THE DENSITY OF COMPLEX ZEROS OF RANDOM SUMS

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ABSTRACT

A random polynomial is a polynomial whose coefficients follow some probability distribution. The fundamental questions that need to be studied are the distribution and correlations between zeros, pairing between zeros and critical points, distribution values, and nodal surfaces. The computation of the average distribution of real zeros of random polynomials was studied by Bloch and Pólya, Littlewood and Offord, Erdős, Kac and others. For standard normally distributed coefficients, the expected density of real zeros is given by Kac's exact formula. The famous result due to Hammersley asserts that, when the coefficients are complex independent standard normal random variables, the zeros of a random complex polynomial largely tend towards the unit circle as the degree approaches infinity. For complex zeros, the expected density was dealt with by Shepp and Vanderbei for real independent and identically distributed normal coefficients. Their technique exploits the argument principle and Cholesky factorization to reduce the question to the evaluation of a holomorphic function of four correlated normal random variables. Their results were generalized by Ibragimov and Zeitouni to a wide class of distribution of coefficients. Recently, Vanderbei extended the results he obtained with Shepp to random sums with holomorphic functions that are real-valued on the real line as the basis functions. Our interest in this dissertation is to refine the techniques of random fields pioneered by Rice in his treatment of the questions on real zeros to obtain exact formulas for the expected density of the distribution of complex zeros of a family of random sums, such as truncated random trigonometric series and random orthogonal polynomials on the unit circle. We further study the level crossings

and answer the question about the expected number of complex zeros for coefficients with nonvanishing mean values and distinct variances.

DEDICATION

To Tory. You have supported me and believed in me from start to finish. I could not have done this without you.

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CHAPTER 1

INTRODUCTION

A random polynomial is a polynomial whose coefficients follow some probability distribution. Since the coefficients are random variables, it is of interest to study how the zeros of the polynomial are distributed. The problem of characterizing the distribution of zeros of random polynomials has a long history, starting with the work of Bloch and Pólya [10]. Kac [34] was the first to study the distribution of real zeros of random polynomials whose coefficients are real standard normal independent random variables. He obtained an exact formula for the expected value of the number of its zeros in measurable subsets of the reals. The distribution of the number of real zeros of random polynomials was further studied by Bharucha-Reid and Sambandham [9], Edelman and Kostlan [20], Farahmand [26,27], Farahmand and Jahangiri [29], Kostlan [35], Mezincescu, Bessis, Fournier, Mantica, and Aaron [41] and many others.

Shepp and Vanderbei [48] developed a remarkable technique based on the argument principle and Cholesky factorizarion for extending Kac's result to the complex plane. They obtained exact formulas for the expected number of complex zeros in measurable subsets of the complex plane, when the coefficients are real standard normal independent random variables. Their computer plots of the density functions and empirical distributions from randomly generated polynomials show that, as the degrees of the random polynomials become large, the zeros appear to be approximately uniformly distributed around the circle. Their asymptotic analysis of the density functions confirm the classical result due to Hammersley [32]. Ibragimov and Zeitouni [33] employed a different method, based on the mathematical theory of random noise developed in a notable paper by Rice [46], to obtain the results in [48]. They also showed the limiting distributions of the density functions under more general distributional assumptions.

In recent years, research on random polynomials has branched off in a number of directions. The zeros of many ensembles of random polynomials have been found to be asymptotically equidistributed near the unit circumference. Pritsker and Yeager [43] provided quantitative estimates for such equidistribution in terms of the expected discrepancy of a certain zero counting measure and the expected number of zeros in various subsets of the complex plane. The random polynomials they studied have coefficients which may be dependent and need not have identical distributions.

Vanderbei [52] later introduced a modest generalization to the central assumptions underlying the results in [48]. He showed that comparable exact formulas for the distribution of the zeros in the complex plane can still be obtained for any value of the degree of the random polynomial. For many years, most authors establish certain properties of the zeros under very general distributional assumptions at the cost that most results hold asymptotically only as the degree of the random polynomial tends to infinity.

Inspired by these works the aim of this dissertation is to study the expected density of the complex zeros and level crossings of these random sums. The main device for treating the expected density function throughout the complex plane is the Rice formula, which provides a representation for the expected number of zeros of certain random fields. Our computations and method of proof [11–14] are done in the spirit of the study conducted by Ibragimov and Zeitouni [33].

In Chapter 2 we compute the expected density of complex zeros and level crossings of a family of random sums constructed from sequences of independent and identically distributed random complex standard normal variables and sequences of given holomorphic functions that are real-valued on the real line as the basis functions. Several practical examples are considered, such as random Weyl polynomials, random root-binomial polynomials, random truncated Fourier sine and cosine series. In addition, we consider random sums whose basis functions are polynomials orthogonal on the real line and unit circle. We then obtain the limiting behavior of the expected density function and produce numerical computations for the density function and empirical distributions. In Chapter 3 we compute the expected density for the case of mean zero and general variances. We apply this result to random sums constructed from sequences of successive observations of a Brownian motion. In Chapter 4 we consider the basic question about the expected number of complex zeros for coefficients of nonvanishing mean values and general variances, thereby generalizing the key results from Chapters 2 and 3. This process can get very involved technically. Only the main steps are provided. In Chapter 5 we consider the expected density and level crossings for a certain form of a random complex trigonometric polynomials. Finally, the expected number of complex zeros in a measurable region of the complex plane can be obtained from these results.

CHAPTER 2

THE DENSITY OF COMPLEX ZEROS OF RANDOM SUMS

Let $\{a_j\}_{j=0}^N$ and $\{b_j\}_{j=0}^N$ be sequences of mutually independent and identically distributed (i.i.d.) random real Gaussian variables defined on the complete probability space $(\Omega, \mathscr{F}, P)$, with each sequence normally distributed with mean zero and variance one. As usual, Ω is a set with generic elements ω, $\mathscr{F}$ is a σ-field of subsets of Ω, and P is a probability measure on $\mathscr{F}$. Throughout this chapter, we shall assume that all sub σ-fields contain all sets of measure zero (see [19]). Let $\{f_j(z)\}_{j=0}^N$ be a sequence of given holomorphic functions that are real-valued on the real line. Furthermore, let us define the random sum

$$S_N(z) = \sum_{j=0}^{N} \eta_j f_j(z), \tag{2.1}$$

where z is the complex variable $x + iy$, and the η_j are i.i.d. random complex Gaussian variables (with density $e^{-z\bar{z}}/\pi$) given by $\eta_j = a_j + ib_j$ for $0 \leq j \leq N$. Suppose that Φ is a compact subset in the complex plane $\mathbb{C}$. We denote by $\nu_{N,\boldsymbol{K}}(\Phi)$ the number of complex zeros in Φ of the random sum $S_N(z)$ with respect to the complex level $\boldsymbol{K} = K_1 + iK_2$, where K_1 and K_2 are constants independent of z. We do not assume necessarily that the scalars K_1 and K_2 are equal. From [26, 29, 36, 54] (see, also, Equation (8.17) in [27]), we see that, with probability one, the value of the density function $h_{N,\boldsymbol{K}}(z)$ for multivariate Gaussian coefficients is given by

$$\mathscr{E}\nu_{N,\boldsymbol{K}}(\Phi) = \int_\Phi h_{N,\boldsymbol{K}}(z)\, dz, \tag{2.2}$$

where $\mathscr{E}\nu_{N,\boldsymbol{K}}(\Phi)$ is the mathematical expectation of $\nu_{N,\boldsymbol{K}}(\Phi)$. Thus, $h_{N,\boldsymbol{K}}(z)$ is the expected density of the complex zeros of the random equation

$$S_N(z) = \boldsymbol{K}.$$

Shepp and Vanderbei [48] wrote a beautiful paper on the complex zeros of the random polynomial $\sum_{j=0}^{N} \eta_j z^j$, where the η_j are i.i.d. random real Gaussian coefficients. In their paper, Shepp and Vanderbei introduced a sophisticated method based on Cauchy's argument principle for producing an explicit density function for the complex zeros. The method uses the Cholesky decomposition for representing correlated random Gaussian variables in terms of uncorrelated (and hence independent) random Gaussian variables. Shepp and Vanderbei generated computer plots of this density function and hundreds of thousands of zeros from randomly generated polynomials that show that, as the degree N becomes large, the zeros tend to lie very close to the unit circle and, when the real zeros are ignored, appear to be approximately uniformly distributed around the unit circle. Their asymptotics for the density function confirm the classical result due to Hammersley [32].

Ibragimov and Zeitouni [33] obtained the results in [48] using a different method, based on an integral representation of the average number of zeros of a random field. Furthermore, Ibragimov and Zeitouni demonstrated the limiting distributions for the density function under more general distributional assumptions.

In later work, Vanderbei [52] introduced a modest generalization to the central assumptions underlying the results in [48] and showed that comparable explicit formulas for the distribution of complex zeros can still be obtained for any N. Following the same general methodology given in [48], Vanderbei derived analogous explicit formulas for the density of complex zeros of the random sum $S_N(z)$ for the case when the η_j are i.i.d. random real Gaussian coefficients, and the $f_j(z)$ are given holomorphic functions that are real-valued on the real line.

In this chapter, we shall continue the line of investigation begun by Vanderbei and study the number of level crossings of the random sum $S_N(z)$ when the η_j are assumed to be i.i.d. random complex Gaussian variables. To consider this challenging general case, we shall employ a multivariate analysis approach based on results due to Adler [1], which provide a representation for the expected number of zeros of certain random fields. The method of proof was first applied by Ibragimov and Zeitouni [33]. Our main result generalizes the density function obtained independently by Yeager [54] and one of the authors [36] to nonzero $\boldsymbol{K}$. (See the remarks in [48, Section 6] and [52, Section 4].) Its proof exploits the assumption that the holomorphic functions $f_j(z)$ are real-valued on the real line. By Schwarz's reflection principle (see Ahlfors's classical book [3, pages 172–173]), these holomorphic functions have the property that $f_j(\overline{z}) = \overline{f_j(z)}$ for $0 \leq j \leq N$ and all $z \in \mathbb{C}$. Their derivatives also have this property.

