j+1\}$ is 0. Then v_j is adjacent in G to each vertex in the subgraph of G induced by $\{v_i : i < j+1\}$. So, in order for there to be a vector u_j such that there is an orthogonal representation r_{j+1} with domain $U_j \cup \{u_j\}$ of the subgraph of G induced by $\{v_i : i < j+1\}$ in which each trio of domain vectors is independent,

$u_j \in R^4 \setminus \{(\bigcup_{i=1}^{j-1} < u_i >^{\perp}) \bigcup (\bigcup_{i_1=1}^{j-1} \bigcup_{i_2=1}^{j-1} < u_{i_1}.u_{i_2} >)\}$. Note that the subspaces $< v_i >^{\perp}$ are 3-dimensional, and the subspaces $< v_{i_1}, v_{i_2} >$ are 2-dimensional (because v_{i_1}, v_{i_2} are independent by assumption). Since, by Observation 6.1, removing a finite number of 2- and 3-dimensional subspaces from a 4-dimensional vector space over $\mathbb{R}^4$ cannot result in an empty space, it follows that an appropriate vector u_j can be chosen.

Case 2: Now assume that the degree of v_j in the subgraph of G_c induced by $\{v_i : i < j+1\}$ is 1, and assume without loss of generality that v_1 is the vertex to which v_j is adjacent. Then in order for a vector u_j to map to v_j by an orthogonal representation $r_{j+1} = r_j \cup \{(u_j, v_j)\}$, u_j must be orthogonal to u_1, and not be orthogonal to any other u_i, for $i < j$. Also, in order for r_{j+1} to be such that each set of three domain vectors is linearly independent, u_j must not be in the span of any pair of u_i, for $1 \leq i \leq j-1$. Thus, it must be that $u_j \in < u_1 >^{\perp} \setminus \{(\bigcup_{i=2}^{j-1} < u_i >^{\perp}) \bigcup (\bigcup_{i_1=1}^{j-1} \bigcup_{i_2=1}^{j-1} < u_{i_1}, u_{i_2} >)\}$. Note that $< u_1 >^{\perp}$ is a 3-dimensional space, and that because u_1 is independent of each of the u_i, for $2 \leq i \leq j-1$, $< u_1 >^{\perp} \cap < u_i >^{\perp}$ can be at most a 2-dimensional space. Note also that each of the $< u_{i_1}, u_{i_2} >$ are 2-dimensional spaces, and so the space from which u_j must be chosen is nonempty by Observation 6.1. Thus, u_j can be chosen so that r_{j+1} is a 4-dimensional orthogonal representation of the subgraph of G_c induced by $\{v_i : i < j+1\}$ in which each set of three domain vectors is linearly independent.

Case 3: Finally, assume that v_j has degree 2 in the subgraph of G_c induced by $\{v_i : i < j+1\}$, and assume without loss of generality that v_1, v_2 are the vertices to which v_j is adjacent in G_c. Then v_j must be adjacent in G to all v_i for $i < j$ except v_1 and v_2. So, in order for u_j to be a vector such that $r_{j+1} = r_j \cup \{(u_j, v_j)\}$ is an orthogonal representation in which each set of three domain vectors is linearly independent, it must be the case that $u_j \in (< u_1 >^{\perp} \cap < u_2 >^{\perp}) \setminus \{(\bigcup_{i=3}^{j-1} < u_i >^{\perp}) \bigcup (\bigcup_{i_1=1}^{j-1} \bigcup_{i_2=1}^{j-1} < u_{i_1}, u_{i_2} >)\}$. Because each set of three vectors in the domain of r_j is linearly independent, the space $< u_1 >^{\perp} \cap < u_2 >^{\perp}$ is 2-dimensional, and the spaces $< u_1 >^{\perp} \cap < u_2 >^{\perp} \cap < u_i >^{\perp}$ are 1-dimensional. So the spaces $< u_1 >^{\perp} \cap < u_2 >^{\perp} \cap < u_i >^{\perp}$ do not cover $< u_1 >^{\perp} \cap < u_2 >^{\perp}$. Furthermore, the spaces $< u_{i_1}, u_{i_2} >$ are 2-dimensional. It remains to be shown that the spaces $(< u_1 >^{\perp} \cap < u_2 >^{\perp}) \cap (< u_{i_1}, u_{i_2} >)$ are at most 1-dimensional. Note that, because $< u_1 >^{\perp} \cap < u_2 >^{\perp}$ and each $< u_{i_1}, u_{i_2} >$ are 2-dimensional, the only way that $(< u_1 >^{\perp} \cap < u_2 >^{\perp}) \cap (< u_{i_1}, u_{i_2} >)$, for some i_1, i_2, can have dimension > 1, is that $(< u_1 >^{\perp} \cap < u_2 >^{\perp}) = < u_{i_1}, u_{i_2} >$. Let $1 \leq i', i'' \leq j-1$.

Case 3a: Let i' be 1, and i'' be 2. Note that, by definition of orthogonal representation, neither of

u_1,u_2 is the zero vector. Hence u_1 is a vector in $< u_{i'}, u_{i''} >$ that is not in $< u_1 >^{\perp} \cap < u_2 >^{\perp}$, and so $(< u_1 >^{\perp} \cap < u_2 >^{\perp}) \neq < u_{i'}, u_{i''} >$.

Case 3b: Let $i' = 1$, and $i'' > 2$. Again, since u_1 is not the zero vector, u_1 is a vector in $< u_{i'}, u_{i''} >$ that is not in $< u_1 >^{\perp} \cap < u_2 >^{\perp}$, and thus $(< u_1 >^{\perp} \cap < u_2 >^{\perp}) \neq < u_{i'}, u_{i''} >$.

Case 3c: Assume that $i', i'' > 2$. Suppose for contradiction that $u_{i'}, u_{i''} \in < u_1 >^{\perp} \cap < u_2 >^{\perp}$. Then, by definition of orthogonal representation, $v_{i'}$ is adjacent to neither v_1 nor v_2 in G, and $v_{i''}$ is adjacent to neither v_1 nor v_2 in G. Thus, $v_{i'}$,$v_{i''}$ are both adjacent to both v_1,v_2 in G_c. Hence, if $V_1 = \{v_{i'}, v_{i''}, v_j\}$ and $V_2 = \{v_1, v_2\}$, then $\{V_1, V_2\}$ is a partition of a complete bipartite subgraph $K_{3,2}$ of G_c. This contradicts the assumption that G_c does not have $K_{3,2}$ as a subgraph. Thus, one of $u_{i'}$, $u_{i''}$ is not $\in < u_1 >^{\perp} \cap < u_2 >^{\perp}$. Hence, there is a vector that is in $< u_{i'}, u_{i''} >$ but is not in $< u_1 >^{\perp} \cap < u_2 >^{\perp}$. So in this case, $(< u_1 >^{\perp} \cap < u_2 >^{\perp}) \neq < u_{i'}, u_{i''} >$.

So, since i', i'' were arbitrary indices, $(< u_1 >^{\perp} \cap < u_2 >^{\perp}) \neq < u_{i_1}, u_{i_2} >$ for any i_1,i_2. Therefore, $(< u_1 >^{\perp} \cap < u_2 >^{\perp}) \cap < u_{i_1}, u_{i_2} >$ has dimension 1 for any i_1, i_2, and so $(< u_1 >^{\perp} \cap < u_2 >^{\perp}) \setminus (\bigcup_{i_1=1}^{j-1} \bigcup_{i_2=1}^{j-1} < u_{i_1}, u_{i_2} >)$ is nonempty. Because, as noted earlier in the proof for Case 3, the spaces $< u_1 >^{\perp} \cap < u_2 >^{\perp} \cap < u_i >^{\perp}$ have dimension 1, it follows from Observation 6.1 that $u_j \in (< u_1 >^{\perp} \cap < u_2 >^{\perp}) \setminus \{(\bigcup_{i=3}^{j-1} < u_i >^{\perp}) \bigcup (\bigcup_{i_1=1}^{j-1} \bigcup_{i_2=1}^{j-1} < u_{i_1}, u_{i_2} >)\}$ is nonempty also. So a vector u_j can be chosen such that $r_{j+1} = r_j \cup \{(v_j, u_j)\}$ is an orthogonal representation of the subgraph of G induced by $\{v_i : i < j+1\}$ in which each set of three distinct domain vectors is linearly independent. Since j was chosen as an arbitrary index $\leq n = |G|$, it follows that an orthogonal representation of dimension 4 of G in which each set of three domain vectors is independent may be constructed. Hence, $d(G) \leq 4$, and so $mr_+(G) \leq 4$. □

Corollary 6.3: If G is a graph such that G_c is cyclic and G_c contains neither $K_{3,2}$ nor a subgraph of minimum degree 3, then $mr_+(G) \leq 4$.

Proof. Because G_c is cyclic and does not contain a subgraph of minimum degree 3, it follows from Corollary 4.6 that $mld(G_c) = 2$. Since G_c does not contain $K_{3,2}$, Proposition 6.2 implies that $mr_+(G) \leq 4$. □

6.2 Applications of Proposition 6.2

In this section classes of graphs for which mld is 2 are identified, and Proposition 6.2 is applied to each class to prove that the complements of certain graphs in these classes have $mr_+ \leq 4$.

Let G be a planar graph. If G is embedded in the plane, then the set of faces $S_f = \{f_i : 1 \leq i \leq k\}$ of G is a minimal set of regions of the plane such that, for $i \neq j$, there is no pair of points $P_i \in f_i$, $P_j \in f_j$ such that a segment may be drawn in the plane between P_i and P_j without crossing an edge of G. The cardinality of the set of faces of a planar graph G is denoted $N_f(G)$. For connected planar graphs, the number of faces is related to the order and size.

Theorem 6.4: If G is connected and planar, then $|G| + N_f(G) = size(G) + 2$. [CL]

Proposition 6.5: If G is a cyclic planar graph in which each cycle subgraph has at least 6 vertices, then $mld(G) = 2$.

Proof. Consider two cases defined according to whether G is connected:

Case 1: Assume that G is connected. Suppose for contradiction that $\delta(G) \geq 3$. Note that, since G is planar, each edge that is part of a cycle subgraph H of G is between two faces of G, one containing points inside of the region of the plane bounded by H, and one containing (not necessarily all) points outside of the region bounded by H. So, each edge on a cycle in G corresponds to exactly two faces. Because each face of G is bounded by at least 6 edges, each face of G corresponds to at least 6 edges, and thus it follows that $N_f(G) \leq 2 * size(G)/6$. Also, by definition of simple graph, each edge corresponds to exactly two distinct vertices. Because $\delta(G) = 3$, each vertex is associated with at least three edges. Hence $|G| \leq 2 * size(G)/3$. By Theorem 6.4, we can obtain $size(G) + 2 \leq 2 * size(G)/6 + 2 * size(G)/3 = (1/3 + 2/3) * size(G) = size(G)$, which cannot be the case. So $\delta(G) \leq 2$. It is clear that, because G is planar and does not have any cycle subgraphs of length < 6, each subgraph of G is planar and does not have any cycle subgraphs of length < 6. Hence, each subgraph of G has minimum degree ≤ 2, and so $mld(G) \leq 2$. Since G is cyclic, $mld(G) \geq 2$. Therefore, $mld(G) = 2$.

Case 2: Assume that G is disconnected with components $G_1, G_2 ..., G_k$. Then each component of G is planar and does not contain any cycles on fewer than 6 vertices. Thus, it can be obtained that $mld(G_i) \leq 2$ for all i, and so by Proposition 4.3, $mld(G) \leq 2$. Because G is cyclic, at least one of the G_i must contain a cycle. By Propositions 4.3 and 4.5, $mld(G) \geq 2$, and thus $mld(G) = 2$. □

Corollary 6.6: If G is a graph such that G_c is a cyclic planar graph in which each cycle subgraph has at least 6 vertices, then $mr_+(G) \leq 4$.