Theorem 2.1 *Let the density function $h_{N,\boldsymbol{K}}(z)$ be defined by* (2.2)*. Under the conditions imposed on the random sum $S_N(z)$ and the sequences $\{\eta_j\}_{j=0}^N$ and $\{f_j(z)\}_{j=0}^N$ in* (2.1)*, for all integers $N > 1$ we have*

$$h_{N,\boldsymbol{K}}(z) = \frac{e^{-(K_1^2+K_2^2)/2B_{0,N}(z)}}{\pi B_{0,N}(z)} \left\{ B_{2,N}(z) - \frac{|B_{1,N}(z)|^2}{B_{0,N}(z)} \left(1 - \frac{K_1^2+K_2^2}{2B_{0,N}(z)}\right) \right\},$$

where the kernels $B_{r,N}(z)$ for $0 \leq r \leq 2$ are given by

$$B_{0,N}(z) = \sum_{j=0}^N |f_j(z)|^2, \quad B_{1,N}(z) = \sum_{j=0}^N \overline{f_j(z)} f_j'(z), \quad B_{2,N}(z) = \sum_{j=0}^N |f_j'(z)|^2.$$

The value of the density function $h_{N,\boldsymbol{K}}(z)$ is expressed in fairly simple terms, in that we can clearly see its form of dependence on $\boldsymbol{K}$. Its form is reminiscent of Farahmand and Jahangiri's [29, Theorem 1] expected density for the random polynomial $\sum_{j=0}^N \eta_j g_j z^j$ with respect to $\boldsymbol{K}$, where the g_j are given real constants. (See [26, 27] for the case when $g_j = 1$.) From Theorem 2.1, we have the following consequence.

Corollary 2.1.1 *For any vector $\boldsymbol{K}$ restricted to a circle of radius $K > 0$ and for all integers $N > 1$ we have*

$$h_{N,\boldsymbol{K}}(z) = \frac{e^{-K^2/B_{0,N}(z)}}{\pi B_{0,N}(z)} \left\{ B_{2,N}(z) - \frac{|B_{1,N}(z)|^2}{B_{0,N}(z)} \left(1 - \frac{K^2}{B_{0,N}(z)} \right) \right\}.$$

Immediate by Corollary 2.1.1 is the following consequence, which was proved independently by Yeager [54] and one of the authors [36].

Corollary 2.1.2 *If $\boldsymbol{K}$ is the zero vector, then for all integers $N > 1$ we have*

$$h_{N,\mathbf{0}}(z) = \frac{B_{0,N}(z)B_{2,N}(z) - |B_{1,N}(z)|^2}{\pi B_{0,N}(z)^2}.$$

In Section 2.1, we shall prove Theorem 2.1. In Section 2.2, we shall use the formula for the density function $h_{N,\boldsymbol{K}}(z)$ in Theorem 2.1 for the special choices of $f_j(z)$ to study its limiting behaviour as N tends to infinity. This shall demonstrate how the zeros of the random equation $S_N(z) = \boldsymbol{K}$ are clustered in the limit. In Section 2.3, using the appropriate forms of the Christoffel–Darboux formulas, we derive the density functions for the complex zeros of orthogonal polynomials, with the orthogonality relation being satisfied on the real line and the unit circle. These random polynomials have been studied by many authors, including Das [15], Lubinsky, Pritsker, and Xie [39, 40], Yattselev and Yeager [53], and Yeager [54, 55]. In connection to these works, we are led to study the density functions for their complex zeros with respect to $\boldsymbol{K}$.

Finally, we remark that the method introduced by Shepp and Vanderbei [48] could be applied in many circumstances. Their method could be modified to produce the number of $\boldsymbol{K}$ complex level crossings. Furthermore, working with real coefficients, in fact, makes the analysis more complicated. This will be addressed in a future work.

2.1 The Evaluation of the Density Function

We shall begin the proof of Theorem 2.1 by letting $X_{1,N}$ and $X_{2,N}$ be the real and imaginary parts of the random sum $S_N(z)$, respectively. For convenience in computation, we shall write

$$f_j(z) = u_j(x, y) + iv_j(x, y),$$

where $u_j(x, y)$ and $v_j(x, y)$ are real-valued functions of $(x, y) \in \mathbb{R}^2$. We have

$$S_N(z) = X_{1,N} + iX_{2,N},$$

where

$$X_{1,N} = \sum_{j=0}^{N}(a_j u_j - b_j v_j)$$

and

$$X_{2,N} = \sum_{j=0}^{N}(a_j v_j + b_j u_j).$$

In our application of Adler's theorem, we need to find all real and complex zeros of $S_N(z) = \boldsymbol{K}$. They are the zeros of the random equations $X_{1,N} = K_1$ and $X_{2,N} = K_2$ for $(x, y) \in \mathbb{R}^2$.

For the sake of brevity, we let $\boldsymbol{X}_N$ be the two-dimensional random field of the real and imaginary parts of the random sum $S_N(z)$ defined by the column vector $\boldsymbol{X}_N = (X_{1,N}, X_{2,N})'$, and we denote the Jacobian matrix of the transformation $(x, y) \longrightarrow (X_{1,N}, X_{2,N})$ by the matrix $\nabla \boldsymbol{X}_N$ of the first-order partial derivatives of $\boldsymbol{X}_N$ with respect to x and y, namely,

$$\nabla \boldsymbol{X}_N = \frac{\partial(X_{1,N}, X_{2,N})}{\partial(x, y)} = \begin{pmatrix} \dfrac{\partial X_{1,N}}{\partial x} & \dfrac{\partial X_{2,N}}{\partial x} \\ \dfrac{\partial X_{1,N}}{\partial y} & \dfrac{\partial X_{2,N}}{\partial y} \end{pmatrix}.$$

Let Φ be a compact subset in the complex plane $\mathbb{C}$ containing not more than a finite number of points such that $\boldsymbol{X}_N = \boldsymbol{K}$, where $\boldsymbol{K} = (K_1, K_2)'$. We assume that the boundary $\partial\Phi$ of Φ does not contain any points for which $\boldsymbol{X}_N = \boldsymbol{K}$ and Φ does not contain any points satisfying $\boldsymbol{X}_N = \boldsymbol{K}$ and $\det\nabla\boldsymbol{X}_N = \mathbf{0}$ at the same time. It is clear that the number of points is finite, as N is fixed. The two former conditions are satisfied in the problem setup. Since the set of points for which $\boldsymbol{X}_N = \boldsymbol{K}$ is of measure zero, the latter two conditions are satisfied almost surely. It is easy to check that the conditions in Theorem 5.1.1 and its Corollary in Adler's classical book [1, pages 95–97] hold (see, also, the papers by Azaïs and Wschebor [6] and Ibragimov and Zeitouni [33]). Hence, the density function $h_{N,\boldsymbol{K}}(z)$ for multivariate Gaussian coefficients given by (2.2) can be expressed through a conditioned expected value given by

$$\begin{aligned} h_{N,\boldsymbol{K}}(z) &= \mathscr{E}(|\det\nabla\boldsymbol{X}_N| \mid X_{1,N} = K_1, X_{2,N} = K_2)\, p_{X_{1,N},X_{2,N}}(K_1, K_2) \\ &= \mathscr{E}(|\det\nabla\boldsymbol{X}_N| \mid \boldsymbol{X}_N = \boldsymbol{K})\, p_{X_{1,N},X_{2,N}}(\boldsymbol{K}'), \end{aligned} \tag{2.3}$$

where $p_{X_{1,N},X_{2,N}}(\boldsymbol{K}')$ is the two-dimensional joint density function of the random vector $\boldsymbol{X}_N$. Since $\det\nabla\boldsymbol{X}_N$ is always nonnegative, let us eliminate the absolute value sign from future occurrences of the extreme right side of (2.3) in the evaluation of the density function $h_{N,\boldsymbol{K}}(z)$.

We now find the determinant of $\nabla\boldsymbol{X}_N$. It will be convenient to first use the Cauchy–Riemann equations to rewrite the expressions for $\partial X_{1,N}/\partial y$ and $\partial X_{2,N}/\partial y$. In order to obtain the conditional expectation of $|\det\nabla\boldsymbol{X}_N|$ on the extreme right side of (2.3), we separate the diagonal terms from the cross terms in the random determinant $\det\nabla\boldsymbol{X}_N$. It is an easy computation, and with a little algebra we find that

$$\begin{aligned}
\det\nabla\boldsymbol{X}_N &= \begin{vmatrix} \sum_{j=0}^{N}\left(a_j\frac{\partial u_j}{\partial x} - b_j\frac{\partial v_j}{\partial x}\right) & \sum_{j=0}^{N}\left(a_j\frac{\partial v_j}{\partial x} + b_j\frac{\partial u_j}{\partial x}\right) \\ \sum_{j=0}^{N}\left(-a_j\frac{\partial v_j}{\partial x} - b_j\frac{\partial u_j}{\partial x}\right) & \sum_{j=0}^{N}\left(a_j\frac{\partial u_j}{\partial x} - b_j\frac{v_j}{\partial x}\right) \end{vmatrix} \\
&= \sum_{j=0}^{N}\sum_{k=0}^{N}\left\{(a_ja_k + b_jb_k)\left(\frac{\partial u_j}{\partial x}\frac{\partial u_k}{\partial x} + \frac{\partial v_j}{\partial x}\frac{\partial v_k}{\partial x}\right)\right. \\
&\qquad\qquad \left. +(a_jb_k - b_ja_k)\left(\frac{\partial v_j}{\partial x}\frac{\partial u_k}{\partial x} - \frac{\partial u_j}{\partial x}\frac{\partial v_k}{\partial x}\right)\right\} \\
&= \sum_{j=0}^{N}(a_j^2 + b_j^2)\left\{\left(\frac{\partial u_j}{\partial x}\right)^2 + \left(\frac{\partial v_j}{\partial x}\right)^2\right\} \\
&\quad + \sum_{j=0}^{N}\sum_{\substack{k=0\\k\neq j}}^{N}\left\{(a_ja_k + b_jb_k)\left(\frac{\partial u_j}{\partial x}\frac{\partial u_k}{\partial x} + \frac{\partial v_j}{\partial x}\frac{\partial v_k}{\partial x}\right)\right. \\
&\qquad\qquad \left. +(a_jb_k - b_ja_k)\left(\frac{\partial v_j}{\partial x}\frac{\partial u_k}{\partial x} - \frac{\partial u_j}{\partial x}\frac{\partial v_k}{\partial x}\right)\right\}.
\end{aligned} \tag{2.4}$$

Thus, the evaluation of the density function $h_{N,\boldsymbol{K}}(z)$ leads to the computation of the expected value of a quadratic form $\det\nabla\boldsymbol{X}_N$ of i.i.d. Gaussian random variables, conditioned on two linear combinations.