Proof. By Proposition 6.5, $mld(G_c) = 2$. Since each cycle subgraph of G_c has at least 6 vertices, G_c does not contain $K_4 = K_{2,2}$ as a subgraph, and so G_c does not have $K_{3,2}$ as a subgraph. Thus, by Proposition 6.2, $mr_+(G) \leq 4$. □

So planar graphs in which each cycle has at least 6 vertices have *mld* 2, and so the complements of such graphs have $mr_+ \leq 4$. Note that not all graphs whose complements are planar have *mld* 2. For example, the dodecahedron graph, which is planar, is 3-regular, and thus has *mld* 3. This graph is shown in Figure 20.

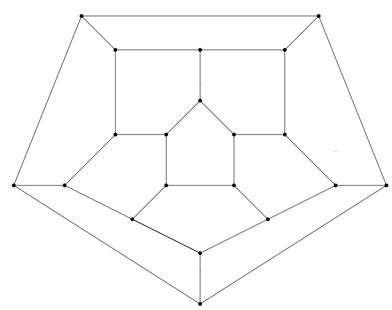

Figure 20 The dodecahedron graph

Proposition 6.7: If G_c is a cyclic cactus that does not contain $K_{3,2}$ as a subgraph, then $mr_+(G) \leq 4$.

Proof. By Example 4.7, $mld(G_c) = 2$, and so the result follows from Proposition 6.2. □

Recall that an LSEAC graph G is one that has k induced cycles C^1,C^2...C^k such that $\bigcup_{i=1}^{k} E(C^i) = E(G)$, of which there is an ordering $\{C^i : 1 \leq i \leq k\}$ such that, for $i \geq 2$, $j \leq i-2$, C^i shares exactly one edge with C^{i-1}, and does not share any vertices with C^j. Define a *beehive* to be any graph G for which there is an ordering of induced cycles of G, $\{C^i : 1 \leq i \leq k\}$, such that:

(a) $\bigcup_{i=1}^{k} E(C^i) = E(G)$.

(b) For j such that $1 \leq j \leq k$, $V(C^j) \setminus \bigcup_{i=1}^{j-1} V(C^i) \neq \emptyset$.

A beehive is shown in Figure 21.

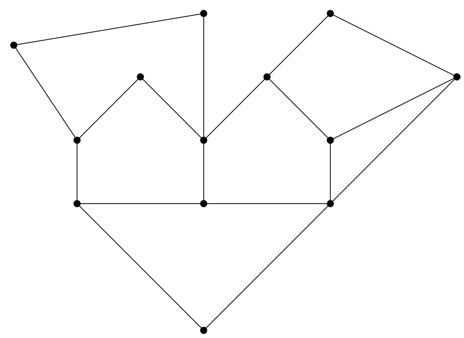

Figure 21 A beehive

Proposition 6.8: If G is a cyclic beehive, then $mld(G) = 2$.

Proof. Let G be a cyclic beehive, and let $\{C^i : 1 \leq i \leq k\}$ be an ordering of induced cycles of G satisfying (a) and (b). Then, by definition of beehive, $V(C^k)$ has a subset V_k of vertices that are not shared with C^j, for $j < k$. So the elements of V_k have degree 2 in G. Because V_k contains all vertices of C^k that are not shared with one of the other induced cycle subgraphs of G, $V(G) \setminus V_k = V(G - C^k)$. By definition of beehive, there is a nonempty set V_{k-1} containing exactly the vertices of $V(C^{k-1})$ that are not shared with C^j, for $j < k-1$. Thus, the elements of V_{k-1} have degree 2 in the subgraph of G induced by $V(G) \setminus V_k$. If we continue in this way, defining, for each i, V_i as the set containing exactly the vertices of C^i that are not shared with C^j, for $j < i$, then each V_i will be nonempty by definition of beehive. Furthermore, since every vertex of C^1 has degree 2, V_1 will equal $V(C^1)$. Thus, a vertex labeling of G in which the first $|V_1|$ vertices are those of V_1, the next $|V_2|$ vertices are those of V_2..., and the last $|V_k|$ vertices are those of V_k has degree 2. Since G is cyclic, it follows that $mld(G) = 2$. □

Proposition 6.9: If G is a graph such that $mld(G) = 2$, then G is a cyclic subgraph of a beehive.

Proof. Assume that G is not a subgraph of a beehive. Assume for simplicity that there is a set S of k induced cycles of G satisfying condition (a) in the definition of beehive. Since G is not a beehive, it must be that there is no ordering $\{C^i : 1 \leq i \leq k\}$ of S such that each C^i has a vertex that is not shared with any C^j, for $j < i$. Let $O = \{Y^i : 1 \leq i \leq k\}$ be an ordering of S, and let i' be the smallest j for which Y^j shares each of its vertices with one of the Y^i, for $i < j$. Let H be the subgraph of G induced by the vertices of the elements of $O^* = \{Y^i : i \leq i'\}$. If $\delta(H) \geq 3$, then $mld(G) \geq 3$ by Proposition 4.5. Assume on the other hand that H has minimum degree 2. Then, if U is the set containing exactly the vertices of H having degree 2 in H, then U is the set of all vertices that are part of only one of the elements of O^*. Thus, the subgraph of H induced by $V(H) \setminus U$ contains only vertices that are part of multiple elements of O^*, and so this subgraph has minimum degree ≥ 3. Hence, G has a subgraph of minimum degree 3 in this case as well, and so $mld(G) \geq 3$. By contraposition, if $mld(G) \leq 2$, then G is a subgraph of a beehive. If $mld(G) = 2$, then G must be cyclic, and so the proposition follows. □

The following proposition gives a characterization of graphs with *mld* 2, and is proved using Propositions 6.8 and 6.9.

Proposition 6.10: If G is a graph, then $mld(G) = 2$ if and only if G is a cyclic subgraph of a beehive.