In the following, we obtain the vectors of conditional expectations, variances, and covariance matrices of the multivariate random vectors $\boldsymbol{a}_N = (a_0, \ldots, a_N)'$ and $\boldsymbol{b}_N = (b_0, \ldots, b_N)'$. From standard methods in multivariate analysis (see the classical books by Anderson [4] and Tong [51]), based on the assumption that all the scalar random variables involved are independent and normally distributed, we define

$$\operatorname{Cov}(\boldsymbol{a}_N, \boldsymbol{b}_N \mid \boldsymbol{X}_N = \boldsymbol{K}) = \begin{pmatrix} \boldsymbol{\Sigma}_{\boldsymbol{a}_N\boldsymbol{a}_N,\boldsymbol{X}_N} & \boldsymbol{\Sigma}_{\boldsymbol{a}_N\boldsymbol{b}_N,\boldsymbol{X}_N} \\ \boldsymbol{\Sigma}_{\boldsymbol{b}_N\boldsymbol{a}_N,\boldsymbol{X}_N} & \boldsymbol{\Sigma}_{\boldsymbol{b}_N\boldsymbol{b}_N,\boldsymbol{X}_N} \end{pmatrix} \tag{2.5}$$

and

$$\mathscr{E}(\boldsymbol{a}_N \mid \boldsymbol{X}_N = \boldsymbol{K}) = \mathscr{E}\boldsymbol{a}_N + \boldsymbol{\Sigma}_{\boldsymbol{a}_N\boldsymbol{X}_N}\boldsymbol{\Sigma}^{-1}_{\boldsymbol{X}_N\boldsymbol{X}_N}(\boldsymbol{K} - \mathscr{E}\boldsymbol{X}_N)', \tag{2.6}$$

where

$$\boldsymbol{\Sigma}_{\boldsymbol{a}_N\boldsymbol{b}_N,\boldsymbol{X}_N} = \boldsymbol{\Sigma}_{\boldsymbol{a}_N\boldsymbol{b}_N} - \boldsymbol{\Sigma}_{\boldsymbol{a}_N\boldsymbol{X}_N}\boldsymbol{\Sigma}^{-1}_{\boldsymbol{X}_N\boldsymbol{X}_N}\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{b}_N} \tag{2.7}$$

and

$$\boldsymbol{\Sigma}_{\boldsymbol{a}_N\boldsymbol{b}_N} = \mathscr{E}(\boldsymbol{a}_N - \mathscr{E}\boldsymbol{a}_N)(\boldsymbol{b}_N - \mathscr{E}\boldsymbol{b}_N)', \tag{2.8}$$

which is a generalized covariance matrix of the vectors $\boldsymbol{a}_N$ and $\boldsymbol{b}_N$. Whereas the distribution of the a_j and b_j is central, we have $\mathscr{E}\boldsymbol{a}_N = 0$, $\mathscr{E}\boldsymbol{b}_N = 0$, and $\mathscr{E}\boldsymbol{X}_N = 0$.

From (2.8) and the assumption of the theorem

$$\boldsymbol{\Sigma}_{\boldsymbol{a}_N\boldsymbol{a}_N} = \mathscr{E}(\boldsymbol{a}_N - \mathscr{E}\boldsymbol{a}_N)(\boldsymbol{a}_N - \mathscr{E}\boldsymbol{a}_N)' = \mathscr{E}\boldsymbol{a}_N\boldsymbol{a}_N' = \boldsymbol{I}_N, \tag{2.9}$$

since the a_j are distributed according to an $\mathscr{N}(0,1)$ distribution and

$$\mathscr{E}a_ja_k = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases} \tag{2.10}$$

In a similar fashion, from (2.8)

$$\boldsymbol{\Sigma}_{\boldsymbol{b}_N\boldsymbol{b}_N} = \mathscr{E}(\boldsymbol{b}_N - \mathscr{E}\boldsymbol{b}_N)(\boldsymbol{b}_N - \mathscr{E}\boldsymbol{b}_N)' = \mathscr{E}\boldsymbol{b}_N\boldsymbol{b}_N' = \boldsymbol{I}_N. \tag{2.11}$$

Since

$$\mathscr{E}a_jb_k = 0, \tag{2.12}$$

we have

$$\boldsymbol{\Sigma}_{\boldsymbol{a}_N\boldsymbol{b}_N} = \mathscr{E}(\boldsymbol{a}_N - \mathscr{E}\boldsymbol{a}_N)(\boldsymbol{b}_N - \mathscr{E}\boldsymbol{b}_N)' = \mathscr{E}\boldsymbol{a}_N\boldsymbol{b}_N' = \boldsymbol{0}. \tag{2.13}$$

It follows that

$$\boldsymbol{\Sigma}_{\boldsymbol{b}_N \boldsymbol{a}_N} = \boldsymbol{\Sigma}'_{\boldsymbol{a}_N \boldsymbol{b}_N} = \mathbf{0}. \tag{2.14}$$

Now, from (2.10) and (2.12)

$$\mathscr{E} a_j X_{1,N} = \sum_{k=0}^{N} (\mathscr{E}(a_j a_k) u_k - \mathscr{E}(a_j b_k) v_k) = u_j \tag{2.15}$$

and

$$\mathscr{E} a_j X_{2,N} = \sum_{k=0}^{N} (\mathscr{E}(a_j a_k) v_k + \mathscr{E}(a_j b_k) u_k) = v_j. \tag{2.16}$$

If we apply (2.15) and (2.16) to (2.8), we obtain

$$\begin{aligned} \boldsymbol{\Sigma}_{\boldsymbol{a}_N \boldsymbol{X}_N} &= \mathscr{E}(\boldsymbol{a}_N - \mathscr{E}\boldsymbol{a}_N)(\boldsymbol{X}_N - \mathscr{E}\boldsymbol{X}_N)' = \mathscr{E}\boldsymbol{a}_N \boldsymbol{X}'_N \\ &= \begin{pmatrix} \mathscr{E} a_0 X_{1,N} & \mathscr{E} a_0 X_{2,N} \\ \mathscr{E} a_1 X_{1,N} & \mathscr{E} a_1 X_{2,N} \\ \vdots & \vdots \\ \mathscr{E} a_N X_{1,N} & \mathscr{E} a_N X_{2,N} \end{pmatrix} = \begin{pmatrix} u_0 & v_0 \\ u_1 & v_1 \\ \vdots & \vdots \\ u_N & v_N \end{pmatrix}. \end{aligned} \tag{2.17}$$

Thus,

$$\boldsymbol{\Sigma}_{\boldsymbol{X}_N \boldsymbol{a}_N} = \boldsymbol{\Sigma}'_{\boldsymbol{a}_N \boldsymbol{X}_N} = \begin{pmatrix} u_0 & u_1 & \dots & u_N \\ v_0 & v_1 & \dots & v_N \end{pmatrix}. \tag{2.18}$$

Proceeding as above, using (2.10) and (2.12) with the obvious substitutions, we obtain

$$\mathscr{E} b_j X_{1,N} = \sum_{k=0}^{N} (\mathscr{E}(b_j a_k) u_k - \mathscr{E}(b_j b_k) v_k) = -v_j \tag{2.19}$$

and

$$\mathscr{E} b_j X_{2,N} = \sum_{k=0}^{N} (\mathscr{E}(b_j a_k) v_k + \mathscr{E}(b_j b_k) u_k) = u_j. \tag{2.20}$$

Then applying (2.19) and (2.20) to (2.8), we get

$$\begin{aligned}\boldsymbol{\Sigma}_{\boldsymbol{b}_N\boldsymbol{X}_N} &= \mathscr{E}(\boldsymbol{b}_N - \mathscr{E}\boldsymbol{b}_N)(\boldsymbol{X}_N - \mathscr{E}\boldsymbol{X}_N)' = \mathscr{E}\boldsymbol{b}_N\boldsymbol{X}_N' \\ &= \begin{pmatrix} \mathscr{E}b_0X_{1,N} & \mathscr{E}b_0X_{2,N} \\ \mathscr{E}b_1X_{1,N} & \mathscr{E}b_1X_{2,N} \\ \vdots & \vdots \\ \mathscr{E}b_NX_{1,N} & \mathscr{E}b_NX_{2,N} \end{pmatrix} = \begin{pmatrix} -v_0 & u_0 \\ -v_1 & u_1 \\ \vdots & \vdots \\ -v_N & u_N \end{pmatrix}. \end{aligned} \tag{2.21}$$

Hence,

$$\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{b}_N} = \boldsymbol{\Sigma}'_{\boldsymbol{b}_N\boldsymbol{X}_N} = \begin{pmatrix} -v_0 & -v_1 & \dots & -v_N \\ u_0 & u_1 & \dots & u_N \end{pmatrix}. \tag{2.22}$$

Again, we use (2.8) to obtain

$$\begin{aligned}\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N} &= \mathscr{E}(\boldsymbol{X}_N - \mathscr{E}\boldsymbol{X}_N)(\boldsymbol{X}_N - \mathscr{E}\boldsymbol{X}_N)' \\ &= \begin{pmatrix} \mathscr{E}X_{1,N}X_{1,N} & \mathscr{E}X_{1,N}X_{2,N} \\ \mathscr{E}X_{2,N}X_{1,N} & \mathscr{E}X_{2,N}X_{2,N} \end{pmatrix}. \end{aligned} \tag{2.23}$$

We compute that

$$\begin{aligned}\mathscr{E}X_{1,N}X_{1,N} &= \mathscr{E}\left(\sum_{j=0}^{N}\sum_{k=0}^{N}(a_ju_j - b_jv_j)(a_ku_k - b_kv_k)\right) \\ &= \sum_{j=0}^{N}\sum_{k=0}^{N}(\mathscr{E}(a_ja_k)u_ju_k + \mathscr{E}(b_jb_k)v_jv_k - \mathscr{E}(a_jb_k)u_jv_k - \mathscr{E}(b_ja_k)v_ju_k) \\ &= \sum_{j=0}^{N}(u_j^2 + v_j^2) = \sum_{j=0}^{N}|f_j(z)|^2. \end{aligned} \tag{2.24}$$

We get, similarly to (2.24),

$$
\begin{aligned}
\mathscr{E} X_{2,N} X_{2,N} &= \mathscr{E}\left(\sum_{j=0}^{N}\sum_{k=0}^{N}(a_j v_j + b_j u_j)(a_k v_k + b_k v_k)\right) \\
&= \sum_{j=0}^{N}\sum_{k=0}^{N}(\mathscr{E}(a_j a_k) v_j v_k + \mathscr{E}(b_j b_k) u_j u_k + \mathscr{E}(a_j b_k) v_j u_k + \mathscr{E}(b_j a_k) u_j v_k) \\
&= \sum_{j=0}^{N}(u_j^2 + v_j^2) = \sum_{j=0}^{N}|f_j(z)|^2. \qquad (2.25)
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
\mathscr{E} X_{2,N} X_{1,N} = \mathscr{E} X_{1,N} X_{2,N} &= \mathscr{E}\left(\sum_{j=0}^{N}\sum_{k=0}^{N}(a_j u_j - b_j v_j)(a_k v_k + b_k u_k)\right) \\
&= \sum_{j=0}^{N}\sum_{k=0}^{N}\left(\mathscr{E}(a_j a_k) u_j v_k - \mathscr{E}(b_j b_k) v_j u_k + \mathscr{E}(a_j b_k) u_j u_k - \mathscr{E}(b_j a_k) v_j v_k\right) \\
&= \sum_{j=0}^{N}(u_j v_j - v_j u_j) = 0. \qquad (2.26)
\end{aligned}
$$

Then combining (2.24)–(2.26) in (2.23), we obtain

$$
\boldsymbol{\Sigma}_{\boldsymbol{X}_N \boldsymbol{X}_N} = \sum_{j=0}^{N}|f_j(z)|^2 \boldsymbol{I}_2.
$$

We note that the existence of the density function $h_{N,\boldsymbol{K}}(z)$ depends on the evaluation of the covariance matrix $\mathrm{Cov}(\boldsymbol{a}_N, \boldsymbol{b}_N \mid \boldsymbol{X}_N = \boldsymbol{K})$, which in turn depends on the existence of the inverse matrix $\boldsymbol{\Sigma}^{-1}_{\boldsymbol{X}_N \boldsymbol{X}_N}$. This is guaranteed, since

$$
\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N \boldsymbol{X}_N}| = \left(\sum_{j=0}^{N}|f_j(z)|^2\right)^2.
$$

Thus,

$$\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}^{-1} = \frac{\boldsymbol{I}_2}{\sqrt{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}|}}. \tag{2.27}$$

Moving now to the components of the covariance matrix given by (2.5), we obtain the results for the jth row and the kth column. Let δ_{jk} denote the Kronecker delta, that is,

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

From (2.7), (2.9), (2.17), (2.18), and (2.27)

$$\begin{aligned} (\boldsymbol{\Sigma}_{\boldsymbol{a}_N\boldsymbol{a}_N,\boldsymbol{X}_N})_{\substack{1\le j\le N \\ 1\le k\le N}} &= (\boldsymbol{\Sigma}_{\boldsymbol{a}_N\boldsymbol{a}_N})_{\substack{1\le j\le N \\ 1\le k\le N}} - (\boldsymbol{\Sigma}_{\boldsymbol{a}_N\boldsymbol{X}_N}\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{a}_N})_{\substack{1\le j\le N \\ 1\le k\le N}} \\ &= \delta_{jk} - \frac{u_j u_k + v_j v_k}{\sqrt{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}|}}. \end{aligned} \tag{2.28}$$

From (2.7), (2.11), (2.21), (2.22), and (2.27)

$$\begin{aligned} (\boldsymbol{\Sigma}_{\boldsymbol{b}_N\boldsymbol{b}_N,\boldsymbol{X}_N})_{\substack{1\le j\le N \\ 1\le k\le N}} &= (\boldsymbol{\Sigma}_{\boldsymbol{b}_N\boldsymbol{b}_N})_{\substack{1\le j\le N \\ 1\le k\le N}} - (\boldsymbol{\Sigma}_{\boldsymbol{b}_N\boldsymbol{X}_N}\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{b}_N})_{\substack{1\le j\le N \\ 1\le k\le N}} \\ &= \delta_{jk} - \frac{u_j u_k + v_j v_k}{\sqrt{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}|}}. \end{aligned} \tag{2.29}$$

From (2.7), (2.13), (2.17), (2.22), and (2.27)

$$\begin{aligned} (\boldsymbol{\Sigma}_{\boldsymbol{a}_N\boldsymbol{b}_N,\boldsymbol{X}_N})_{\substack{1\le j\le N \\ 1\le k\le N}} &= (\boldsymbol{\Sigma}_{\boldsymbol{a}_N\boldsymbol{b}_N})_{\substack{1\le j\le N \\ 1\le k\le N}} - (\boldsymbol{\Sigma}_{\boldsymbol{a}_N\boldsymbol{X}_N}\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{b}_N})_{\substack{1\le j\le N \\ 1\le k\le N}} \\ &= \frac{u_j v_k - v_j u_k}{\sqrt{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}|}}. \end{aligned} \tag{2.30}$$

From (2.7), (2.14), (2.18), (2.21), and (2.27)

$$(\boldsymbol{\Sigma}_{\boldsymbol{b}_N\boldsymbol{a}_N,\boldsymbol{X}_N})_{\substack{1\le j\le N \\ 1\le k\le N}} = (\boldsymbol{\Sigma}_{\boldsymbol{b}_N\boldsymbol{a}_N})_{\substack{1\le j\le N \\ 1\le k\le N}} - (\boldsymbol{\Sigma}_{\boldsymbol{b}_N\boldsymbol{X}_N}\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{a}_N})_{\substack{1\le j\le N \\ 1\le k\le N}}$$

$$= -\frac{u_j v_k - v_j u_k}{\sqrt{\det|\mathbf{\Sigma}_{\boldsymbol{X}_N \boldsymbol{X}_N}|}}. \tag{2.31}$$

We next find the necessary conditional expectations for computing $\mathscr{E}(\det\nabla\boldsymbol{X}_N \mid \boldsymbol{X}_N = \boldsymbol{K})$. The conditional expectations of $\boldsymbol{a}_N$ and $\boldsymbol{b}_N$ are easily derived, respectively, from (2.6), (2.17), and (2.27) as

$$\begin{aligned}(\mathscr{E}(a_j \mid \boldsymbol{X}_N = \boldsymbol{K}))_{1\le j\le N} &= (\mathbf{\Sigma}_{\boldsymbol{a}_N \boldsymbol{X}_N} \mathbf{\Sigma}^{-1}_{\boldsymbol{X}_N \boldsymbol{X}_N} \boldsymbol{K})_{1\le j\le N} \\ &= \frac{K_1 u_j + K_2 v_j}{\sqrt{\det|\mathbf{\Sigma}_{\boldsymbol{X}_N \boldsymbol{X}_N}|}}\end{aligned} \tag{2.32}$$

and from (2.6), (2.21), and (2.27) as

$$\begin{aligned}(\mathscr{E}(b_j \mid \boldsymbol{X}_N = \boldsymbol{K}))_{1\le j\le N} &= (\mathbf{\Sigma}_{\boldsymbol{b}_N \boldsymbol{X}_N} \mathbf{\Sigma}^{-1}_{\boldsymbol{X}_N \boldsymbol{X}_N} \boldsymbol{K})_{1\le j\le N} \\ &= \frac{K_2 u_j - K_1 v_j}{\sqrt{\det|\mathbf{\Sigma}_{\boldsymbol{X}_N \boldsymbol{X}_N}|}}.\end{aligned} \tag{2.33}$$

We derive from (2.32)

$$\begin{aligned}(\mathscr{E}(a_j^2 \mid \boldsymbol{X}_N = \boldsymbol{K}))_{1\le j\le N} &= (\mathscr{E}(a_j \mid \boldsymbol{X}_N = \boldsymbol{K}))^2_{1\le j\le N} + (\mathrm{Var}(a_j \mid \boldsymbol{X}_N = \boldsymbol{K}))_{1\le j\le N} \\ &= \frac{K_1^2 u_j^2 + K_2^2 v_j^2 + 2K_1 K_2 v_j v_j}{\det|\mathbf{\Sigma}_{\boldsymbol{X}_N \boldsymbol{X}_N}|} + 1 - \frac{u_j^2 + v_j^2}{\sqrt{\det|\mathbf{\Sigma}_{\boldsymbol{X}_N \boldsymbol{X}_N}|}}\end{aligned} \tag{2.34}$$

and from (2.33)

$$\begin{aligned}(\mathscr{E}(b_j^2 \mid \boldsymbol{X}_N = \boldsymbol{K}))_{1\le j\le N} &= (\mathscr{E}(b_j \mid \boldsymbol{X}_N = \boldsymbol{K}))^2_{1\le j\le N} + (\mathrm{Var}(b_j \mid \boldsymbol{X}_N = \boldsymbol{K}))_{1\le j\le N} \\ &= \frac{K_2^2 u_j^2 + K_1^2 v_j^2 - 2K_1 K_2 v_j v_j}{\det|\mathbf{\Sigma}_{\boldsymbol{X}_N \boldsymbol{X}_N}|} + 1 - \frac{u_j^2 + v_j^2}{\sqrt{\det|\mathbf{\Sigma}_{\boldsymbol{X}_N \boldsymbol{X}_N}|}}.\end{aligned} \tag{2.35}$$

By virtue of (2.34) and (2.35)

$$(\mathscr{E}(a_j^2 + b_j^2 \mid \boldsymbol{X}_N = \boldsymbol{K}))_{1\le j\le N} = \frac{(K_1^2 + K_2^2)(u_j^2 + v_j^2)}{\det|\mathbf{\Sigma}_{\boldsymbol{X}_N \boldsymbol{X}_N}|} + 2 - \frac{2(u_j^2 + v_j^2)}{\sqrt{\det|\mathbf{\Sigma}_{\boldsymbol{X}_N \boldsymbol{X}_N}|}}. \tag{2.36}$$

Next, using (2.5) and (2.28)–(2.32), we get

$$
\begin{aligned}
(\mathscr{E}(a_j a_k \mid \boldsymbol{X}_N = \boldsymbol{K}))_{\substack{1\le j\le N\\1\le k\le N}} &= (\mathscr{E}(a_j \mid \boldsymbol{X}_N = \boldsymbol{K}))_{1\le j\le N}\,(\mathscr{E}(a_k \mid \boldsymbol{X}_N = \boldsymbol{K}))_{1\le k\le N} \\
&\quad + (\mathrm{Cov}(a_j, a_k \mid \boldsymbol{X}_N = \boldsymbol{K}))_{\substack{1\le j\le N\\1\le k\le N}} \\
&= \frac{K_1^2 u_j u_k + K_2^2 v_j v_k + K_1 K_2 (u_j v_k + v_j u_k)}{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N \boldsymbol{X}_N}|} \\
&\quad - \frac{u_j u_k + v_j v_k}{\sqrt{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N \boldsymbol{X}_N}|}}.
\end{aligned}
\tag{2.37}
$$

Using (2.5), (2.28)–(2.31), and (2.33), we find

$$
\begin{aligned}
(\mathscr{E}(b_j b_k \mid \boldsymbol{X}_N = \boldsymbol{K}))_{\substack{1\le j\le N\\1\le k\le N}} &= (\mathscr{E}(b_j \mid \boldsymbol{X}_N = \boldsymbol{K}))_{1\le j\le N}\,(\mathscr{E}(b_k \mid \boldsymbol{X}_N = \boldsymbol{K}))_{1\le k\le N} \\
&\quad + (\mathrm{Cov}(b_j, b_k \mid \boldsymbol{X}_N = \boldsymbol{K}))_{\substack{1\le j\le N\\1\le k\le N}} \\
&= \frac{K_2^2 u_j u_k + K_1^2 v_j v_k - K_1 K_2 (u_j v_k + v_j u_k)}{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N \boldsymbol{X}_N}|} \\
&\quad - \frac{u_j u_k + v_j v_k}{\sqrt{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N \boldsymbol{X}_N}|}}.
\end{aligned}
\tag{2.38}
$$