Proof. : $(->)$ This direction is Proposition 6.9.
$(<-)$ Let G^* be a beehive of which G is a subgraph. By Proposition 6.8, $mld(G^*) = 2$. Then G cannot have a subgraph of minimum degree ≥ 3, since otherwise G^* has a subgraph of minimum degree ≥ 3, which would contradict the fact that $mld(G^*) = 2$. But because G is cyclic, G has a subgraph of minimum degree 2. So $mld(G) = 2$. □

Corollary 6.11: If G_c is a cyclic subgraph of a beehive that does not contain $K_{3,2}$ as a subgraph, then $mr_+(G) \leq 4$.

Proof. By Proposition 6.10, $mld(G) = 2$, and so the result follows from Proposition 6.2. □

Because, as noted above, Proposition 6.10 is a biconditional statement, it follows that the class of graphs whose complements are cyclic subgraphs of beehives that do not contain $K_{3,2}$ as a subgraph is the maximal class that can be shown to have $mr_+ \leq 4$ using Proposition 6.2.

6.3 Graphs Whose Complements Are Unicyclic and Do Not Contain A 4-cycle

In this section, we show that the mr_+ of certain graphs whose complements are unicyclic is 3. The following proposition is proved first.

Proposition 6.12: If G is a graph such that $mld(G) = m$, and H is the induced subgraph of G having maximal order among all subgraphs of G with minimum degree m, then there is a vertex labeling f of $G-H$ such that, if $v \in V(G-H)$, then the degree of v in the subgraph of G induced by $V(H) \cup \{u \in V(G-H) : f(u) \leq f(v)\}$ is $< m$.

Proof. Let $mld(G) = m$. Let H be the induced subgraph of G of maximal order among those having minimum degree m. If $V(G-H)$ is nonempty, then since $mld(G) = m$ and H is the induced subgraph of G of maximal order among those having minimum degree m, there must be a nonempty set V_1 of vertices in $V(G-H)$ whose degree in G is $< m$. If $V(G-H) - V_1$ is nonempty, then since $mld(G) = m$ and H is the maximal-order subgraph of G having degree m, there must be a nonempty set V_2 containing the vertices of $V(G-H) \setminus V_1$ that have degree $< m$ in the subgraph G_1 of G induced by $V(G-H) \setminus V_1$. Similarly, if $V(G-H) \setminus (V_1 \cup V_2)$ is nonempty, then there must be a nonempty set V_3 containing exactly the vertices of $V(G-H) \setminus (V_1 \cup V_2)$ that have degree $< m$ in the subgraph G_2 of G induced by $V(G-H) \setminus (V_1 \cup V_2)$. If we continue defining sets V_i containing exactly the vertices of $V(G-H) - \bigcup_{j=1}^{i-1} V_j$, that have degree $< m$ in the subgraph G_{i-1} of G induced by $V(G-H) - \bigcup_{j=1}^{i-1} V_j$, a subgraph, say V_{k-1}, will be defined such that $V(G-H) - \bigcup_{j=1}^{k-1} V_j$ induces a subgraph G_{k-1} of G in which every vertex has degree $< m$. If we define V_k as the vertex set of this subgraph, then in any vertex labeling f of $G-H$ in which the first $|V_k|$ vertices are those of V_k, the next $|V_{k-1}|$ are those of V_{k-1}..., and the last $|V_1|$ vertices are those of V_1, every vertex $v \in V(G-H)$ has degree $< m$ in the subgraph of G induced by $V(H) \cup \{u \in V(G-H) : f(u) \leq f(v)\}$. Thus, the proposition is proved. □

The next two results are lemmas that will be used in the main proof for this chapter. The proof of the first is a consequence of Proposition 6.12.

Lemma 6.13: If G is unicyclic with cycle subgraph H, then there is a vertex labeling f of $G-H$ such that, for each $v \in V(G-H)$, the degree of v in the subgraph of G induced by $V(H) \cup \{u \in V(G-H) : f(u) \leq f(v)\}$ is < 2.

Proof. Note that $mld(G) = 2$, and that H is the induced subgraph of maximal order among those having minimum degree 2 in G. By Proposition 6.12, it follows that there is a vertex labeling f of $G-H$ such that, if $v \in V(G-H)$, then the degree of v in the subgraph of G induced by $V(H) \cup \{u \in V(G-H) : f(u) \leq f(v)\}$ is < 2. So, the lemma follows. □

The following result will also be useful.

Lemma 6.14: If H is a cycle that is not a 4-cycle, then (1) every orthogonal representation of H_c satisfies the condition that each pair of domain vectors is linearly independent. Furthermore, (2) there is an orthogonal representation of H_c of dimension 3 in which each pair of domain vectors is linearly independent.

Proof. Let H be a cycle that is not a 4-cycle, and let $|H| = k$. Suppose for contradiction that there is an orthogonal representation r of H_c such that it is not the case that each pair of elements of U is linearly independent. Let $U = \{u_i : 1 \leq i \leq k\}$ be the domain of r, and let $r(u_i) = v_i$ for $1 \leq i \leq k$. Without loss of generality, assume that u_1,u_2 are elements of U that are linearly dependent. Then $< u_1 >^{\perp} = < u_2 >^{\perp}$, and so by definition of orthogonal representation, the sets of vertices to which v_1 is not adjacent in H_c, and to which v_2 is not adjacent in H_c, are the same. That is, $N_H(v_1) = N_H(v_2)$. But since H is not a 4-cycle, this cannot be the case. Hence, by contradiction, (1) follows.

With regard to proving (2), consider two cases. If H is not a 3-cycle, H_c does not have any isolated vertices. Hence, the fact that $mr_+(H_c) = 3$ (by Theorem 2.12) implies that there is an orthogonal representation of H_c of dimension 3. By (1), (2) follows in this case. If H is a 3-cycle, then H only has 3 vertices, and since it is possible to have 3 non-parallel, mutually orthogonal vectors in $\mathbb{R}^3$, (2) follows in this case as well. □

Proposition 6.15: Let G_c be a graph that is unicyclic and does not contain a 4-cycle. Then $mr_+(G) = 3$.

Proof. Let G_c be a unicyclic graph that does not contain a 4-cycle, and let $|G_c| = n$. Let H_c be the cycle subgraph of G_c, and let $|H_c| = k$. By Lemma 6.13, there is a vertex labeling f of $G_c - H_c$ such that, for each $v \in V(G_c - H_c)$, the degree of v in the subgraph of G_c induced by $V(H_c) \cup \{u \in V(G_c - H_c) : f(u) \leq f(v)\}$ is < 2. Denote the vertices of $G_c - H_c$ by $v_1 ..., v_{n-k}$ such that $f(v_i) = i$, for $1 \leq i \leq n-k$. In addition, denote the vertices of H_c by $w_1 ..., w_k$. By Lemma 6.14, there is a 3-dimensional orthogonal representation of H in which each pair of domain vectors is linearly independent. Let r_0 be such an orthogonal representation of H. Assume that there is a 3-dimensional orthogonal representation r_j of the subgraph of G induced by $V(H) \cup \{v_i : i \leq j\}$, where j is some number such that $0 \leq j < n-k$. Let the domain of r_j be given by $U = \{u_i : 1 \leq i \leq k+j\}$, and let $r_j(u_{i_1}) = w_{i_1}$ for $1 \leq i_1 \leq k$ and $r_j(u_{i_2+k}) = v_{i_2}$ for $1 \leq i_2 \leq j$. It will be shown that there is a 3-dimensional orthogonal representation $r_{j+1} = r_j \cup \{(u_{j+1}, v_{j+1})\}$ of the subgraph of G induced by $V(H) \cup \{v_i : i \leq j+1\}$ in which each pair of vectors is linearly independent. Because of the assumption about f, the vertex v_{j+1} has degree either 0 or 1 is the subgraph of G_c induced by $V(H_c) \cup \{v_i : i \leq j+1\}$. Consider two cases:

Case 1: Assume that v_j has degree 0 in this subgraph. Then u_{j+1} is such that $r_{j+1} = r_j \cup \{(u_{j+1}, v_{j+1})\}$ is a 3-dimensional orthogonal representation of the subgraph of G induced by $V(H) \cup \{v_i : i \leq j+1\}$ such that each pair of domain vectors is independent if and only if $u_{j+1} \in \mathbb{R}^3 \setminus (\bigcup_{i=1}^{k+j} < u_i >^{\perp} \cup \bigcup_{i=1}^{k+j} < u_i >)$. The spaces $< u_i >^{\perp}$, for $1 \leq i \leq k+j$, are 2-dimensional, and the spaces $< u_i >$ are 1-dimensional, so by Observation 6.1, it follows that the space from which u_j can be chosen is nonempty. Thus, an orthogonal representation r_{j+1}, defined as above, can be constructed.

Case 2: Now assume that v_{j+1} has degree 1 in this subgraph, and without loss of generality, assume that v_1 is the vertex to which v_{j+1} is adjacent. Then there is a vector u_{j+1} such that $r_{j+1} = r_{j+1} \cup \{(u_{j+1}, v_{j+1})\}$ is an orthogonal representation of the subgraph of G induced by $V(H) \cup \{v_i : i \leq j+1\}$ satisfying the intended assumptions if and only if $u_j \in < u_1 >^{\perp} \setminus (\bigcup_{i=2}^{k+j} < u_i >^{\perp} \cup \bigcup_{i=1}^{k+j} < u_i >)$. Each of the $< u_i >^{\perp}$, for $1 \leq i \leq k+j$, is 2-dimensional. Because u_1 is independent of each vector u_i, for $2 \leq i \leq k+j$, no $< u_i >^{\perp}$, for $2 \leq i \leq k+j$, is equal to $< u_1 >^{\perp}$. Thus, $< u_1 >^{\perp} \cap < u_i >^{\perp}$, for $2 \leq i \leq k+j$, is a 1-dimensional space. Since the $< u_i >$ are all 1-dimensional spaces also, it follows that the space from which u_j can be chosen is nonempty by Observation 6.1. Hence, an orthogonal representation r_{j+1}, defined as above, can be constructed.

Hence, an orthogonal representation r_{j+1} of the subgraph of G induced by $V(H) \cup \{v_i : i \leq j+1\}$ can be constructed such that each pair of domain vectors of r_{j+1} is linearly independent. As noted above, there is an orthogonal representation r_0 of H satisfying the condition that each pair of domain vectors of r_0 is linearly independent. So, since j was chosen such that $j+1$ is an arbitrary label (assigned by f) of a vertex of $G-H$, it follows from the above proof that there is a 3-dimensional orthogonal representation of G. Hence, $mr_+(G) \leq 3$. Because, by Theorem 2.12, $mr_+(H) = 3$, it cannot be the case that G has $mr_+ < 3$, and so $mr_+(G) = 3$. □

CHAPTER 7

TWO ADDITIONAL DEFINITIONS OF *mld*

In Chapter 4, it was proved that the set of graphs G having mld m was the same as the set of graphs G such that $max(\{\delta(H) : H \text{ is a subgraph of } G\}) = m$. Below, two additional conceptions of this class is given.

7.1 A Definition in Terms of a Certain Vertex-Removal Scheme

Let G be a graph and m be a positive integer. Define the *1st m-peeling subgraph* of G, denoted $p_{m,1}(G)$, as that induced by the set of vertices having degree $> m$ in G, and define the *kth m-peeling subgraph* of G, $p_{m,k}(G)$, as $p_{m,1}(p_{m,k-1}(G))$. Say that G is *m-peelable* if and only if there exists k such that $V(p_{m,k}(G)) = \emptyset$. Define the *peeling number* of G, $peel(G)$, as the smallest m for which G is m-peelable.

Proposition 7.1: If G is a graph, then $peel(G) = mld(G)$.