By virtue of (2.37) and (2.38)

$$
(\mathscr{E}(a_j a_k + b_j b_k \mid \boldsymbol{X}_N = \boldsymbol{K}))_{\substack{1\le j\le N\\1\le k\le N}} = \frac{(K_1^2 + K_2^2)(u_j u_k + v_j v_k)}{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N \boldsymbol{X}_N}|} - \frac{2(u_j u_k + v_j v_k)}{\sqrt{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N \boldsymbol{X}_N}|}}. \tag{2.39}
$$

Next, we derive from (2.5) and (2.28)–(2.33)

$$
\begin{aligned}
(\mathscr{E}(a_j b_k \mid \boldsymbol{X}_N = \boldsymbol{K}))_{\substack{1\le j\le N\\1\le k\le N}} &= (\mathscr{E}(a_j \mid \boldsymbol{X}_N = \boldsymbol{K}))_{1\le j\le N}\,(\mathscr{E}(b_k \mid \boldsymbol{X}_N = \boldsymbol{K}))_{1\le k\le N} \\
&\quad + (\mathrm{Cov}(a_j, b_k \mid \boldsymbol{X}_N = \boldsymbol{K}))_{\substack{1\le j\le N\\1\le k\le N}} \\
&= \frac{K_1 K_2 (u_j u_k - v_j v_k) - K_1^2 u_j v_k + K_2^2 v_j u_k}{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N \boldsymbol{X}_N}|} \\
&\quad + \frac{u_j v_k - v_j u_k}{\sqrt{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N \boldsymbol{X}_N}|}}
\end{aligned}
\tag{2.40}
$$

and

$$
\begin{aligned}
(\mathscr{E}(b_j a_k \mid \boldsymbol{X}_N = \boldsymbol{K}))_{\substack{1\le j\le N\\1\le k\le N}} &= (\mathscr{E}(b_j \mid \boldsymbol{X}_N = \boldsymbol{K}))_{1\le j\le N}\,(\mathscr{E}(a_k \mid \boldsymbol{X}_N = \boldsymbol{K}))_{1\le k\le N}\\
&\quad + (\mathrm{Cov}(b_j, a_k \mid \boldsymbol{X}_N = \boldsymbol{K}))_{\substack{1\le j\le N\\1\le k\le N}}\\
&= \frac{K_1K_2(u_ju_k - v_jv_k) + K_2^2 u_jv_k - K_1^2 v_ju_k}{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}|}\\
&\quad - \frac{u_jv_k - v_ju_k}{\sqrt{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}|}}.
\end{aligned}
\tag{2.41}
$$

By virtue of (2.40) and (2.41)

$$
(\mathscr{E}(a_jb_k - b_ja_k \mid \boldsymbol{X}_N = \boldsymbol{K}))_{\substack{1\le j\le N\\1\le k\le N}} = \frac{(K_1^2 + K_2^2)(v_ju_k - u_jv_k)}{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}|} + \frac{2(u_jv_k - v_ju_k)}{\sqrt{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}|}}. \tag{2.42}
$$

It remains to evaluate the conditional expectation of the random determinant $\det\nabla\boldsymbol{X}_N$. From (2.4), (2.36), (2.39), and (2.42), it emerges from an arduous calculation that

$$
\begin{aligned}
&\mathscr{E}(\det\nabla\boldsymbol{X}_N \mid \boldsymbol{X}_N = \boldsymbol{K})\\
&= \sum_{j=0}^{N} \mathscr{E}(a_j^2 + b_j^2 \mid \boldsymbol{X}_N = \boldsymbol{K}) \left\{ \left(\frac{\partial u_j}{\partial x}\right)^2 + \left(\frac{\partial v_j}{\partial x}\right)^2 \right\}\\
&\quad + \sum_{j=0}^{N}\sum_{\substack{k=0\\k\ne j}}^{N} \left\{ \mathscr{E}(a_ja_k + b_jb_k \mid \boldsymbol{X}_N = \boldsymbol{K}) \left(\frac{\partial u_j}{\partial x}\frac{\partial u_k}{\partial x} + \frac{\partial v_j}{\partial x}\frac{\partial v_k}{\partial x}\right)\right\}\\
&\quad + \sum_{j=0}^{N}\sum_{\substack{k=0\\k\ne j}}^{N} \left\{ \mathscr{E}(a_jb_k - b_ja_k \mid \boldsymbol{X}_N = \boldsymbol{K}) \left(\frac{\partial v_j}{\partial x}\frac{\partial u_k}{\partial x} - \frac{\partial u_j}{\partial x}\frac{\partial v_k}{\partial x}\right)\right\}\\
&= 2\sum_{j=0}^{N} |f_j'(z)|^2 - \left(\frac{2}{\sqrt{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}|}} - \frac{K_1^2 + K_2^2}{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}|}\right)\\
&\qquad\qquad \times \sum_{j=0}^{N}\sum_{k=0}^{N} \left\{ (u_ju_k + v_jv_k)\left(\frac{\partial u_j}{\partial x}\frac{\partial u_k}{\partial x} + \frac{\partial v_j}{\partial x}\frac{\partial v_k}{\partial x}\right)\right.\\
&\qquad\qquad\qquad \left. +(v_ju_k - u_jv_k)\left(\frac{\partial v_j}{\partial x}\frac{\partial u_k}{\partial x} - \frac{\partial u_j}{\partial x}\frac{\partial v_k}{\partial x}\right)\right\}.
\end{aligned}
$$

An uninspired calculation then shows that the double sum on the extreme right side can be reduced to

$$\begin{aligned}
&\sum_{j=0}^{N}\sum_{k=0}^{N}\left(u_j\frac{\partial u_j}{\partial x}+v_j\frac{\partial v_j}{\partial x}\right)\left(u_k\frac{\partial u_k}{\partial x}+v_k\frac{\partial v_k}{\partial x}\right)\\
&\qquad+\sum_{j=0}^{N}\sum_{k=0}^{N}\left(u_j\frac{\partial v_j}{\partial x}-v_j\frac{\partial u_j}{\partial x}\right)\left(u_k\frac{\partial v_k}{\partial x}-v_k\frac{\partial u_k}{\partial x}\right)\\
&\qquad=\left\{\sum_{j=0}^{N}\left(u_j\frac{\partial u_j}{\partial x}+v_j\frac{\partial v_j}{\partial x}\right)\right\}^2+\left\{\sum_{j=0}^{N}\left(u_j\frac{\partial v_j}{\partial x}-v_j\frac{\partial u_j}{\partial x}\right)\right\}^2\\
&\qquad=\left|\sum_{j=0}^{N}\left\{\left(u_j\frac{\partial u_j}{\partial x}+v_j\frac{\partial v_j}{\partial x}\right)+i\left(u_j\frac{\partial v_j}{\partial x}-v_j\frac{\partial u_j}{\partial x}\right)\right\}\right|^2\\
&\qquad=\left|\sum_{j=0}^{N}(u_j-iv_j)\left(\frac{\partial u_j}{\partial x}+i\frac{\partial v_j}{\partial x}\right)\right|^2=\left|\sum_{j=0}^{N}\overline{f_j(z)}f_j'(z)\right|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathscr{E}(\det\nabla\boldsymbol{X}_N\mid\boldsymbol{X}_N=\boldsymbol{K})\\
&\quad=\frac{2}{\sqrt{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}|}}\left(\sqrt{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}|}\sum_{k=0}^{N}|f_k'(z)|^2-\left|\sum_{j=0}^{N}\overline{f_j(z)}f_j'(z)\right|^2\right)\\
&\quad+\frac{K_1^2+K_2^2}{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}|}\left|\sum_{j=0}^{N}\overline{f_j(z)}f_j'(z)\right|^2.
\end{aligned}\tag{2.43}$$

For definiteness, we recall from [31, Chapter 10] (see, also, [51, Chapter 2]) that the joint density of two random real Gaussian variables $X_{1,N}$ and $X_{2,N}$ at the points K_1 and K_2, respectively, is equal to

$$p_{X_{1,N},X_{2,N}}(K_1,K_2)=\frac{1}{2\pi\sigma^2\sqrt{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}|}}\exp\left(-\frac{(K_1-\mathscr{E}X_1)^2+(K_2-\mathscr{E}X_2)^2}{2\sigma^2\sqrt{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}|}}\right).$$

In our case, the conditions $\mathscr{E}X_1 = 0, \mathscr{E}X_2 = 0$, and $\sigma^2 = 1$ apply. Thus, we have

$$p_{X_{1,N},X_{2,N}}(\boldsymbol{K}') = \frac{1}{2\pi\sqrt{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}|}} \exp\left(-\frac{K_1^2 + K_2^2}{\sqrt{\det|\boldsymbol{\Sigma}_{\boldsymbol{X}_N\boldsymbol{X}_N}|}}\right). \tag{2.44}$$

By virtue of (2.3), (2.43), and (2.44), upon simplifying and applying the formulas for the kernels $B_{r,N}(z)$ for $0 \leq r \leq 2$ in Theorem 2.1, the required result follows.

2.2 The Asymptotic Analysis

It is well known and, for example, Farahmand [21] has shown that, for large values of N, the real zeros of random polynomials with real coefficients are clustered about ± 1. (See, also, Bharucha-Reid and Sambandham's book [9].) In order to understand better the behaviour of the density function $h_{N,\boldsymbol{K}}(z)$ in Theorem 2.1 as N tends to infinity, we define special values of the functions $f_j(z)$. Indeed, we are restricted to the cases that the evaluation of sums in Theorem 2.1 becomes analytically feasible. To exhibit the numerical behaviour of the density function $h_{N,\boldsymbol{K}}(z)$ and the zeros of the random equation $S_N(z) = \boldsymbol{K}$ for various values of N numerically, we used the general computing environment Wolfram Mathematica® version number 12.0.0.0 developed by Wolfram Research for the platform Mac OS X x86 (64-bit), which ran on the Apple Mac Pro (late 2013) with the 2.7 GHz 12-core Intel® Xeon® Processor E5-2697 v2.

The simplest example of random sums is when

$$f_j(z) = z^j. \tag{2.45}$$

The resulting limits are best expressed in terms of the function

$$B(z) = \frac{1}{1 - |z|^2}.$$

Using the notation given in Theorem 2.1, we have

$$B_{0,N}(z) = \sum_{j=0}^{N} |z|^{2j} = (1 - |z|^{2N+2})B(z).$$

By repeated differentiation, we obtain

$$zB_{1,N}(z) = \sum_{j=0}^{N} j|z|^{2j} = (N|z|^{2N+2} - (N+1)|z|^{2N} + 1)|z|^2 B(z)^2$$

and

$$B_{2,N}(z) = \sum_{j=0}^{N} j^2|z|^{2j-2} = (1 + |z|^2 - |z|^{2N}(N^2|z|^4 - (2N^2 + 2N - 1)|z|^2 + (N+1)^2))B(z)^3.$$

Clearly, if $|z| < 1$, then

$$\lim_{N\to\infty} B_{0,N}(z) = B(z)$$

and

$$\lim_{N\to\infty} zB_{1,N}(z) = |z|^2 B(z)^2,$$

as well as

$$\lim_{N\to\infty} B_{2,N}(z) = (1 + |z|^2)B(z)^3.$$

The following follows from Theorem 2.1.

THEOREM 2.2 *Let the sequence of functions* $\{f_j(z)\}_{j=0}^{N}$ *in the definition of the random sum* $S_N(z)$ *in* (2.1) *be given by* (2.45). *If* $|z| < 1$, *then we have*

$$\lim_{N\to\infty} h_{N,\boldsymbol{K}}(z) = \frac{1}{\pi} e^{-(K_1^2 + K_2^2)/2B(z)} B(z) \left\{ B(z) + \left(\frac{K_1^2 + K_2^2}{2} \right) |z|^2 \right\}.$$

Then for any vector $\boldsymbol{K}$ *restricted to a circle of radius* $K > 0$ *we have*

$$\lim_{N\to\infty} h_{N,\boldsymbol{K}}(z) = \frac{1}{\pi} e^{-K^2/B(z)} B(z)(B(z) + K^2|z|^2).$$

Furthermore, if $\boldsymbol{K}$ *is the zero vector, then we have*

$$\lim_{N\to\infty} h_{N,\mathbf{0}}(z) = \frac{1}{\pi} B(z)^2.$$

We note that, in all the cases considered, the limiting value of the density function $h_{N,\boldsymbol{K}}(z)$ has $|z|^4 - 2|z|^2 + 1$ in its denominators. An exponential factor is present when $\boldsymbol{K}$ is nonzero.

If now $|z| > 1$, then for all sufficiently large N we can write

$$B_{0,N}(z) \sim -|z|^{2N} B(z)$$

and

$$zB_{1,N}(z) \sim |z|^2 B(z)^2,$$

as well as

$$B_{2,N}(z) \sim (1 + |z|^2 - N^2|z|^{2N}(|z|^4 - |z|^2 + 1))B(z)^3.$$

The following follows from Theorem 2.1.

THEOREM 2.3 *Let the sequence of functions* $\{f_j(z)\}_{j=0}^N$ *in the definition of the random sum* $S_N(z)$ *in* (2.1) *be given by* (2.45). *If* $|z| > 1$, *then for all sufficiently large* N *we have*

$$h_{N,\boldsymbol{K}}(z) \sim \frac{1}{\pi} e^{(K_1^2+K_2^2)/2|z|^{2N}B(z)} \\ \times B(z)^2 \left\{ N^2(|z|^4 - |z|^2 + 1) - \frac{|z|^2+1}{|z|^{2N}} - \frac{|z|^2}{|z|^{4N}} \left(1 + \frac{K_1^2+K_2^2}{2|z|^{2N}B(z)}\right) \right\}.$$

Then for any vector $\boldsymbol{K}$ *restricted to a circle of radius* $K > 0$ *we have*

$$h_{N,\boldsymbol{K}}(z) \sim \frac{1}{\pi} e^{K^2/|z|^{2N} B(z)} \times B(z)^2 \left\{ N^2(|z|^4 - |z|^2 + 1) - \frac{|z|^2+1}{|z|^{2N}} - \frac{|z|^2}{|z|^{4N}} \left(1 + \frac{K^2}{|z|^{2N} B(z)}\right) \right\}.$$

Furthermore, if $\boldsymbol{K}$ *is the zero vector, then we have*

$$h_{N,\mathbf{0}}(z) \sim \frac{1}{\pi} B(z)^2 \left(N^2(|z|^4 - |z|^2 + 1) - \frac{|z|^2+1}{|z|^{2N}} - \frac{|z|^2}{|z|^{4N}} \right).$$

Using the appropriate power sum formulas in Theorem 2.1, we obtain the following.

THEOREM 2.4 *Let the sequence of functions* $\{f_j(z)\}_{j=0}^N$ *in the definition of the random sum* $S_N(z)$ *in* (2.1) *be given by* (2.45). *If* $z = \pm 1$, *then we have*

$$h_{N,\boldsymbol{K}}(\pm 1) = \frac{1}{12\pi} e^{-(K_1^2+K_2^2)/(2N+2)} \left\{ 2N + N^2 \left(1 + \frac{3(K_1^2+K_2^2)}{2N+2}\right) \right\}.$$

Then for any vector $\boldsymbol{K}$ *restricted to a circle of radius* $K > 0$ *we have*

$$h_{N,\boldsymbol{K}}(\pm 1) = \frac{1}{12\pi} e^{-K^2/(N+1)} \left\{ 2N + N^2 \left(1 + \frac{3K^2}{N+1}\right) \right\}.$$

Furthermore, if $\boldsymbol{K}$ *is the zero vector, then we have*

$$h_{N,\mathbf{0}}(\pm 1) = \frac{1}{12\pi} N(N+2).$$

In Figure 2.1, the left-hand plot is a grey-scale image of the density function $h_{N,\boldsymbol{K}}(z)$ with $N = 10$ and $\boldsymbol{K} = (10, 10)'$. The right-hand plot shows the zeros obtained by generating 20,000 random polynomials and explicitly finding the zeros of the random equation $S_{10}(z) = \boldsymbol{K}$. The zeros cluster near the unit circle, and the density function does not have mass concentrated on the real axis. There is no jump present near the real axis. For larger values of $\boldsymbol{K}$ the effect is more pronounced.

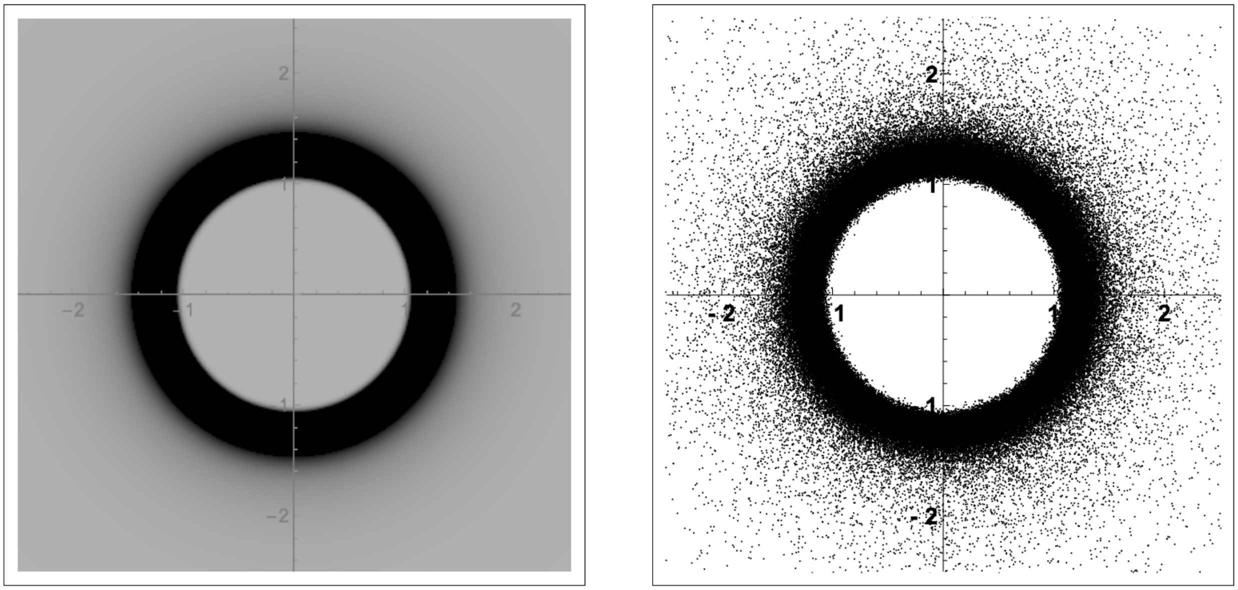

Figure 2.1 Twenty thousand random degree 10 polynomials for the random equation $S_{10}(z) = \eta_0 + \eta_1 z + \eta_2 z^2 + \ldots + \eta_{10} z^{10} = 10 + i10$

Also of interest are random Weyl polynomials in which

$$f_j(z) = \frac{z^j}{\sqrt{j!}}, \tag{2.46}$$

also studied by Farahmand and Jahangiri [29], Littlewood and Offord [37, 38], Offord [42], and Vanderbei [52]. For this case, the limiting forms of the various functions defining $h_{N,\boldsymbol{K}}(z)$ are computed by repeated differentiation. We have

$$\lim_{N\to\infty} B_{0,N}(z) = \lim_{N\to\infty} \sum_{j=0}^{N} \frac{|z|^{2j}}{j!} = e^{|z|^2}$$

and

$$\lim_{N\to\infty} B_{1,N}(z) = \lim_{N\to\infty} \sum_{j=0}^{N} \frac{|z|^{2j}}{(j-1)!z} = \overline{z}e^{|z|^2},$$

as well as

$$\lim_{N\to\infty} B_{2,N}(z) = \lim_{N\to\infty} \sum_{j=0}^{N} \frac{j|z|^{2j-2}}{(j-1)!} = (|z|^2+1)e^{|z|^2}.$$

Substituting these values into Theorem 2.1, we obtain the following result.