Proof. Let $peel(G) = m$. Then there is a k such that $V(p_{m,k}(G)) = \emptyset$. Thus, in $p_{m,k-1}(G)$, each vertex has degree $\leq m$. Also, in $p_{m,k-2}(G)$, each vertex of $p_{m,k-2}(G) - p_{m,k-1}(G)$ has degree $\leq m$. It is clear that if f is a vertex labeling of G such that the first $|p_{m,k-1}(G)|$ vertices are those of $p_{m,k-1}(G)$, the next $|p_{m,k-2}(G) - p_{m,k-1}(G))|$ are those of $p_{m,k-2}(G) - p_{m,k-1}(G)$,...and the last $|p_{m,1}(G) - \bigcup_{i=2}^{k-1} p_{m,i}(G)|$ vertices are those of $p_{m,1}(G) - \bigcup_{i=2}^{k-1} p_{m,i}(G)$ is a vertex labeling of degree $\leq m$. So $mld(G) \leq m$. Suppose for contradiction that $mld(G) = m' < m$. Because $peel(G) = m$, there is no k^* such that $V(p_{m',k^*}(G)) = \emptyset$. Thus, there must exist a k' such that all vertices of $p_{m',k'}(G)$ have degree $> m'$. Then $p_{m',k'}(G)$ is a subgraph of G with minimum degree $> mld(G)$. But by Corollary 4.6, G cannot have such a subgraph. By contradiction, $mld(G) = m$, and so $peel(G) = mld(G)$. □

Note that the conception of *mld* given in Propositon 7.1 gives a algorithm for calculating the value of the parameter that is more efficient for large graphs that that given by both the definition of *mpd* and Corollary 4.6. Consider that, to apply the definition of *mld* to determine its value for a given graph G, all $n!$ vertex labelings of G must be considered. Furthermore, applying the discussion immediately following Corollary 4.6 (which showed that, for any graph G, $mld(G) = \{\delta(H) : H \text{ is an induced subgraph of } G\}$) to find the *mld* of a graph G requires consideration of all induced subgraphs of G, and using Corollary 4.6 itself would clearly require consideration of at least that many subgraphs. However, using Proposition 7.1 to find $mld(G)$ only requires the determination of $peel(G)$. Thus, one may proceed by determining, for each i such that $\delta(G) \leq i \leq mld(G)$, whether G is i-peelable, increasing the value of i by 1 at each iteration. By Proposition 4.8, $mld(G) \leq \Delta(G)$, and so there will not be more than $\Delta(G) - \delta(G) + 1$ values of i to consider. This suggests that the *peel* definition of *mld* provides a more efficent algorithm for calculating *mld* of a graph that any other conception of *mld* that has been stated in this thesis.

7.2 A Treewidth-Like Definition

Recall that a k-tree is a graph that can be constructed from K_{k+1} by adding one vertex at a time, where 1) each added vertex is adjacent to exactly k vertices at the point that it is added, where 2) these k vertices induce K_k. Thus, a k-tree is any graph that can be constructed from K_{k+1} by adding one vertex at a time and such that each of the following two conditions (labeled 1) and 2) in the definition just given) is satisfied: the ”degree condition” (labeled 1)), which regards the degree of each added vertex at the point it is added, and the ”clique condition” (labeled 2)) specifying that the elements of the neghborhood of each added vertex, at the point it is added, must induce a k-clique. Also, recall that a partial k-tree is defined to be a subgraph of a k-tree, and that the treewidth of a graph G, denoted $tw(G)$, can be defined as the minimum k such that G is a k-tree.

Define a *k-degree-tree* to be a graph that can be constructed from K_{k+1} by adding one vertex at a time, where each added vertex is adjacent to exactly k vertices at the point that it is added. Define a *partial k-degree-tree* to be a graph that is a subgraph of a k-degree-tree. Refer to the smallest k such that a graph G is a partial k-degree-tree as the *degree-treewidth* of G, and denote this parameter $dtw(G)$. Note that G is a k-degree-tree if and only if G can be constructed from K_{k+1} by adding one vertex at a time, such that G satisfies the ”degree condition” in the definition of k-tree given above. Thus, every k-tree is a k-degree-tree, and so $tw(G) \leq dtw(G)$ for every graph G.

Proposition 7.2: If G is an m-degree-tree, then $mld(G) = m$.

Proof. Let H be the K_{m+1} subgraph of G from which G is constructed, and denote the ith vertex that is added by u_i. Let f be a labeling of G such that $f(v) \leq m+1$ if and only if $v \in V(H)$ and $f(u_i) = i+m+1$ for each $u_i \in V(G-H)$. Then f has degree m by definition of m-degree-tree, and so $mld(G) \leq m$. Because H has minimum degree m, $mld(G) \geq m$ by Proposition 4.5, and so $mld(G) = m$. □

Proposition 7.3: If G is a graph, then $mld(G) = dtw(G)$.

Proof. Let $dtw(G) = m$. Then G is a subgraph of an m-degree-tree, and since mld of an m-degree-tree is m, it follows that $mld(G) \leq m$. Suppose that $mld(G) = m' < m$, and let f' be a vertex labeling of G of degree m'. Let H be the subgraph of G induced by $\{v : f(v) \leq m'+1\}$. Then H is a subgraph of $K_{m'+1}$. Because each vertex u of $G-H$ has degree $\leq m'$ in the subgraph of G induced by $\{v \in V(G) : f(v) \leq f(u)\}$, G can be constructed from H by adding one vertex at a time, where each added vertex is adjacent to at most m' vertices at the point that it is added. Thus, it follows that G is a subgraph of an m'-tree. But this contradicts the assumption that $dtw(G) = m$, where $m > m'$. Therefore, $mld(G) \geq m$, and since $mld(G) \leq m$ also, it follows that $mld(G) = m$. So, $mld(G) = dtw(G)$ for any graph G. □

At this point, a few characterizations of the class of graphs having *mld* m, either for general or particular m, have been proven. The last result of this thesis (before the Appendix) summarizes these characterizations.

Corollary 7.4: If G is a graph and m is any positive integer, then the following are equivalent:

(1) $mld(G) = m$.

(2) $max(\{\delta(H) : H \text{ is a subgraph of } G\}) = m$.

(3) $peel(G) = m$.