THEOREM 2.5 *Let the sequence of functions* $\{f_j(z)\}_{j=0}^N$ *in the definition of the random sum* $S_N(z)$ *in* (2.1) *be given by* (2.46). *Then we have*

$$\lim_{N\to\infty} h_{N,\boldsymbol{K}}(z) = \frac{1}{\pi} e^{-(K_1^2+K_2^2)/2e^{|z|^2}} \left(1 + \frac{(K_1^2+K_2^2)|z|^2}{2e^{|z|^2}}\right).$$

Then for any vector $\boldsymbol{K}$ *restricted to a circle of radius* $K>0$ *we have*

$$\lim_{N\to\infty} h_{N,\boldsymbol{K}}(z) = \frac{1}{\pi} e^{-K^2/e^{|z|^2}} \left(1 + \frac{K^2|z|^2}{e^{|z|^2}}\right).$$

Furthermore, if $\boldsymbol{K}$ *is the zero vector, then we have*

$$\lim_{N\to\infty} h_{N,\mathbf{0}}(z) = \frac{1}{\pi}.$$

We note that the distribution of the real zeros becomes uniform over the real line. The complex zeros are much more uniformly distributed than was the case when the factor $1/\sqrt{j!}$ was not present. The pictures in Figure 2.2 show the density function $h_{10,\boldsymbol{K}}(z)$ and the empirical distribution for 20,000 random sums when $f_j(z) = z^j/\sqrt{j!}$ and $\boldsymbol{K} = (10,10)'$, that is, random degree 10 Weyl polynomials. The behaviour of the density function and the empirical distribution for the random sums becomes very noticeable and intensified when $\boldsymbol{K}$ is increased.

Next, let us assume that

$$f_j(z) = \sqrt{\binom{N}{j}\frac{1}{j+1}}\, z^j. \tag{2.47}$$

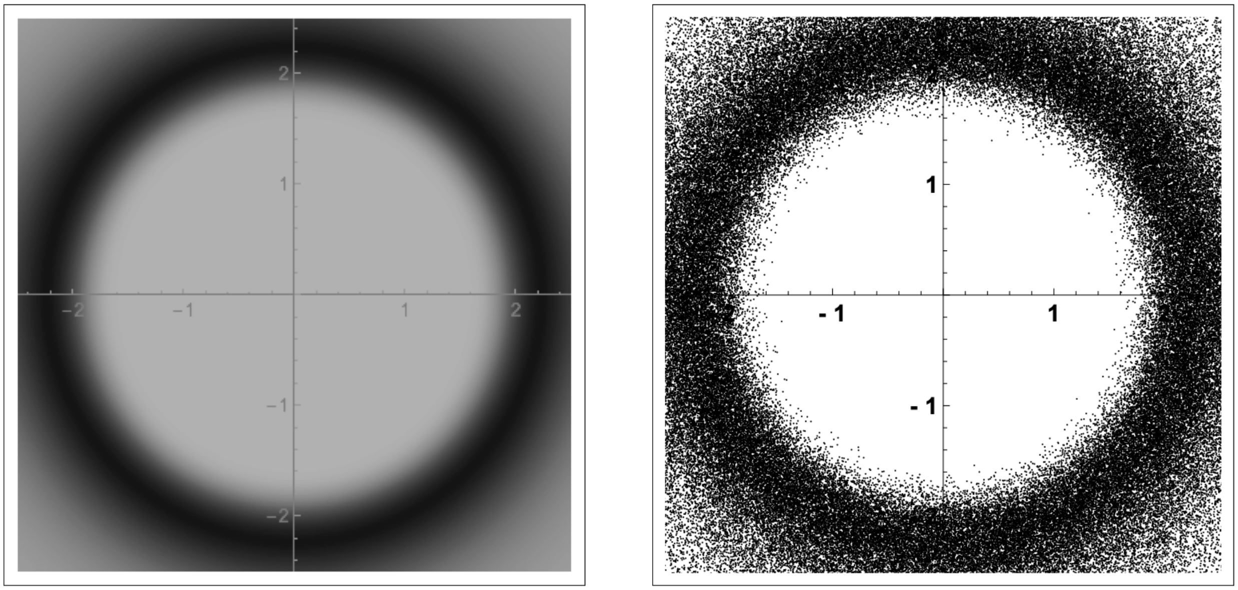

Figure 2.2 Twenty thousand random degree 10 Weyl polynomials for the random equation $S_{10}(z) = \eta_0 + \eta_1 z + \eta_2 z^2/\sqrt{2!} + \ldots + \eta_{10} z^{10}/\sqrt{10!} = 10 + i10$

The random root-binomial polynomials were also studied by Farahmand and Jahangiri [29], Littlewood and Offord [37, 38], Offord [42], and Vanderbei [52]. The pictures in Figure 2.3 show the density function $h_{10,\boldsymbol{K}}(z)$ and the empirical distribution for 20,000 random degree 10 root-binomial polynomials with $\boldsymbol{K} = (50, 50)'$.

By repeated differentiation, we can easily verify that

$$B_{0,N}(z) = \sum_{j=0}^{N} \binom{N}{j} \frac{|z|^{2j}}{j+1} = \frac{(|z|^2+1)^{N+1} - 1}{(N+1)|z|^2}$$

and

$$B_{1,N}(z) = \sum_{j=0}^{N} \binom{N}{j} \frac{j|z|^{2j}}{(j+1)z} = \frac{(N|z|^2-1)(|z|^2+1)^N - 1}{(N+1)z|z|^2},$$

as well as

$$B_{2,N}(z) = \sum_{j=0}^{N} \binom{N}{j} \frac{j^2|z|^{2j-2}}{j+1} = \frac{(|z|^2+1)^N(|z|^2(N^2|z|^2 - N + 1) + 1) - |z|^2 - 1}{(N+1)|z|^4(|z|^2+1)}.$$

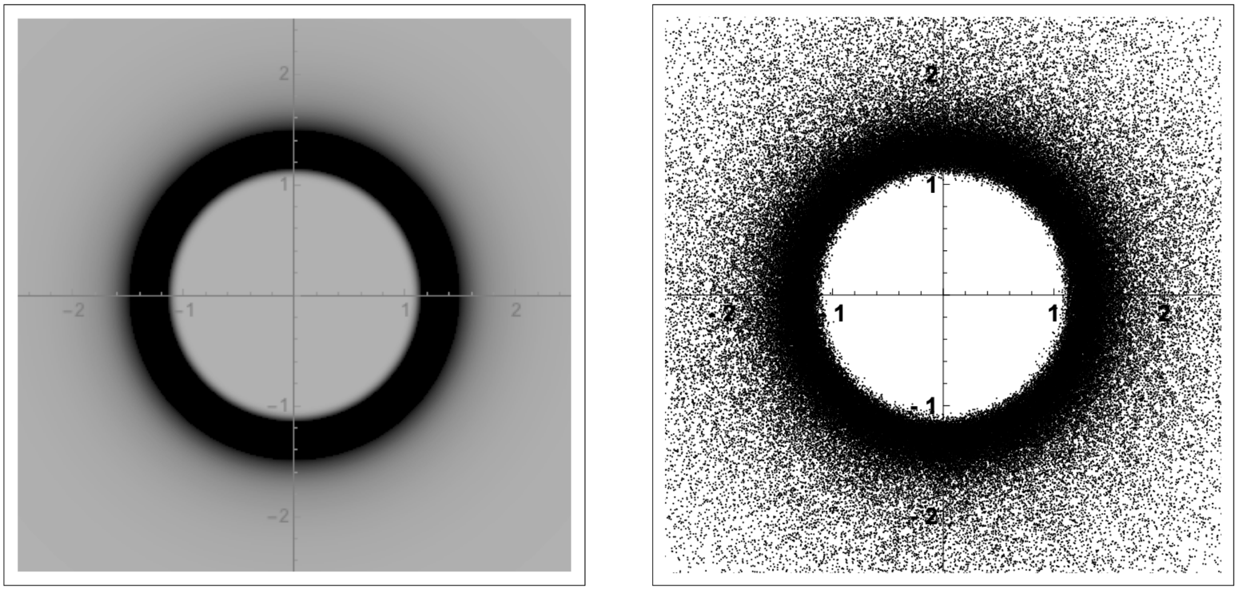

Figure 2.3 Twenty thousand random degree 10 root-binomial polynomials for the random equation $S_{10}(z) = \eta_0 + \sqrt{5}\eta_1 z + \sqrt{15}\eta_2 z^2 + \ldots + \sqrt{1/11}\eta_{10} z^{10} = 50 + i50$

If we now assume that $|z| > 0$, that is, except at the origin, for all sufficiently large N we can write

$$B_{0,N}(z) \sim \frac{(|z|^2+1)^N}{N|z|^2}$$

and

$$B_{1,N}(z) \sim \frac{(|z|^2+1)^N}{z},$$

as well as

$$B_{2,N}(z) \sim N(|z|^2+1)^{N-1}.$$

We immediately get the following from Theorem 2.1.

THEOREM 2.6 *Let the sequence of functions $\{f_j(z)\}_{j=0}^N$ in the definition of the random sum $S_N(z)$ in* (2.1) *be given by* (2.47). *Then for $|z| > 0$ and all sufficiently large N we have*

$$h_{N,\boldsymbol{K}}(z) \sim e^{-(K_1^2+K_2^2)N|z|^2/2(|z|^2+1)^N} \frac{N^2|z|^4((K_1^2+K_2^2)N - 2(|z|^2+1)^{N-1})}{2\pi(|z|^2+1)^N}.$$

Then for any vector $\boldsymbol{K}$ restricted to a circle of radius $K > 0$ we have

$$h_{N,\boldsymbol{K}}(z) \sim e^{-K^2N|z|^2/(|z|^2+1)^N} \frac{N^2|z|^4(K^2N - (|z|^2+1)^{N-1})}{\pi(|z|^2+1)^N}.$$

Furthermore, if $\boldsymbol{K}$ is the zero vector, then we have

$$h_{N,\mathbf{0}}(z) \sim -\frac{N^2|z|^4}{\pi(|z|^2+1)}.$$

Finally, we consider examples of random trigonometric sums. The behaviour of the density function $h_{10,\boldsymbol{K}}(z)$ and the empirical distribution for a family of 20,000 random sums with $f_j(z) = \cos jz$ for $0 \leq j \leq 10$ and $\boldsymbol{K} = (50, 50)'$ can be seen in Figure 2.4. Since $\cos iy = \cosh y$, these random truncated Fourier cosine series are real-valued on both the real axis and the orthogonal imaginary axis. Figure 2.5 shows the corresponding behaviour for the random truncated Fourier sine/cosine series defined by

$$f_j(z) = \begin{cases} \cos\left(\dfrac{jz}{2}\right) & \text{if } j \text{ is even,} \\ \sin\left(\dfrac{(j+1)z}{2}\right) & \text{if } j \text{ is odd.} \end{cases}$$

The right-hand plot requires 10 days, 16 hours, and 48 minutes of Central Processing Unit time to generate. The density functions for these examples have been studied by Vanderbei [52] solely for the case when the η_j are i.i.d. random real Gaussian coefficients and $\boldsymbol{K} = (0, 0)'$. For this case, Vanderbei verified that the density function for the random truncated Fourier sine/cosine series depends on the imaginary part of z only.