(4) $dtw(G) = m$.

Furthermore, in the case that $m = 2$, the following is equivalent to the above statements:

(5) G is a cyclic subgraph of a beehive.

Proof. The equivalence of (1), (2), (3), and (4) follows from Corollary 4.6, Proposition 7.1, and Proposition 7.3. That (5) is equivalent to (1)-(4) if $m = 2$ is Proposition 6.10. □

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APPENDIX A

THE BEHAVIOR OF $mld(G)$ WHEN VERTICES ARE DUPLICATED IN G

In Chapter 5, it was shown that the expression $|G| - mld(G)$ is an upper bound of $mr(G)$ for several graphs G. This is the only expression in terms of minimum labeling degree whose value was compared with that of minimum rank in this thesis. Initially, however, it was thought that we might investigate whether an expression of the form $mld(G_c) + C$, where C is some constant, was an upper bound for $mr(G)$ in some set of cases. For this reason, we considered whether or not the parameter mld has a certain property, where the fact that that Δ does not have this property causes the bound in the δ-conjecture to not be sharp in certain cases (and in fact, to be arbitrarily unsharp). In the first subsection of this Appendix, we discuss why the property does not hold for Δ, and why this causes the bound in the δ-conjecture to be arbitrarily unsharp. In the last three sections, it is shown that mld does not have the property either, and we consider which cases this causes an expression of the form $mld(G_c) + C$ to grow arbitrarily larger than $mr(G)$.

A.1 Duplicate Vertices and the Non-Sharpness of the δ-Conjecture

In a graph G, two distinct vertices v', v'' are said to be *duplicate vertices* (or *twin vertices*) in G if and only $N(v') = N(v'')$. Note that this definition implies that, if $v', v'' \in V(G)$ are duplicate vertices, then $\{v', v''\} \notin E(G)$.

If G is a graph such that $v', v'' \in V(G)$ are duplicate vertices in G_c, then for any element of $S(G)$, the two columns corresponding to v' and v'' have zero entries in the exact same rows. Thus, these two columns can be made to be exactly the same. As will be seen from the following discussion, this fact causes the difference between $\Delta(G_c) + 1$ and $mr(G)$ to be very large for certain G for which G_c has a set of duplicate vertices.

Let G be a graph, let $U \subseteq V(G)$ be the set of vertices attaining maximal degree in G_c, and suppose that there is a vertex $v \in N_{G_c}(u)$ for some $u \in U$. Let G^* be the graph obtained from G by adding a vertex v^* to G such that v, v^* are duplicate vertices in ${G^*}_c$. Then $\Delta({G^*}_c) = \Delta(G_c) + 1$. Since v and v^* are duplicates in G_c, the columns corresponding to v and v^* in each element of $S(G)$ can be made to be the same. So $mr(G*) = mr(G)$. Thus, duplicating the vertex v to the graph causes the maximum degree of the complement to increase, but the minimum rank of the graph is unaffected. Hence, the positive difference between the minimum rank of the graph and the maximum degree of its complement is caused to increase by one by the duplication of v.

In the general example discussed in the last paragraph, only one duplicate of v was added to

G_c, and the difference between the maximum degree of G_c and the minimum rank of the graph for the resulting graph was one greater than that difference for the original graph. Clearly, each subsequent duplication of v, and indeed of any vertex that is in the neighborhood of each element of U in G_c, results in this difference increasing by one more.

It was mentioned in subsection 3.2 that, if G_c is a k-star, the difference between $mr(G)$ and $\Delta(G_c)$ gets infinitely large as k gets infinitely large. This result makes sense in light of the above discussion. A k-star can be constructed from the graph P_2 by choosing one of the two vertices of the graph, and duplicating it $k-1$ times. In P_2, both vertices attain maximal degree. Thus, if one of the vertices is duplicated $k-1$ times, the result is a graph on $k+1$ vertices, one of which (the vertex of the original graph P_2 that was not duplicated) has degree k, and the rest of which have degree 1. That is, duplicating either of the vertices of P_2 $k-1$ times results in a k-star. Note that the minimum rank of the complement of P_2 is 1, and the maximum degree of P_2 is 1. By the above discussion, one would expect the difference between the minimum rank of the complement of the k-star and the maximum degree of the k-star to be $(1-1)+(k-1) = k-1$. In subsection 3.2, it was noted that, if G is the complement of the k-star, then $mr(G) = 1$ and $\Delta(G_c) = k$, so this is indeed the difference.

So, for any graph G, duplication of a vertex that is adjacent to a vertex of maximal degree in G_c results in an increase in the parameter $\Delta(G_c)$, but does not cause any change in the parameter $mr(G)$. As a consequence, the value of $\Delta(G_c)$ can be arbitrarily larger than that of $mr(G)$.

In what follows, we examine how mld is affected by the duplication of vertices in a graph, to determine if a potential upper bound of $mr(G)$ of the form $mld(G_c)+C$ would escape the non-sharpness issue with that in terms of $\Delta(G_c)$ given in Form 2 of the δ-conjecture.

A.2 Characterization of Vertices v such that the Repeated Duplication of v Does Not Result in an Increase in mld

We now consider the effect that duplicating a single vertex v has on the value of minimum labeling degree. Note that, if v is the vertex with degree > 1 in a k-star, any vertex labeling f for which $f(v) = 1$ has degree 1. As noted above, a k-star can be constructed from the graph P_2 by choosing one of the vertices of P_2 and duplicating it $k-1$ times. The value of $mld(P_2)$ is also 1, so duplicating one of the vertices of P_2 an arbitrary number of times in P_2 does not increase the value of mld.

However, consider the graph G shown in Figure 22. Clearly, the maximum among minimum degrees of subgraphs of G is 2, the minimum degree of the subgraph C_4 of G. Let G' (shown in Figure 23) be the graph obtained from G by duplicating the vertex of degree 4 (labeled v) in G two times. Then the subgraph of G' induced by v, $N_G(v)$, and the two duplicates of v has minimum degree 3. So duplicating v two times results in an increase in mld.

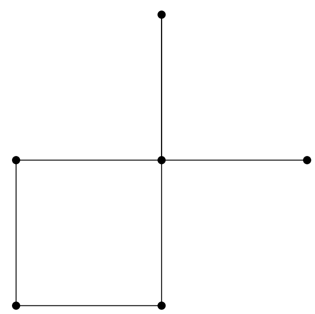

Figure 22 The graph G from second Appendix section

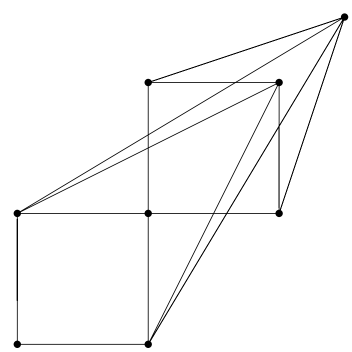

Figure 23 The graph G' from second Appendix section

It was shown that, beginning with P_2, repeatedly duplicating a vertex of degree one in each resulting graph does not affect the value of mld, but duplicating the vertex v in G two times causes an increase in the value of mld. The next result characterizes the vertices v in a graph G such that v may be duplicated in G any number of times without increasing the value of mld.

Proposition A1: If G is a graph and $v \in V(G)$, then the following are equivalent:

(a) For every m, the graph G' obtained from G by duplicating v m times satisfies $mld(G') = mld(G)$.

(b) $deg_G(v) \leq mld(G)$.

Proof. Let $deg_G(v) = k > mld(G)$. Let G' be the graph obtained from G be duplicating v $mld(G)$ times.

By definition of duplicate vertices, all of the duplicates of v are adjacent to all of the k vertices in $N_G(v)$. Further, all of the neighbors of v in G are adjacent to v and all of its $mld(G)$ duplicates in G'. Hence, in the subgraph of G' induced by v, the $mld(G)$ duplicates of v, and $N_G(v)$, each vertex has degree greater than $mld(G)$. So $mld(G') > mld(G)$, and so there is a graph obtained from G by duplicating v some number of times whose mld is not equal to that of G. Thus, the fact that (a)$-$ $>$(b) follows by contraposition.

Now let $deg_G(v) = d \leq mld(G)$, and G^* be the graph obtained from G by duplicating v m times for some m. Suppose for contradiction that there is a subgraph H of G^* such that $\delta(H) > mld(G)$. Note that v and all m of its duplicates have degree $d \leq mld(G)$ in G^*, and thus have degree $\leq mld(G)$ in H. So, if $V(H)$ contains v or any of its duplicates as a vertex, then $\delta(H) \leq mld(G)$, contradicting the assumption about H. Thus, $V(H)$ contains only vertices other than v and its duplicates. But then $V(H) \subseteq V(G)$, and since no edges were added to G between vertices of $V(G)$ in the formation of G^*, it follows that H is a subgraph of G. So G has a subgraph whose minimum degree is greater than $mld(G)$, which cannot be the case. By contradiction, G^* does not contain a subgraph with minimum degree greater than $mld(G)$. Hence $mld(G^*) \leq mld(G)$. Because G is a subgraph of G^*, $mld(G) \leq mld(G^*)$ also. As a result, it can be concluded that $mld(G^*) = mld(G)$. This proves that (b)$-$ $>$(a). This completes the proof of the proposition. □

A.3 Proof That *mld* Does Not Increase Arbitrarily with the Repeated Duplication of a Single Vertex

As noted above, the parameter $\Delta(G)$ can increase arbitrarily with infinite duplications of a single vertex. However, this is not the case for mld.

Proposition A2: Let G be a graph and let v be a vertex of G such that $deg_G(v) \geq mld(G)$. If G' is a graph obtained from G by duplicating v $m-1$ times, where $m \geq deg_G(v)$, then $mld(G') = deg_G(v)$.

Proof. In the subgraph of G' induced by v, $N_G(v)$, and the $m-1$ duplicates of v, the vertices of $N_G(v)$ have degree at least m, and v and its duplicates have degree $deg_G(v)$. So this subgraph has minimum degree $deg_G(v)$, and so $mld(G') \geq deg_G(v)$. Let H be a subgraph of G'. If H has v or any of its duplicates as a vertex, then $\delta(H) \leq deg_G(v)$. If H does not have v or any of its duplicates as a vertex, then H is a subgraph of G, and so $\delta(H) \leq mld(G)$ by Proposition 4.5, and so $\delta(H) \leq deg_G(v)$ by assumption. So H has minimum degree $\leq deg_G(v)$. Since H is an arbitrary subgraph of G' it follows from Proposition

4.5 that $mld(G') \leq deg_G(v)$. So $mld(G') = deg_G(v)$. □

A.4 Proof That Duplicating Multiple Vertices in a Graph G Can Result in mld Growing Arbitrarily Larger Than $mld(G)$

In the last section it was seen that the duplication of a single vertex of a graph some number m of times can only increase *mld* to the degree of the vertex in G, and hence, unlike the value of Δ, the value of *mld* will never increase indefinitely with the duplication of a single vertex an arbitrary number of times. However, if multiple vertices are each duplicated an arbitrary number of times, the *mld* of the resulting graph may increase arbitrarily.

Proposition A3: Let G be a graph, $v', v'' \in V(G)$, and $v', v'' \in E(G)$. If G' is the graph obtained from G by duplicating v' n' times and v'' n'' times, then $mld(G') \geq min(\{n'+1, n''+1\})$.

Proof. Note that, since v' and v'' are adjacent in G, G' must have $K_{n'+1,n''+1}$ as an induced subgraph. Thus, G' has a subgraph of minimum degree $min(\{n'+1, n''+1\})$. So, by Proposition 4.8, $mld(G) \geq min(\{n'+1, n''+1\})$. □

Since n' and n'' are arbitrary positive integers, it follows that *mld* can grow arbitrarily large with the repeated duplication of v', v''. Thus, the difference between $mld(G_c)$ and $mr(G)$ can be indefinitely large.

VITA

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