2.3 The Crossings of Random Orthogonal Polynomials

We shall now consider the case when the functions $f_j(z)$ are either polynomials $p_j(z)$ orthogonal on the real line or polynomials $\varphi_j(z)$ orthogonal on the unit circle. These or-

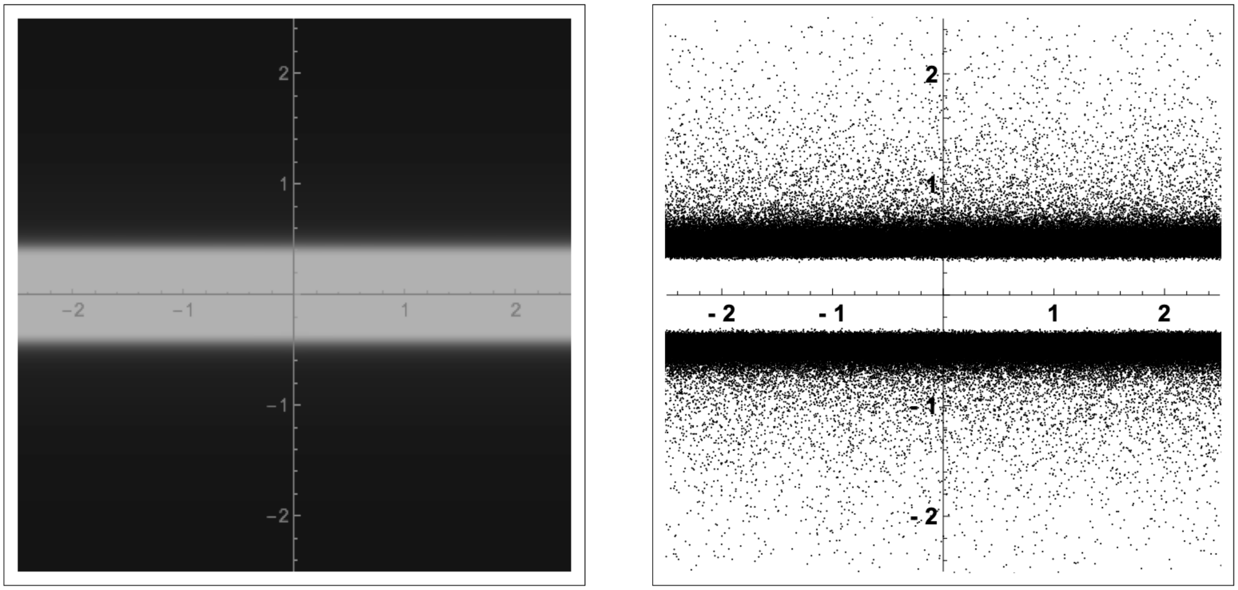

Figure 2.4 Twenty thousand random sums of the first 10 terms in a Fourier cosine series for the random equation $S_{10}(z) = \eta_0 + \eta_1 \cos z + \eta_2 \cos 2z + \ldots + \eta_{10} \cos 10z = 50 + i50$

thogonal polynomials have real coefficients, and are real-valued on the real line. We shall examine these objects in turn. First, we let α denote a nondecreasing function with an infinite number of points of increase in the interval $[a, b]$. Assuming that moments of all orders exist, we say that a sequence of polynomials $\{p_j(z)\}_{j=0}^{\infty}$, where the $p_j(z)$ have degree N, is orthogonal with respect to the distribution $d\alpha$ if

$$\int_a^b p_j(z) p_k(z)\, d\alpha(z) = \delta_{jk}.$$

From [5, Theorem 5.2.4] (see, also, [50, Theorem 3.2.2]), the Christoffel–Darboux formula for orthogonal polynomials $p_j(z)$ on the real line can be stated as follows: Let k_j be the highest coefficient of $p_j(z)$. Suppose that the orthogonal polynomials $p_j(z)$ are normalized. Then for complex variables z and w

$$\sum_{j=0}^{N} p_j(z) p_j(w) = \frac{k_N}{k_{N+1}} \left(\frac{p_{N+1}(z) p_N(w) - p_N(z) p_{N+1}(w)}{z - w} \right). \tag{2.48}$$

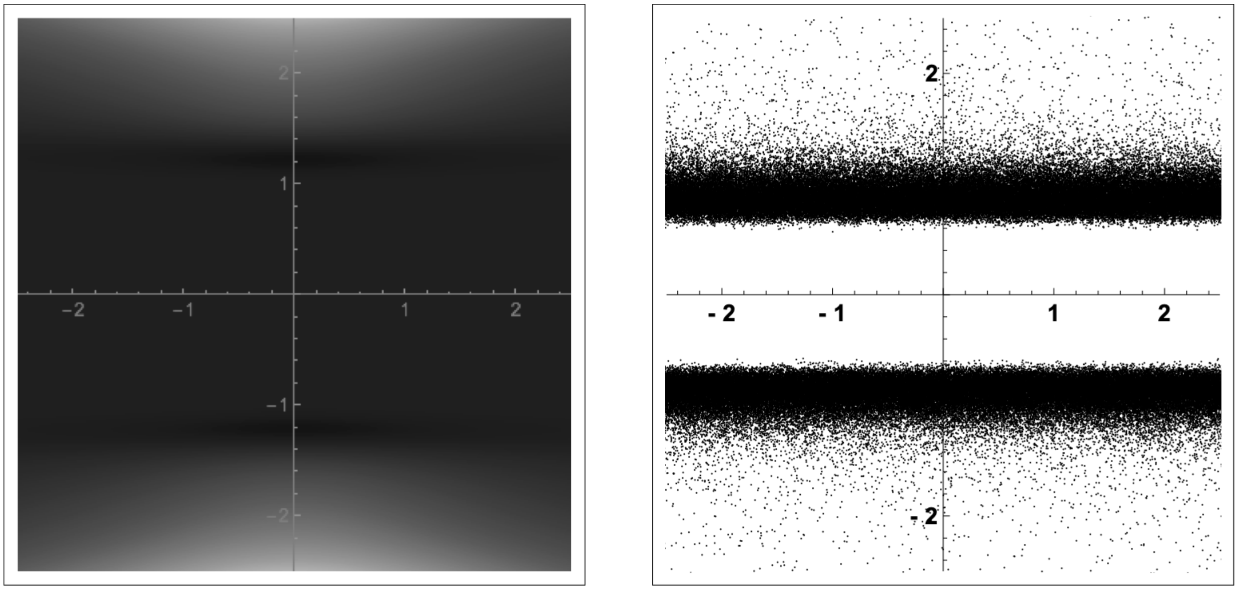

Figure 2.5 Twenty thousand random sums of the first 10 terms in a Fourier sine/cosine series for the random equation $S_{10}(z) = \eta_0+\eta_1 \sin z+\eta_2 \cos z+\ldots+\eta_9 \sin 5z+\eta_{10} \cos 5z = 50 + i50$

We proceed to find the representations for the kernels $B_{r,N}(z)$ for $0 \leq r \leq 2$. We shall utilize the fact that the polynomials $p_j(z)$ are real-valued on the real line. Thus, we have that $p_j(\overline{z}) = \overline{p_j(z)}$ for $j \geq 0$ and all $z \in \mathbb{C}$. First, setting $w = \overline{z}$, so that $p_j(w) = p_j(\overline{z}) = \overline{p_j(z)}$, and $z - \overline{z} = 2i\,\mathrm{Im}(z)$ in (2.48), we obtain

$$B_{0,N}(z) = \sum_{j=0}^{N} p_j(z)\overline{p_j(z)} = \frac{k_N}{k_{N+1}} \left(\frac{\mathrm{Im}(p_{N+1}(z)\overline{p_N(z)})}{\mathrm{Im}(z)} \right). \tag{2.49}$$

Second, for $B_{1,N}(z)$ we first take the derivative of (2.48) with respect to w to achieve

$$\sum_{j=0}^{N} p_j(z)p_j'(w) = \frac{k_N}{k_{N+1}} \left(\frac{p_{N+1}(z)p_N'(w) - p_N(z)p_{N+1}'(w)}{z-w} \right) + \frac{1}{z-w} \sum_{j=0}^{N} p_j(z)p_j(w). \tag{2.50}$$

Then using z in the place of $\overline{z}$ and putting $w = z$, so that $p_j(\overline{z}) = \overline{p_j(z)}$ and $p_j'(w) = p_j'(z)$, and $\overline{z} - z = -2i\,\mathrm{Im}(z)$ in (2.50), we obtain

$$\begin{aligned} B_{1,N}(z) &= \sum_{j=0}^{N} \overline{p_j(z)} p_j'(z) \\ &= \frac{k_N}{k_{N+1}} \left(\frac{\overline{p_N(z)} p_{N+1}'(z) - \overline{p_{N+1}(z)} p_N'(z)}{2i\,\mathrm{Im}(z)} \right) - \frac{B_{0,N}(z)}{2i\,\mathrm{Im}(z)}. \end{aligned} \tag{2.51}$$

Third, for $B_{2,N}(z)$ we first take the derivative of (2.50) with respect to z to attain

$$\begin{aligned} \sum_{j=0}^{N} p_j'(z) p_j'(w) &= \frac{k_N}{k_{N+1}} \left(\frac{p_{N+1}'(z) p_N'(w) - p_N'(z) p_{N+1}'(w)}{z - w} \right) \\ &\quad - \frac{k_N}{k_{N+1}} \left(\frac{p_{N+1}(z) p_N'(z) - p_N(z) p_{N+1}'(w)}{(z-w)^2} \right) \\ &\quad + \frac{1}{z-w} \sum_{j=0}^{N} p_j'(z) p_j(w) - \frac{1}{(z-w)^2} \sum_{j=0}^{N} p_j(z) p_j(w) \\ &= \frac{k_N}{k_{N+1}} \left(\frac{p_{N+1}'(z) p_N'(w) - p_N'(z) p_{N+1}'(w)}{z - w} \right) \\ &\quad - \frac{1}{z-w} \sum_{j=0}^{N} p_j(z) p_j'(w) + \frac{1}{z-w} \sum_{j=0}^{N} p_j'(z) p_j(w). \end{aligned}$$

Thus,

$$\begin{aligned} B_{2,N}(z) &= \sum_{j=0}^{N} p_j'(z) \overline{p_j'(z)} \\ &= \frac{k_N}{k_{N+1}} \left(\frac{\mathrm{Im}(p_{N+1}'(z) \overline{p_N'(z)})}{\mathrm{Im}(z)} \right) - \frac{\overline{B_{1,N}(z)}}{2i\,\mathrm{Im}(z)} + \frac{B_{1,N}(z)}{2i\,\mathrm{Im}(z)}. \end{aligned} \tag{2.52}$$

We apply (2.49), (2.51), and (2.52) to Theorem 2.1 to obtain the following formula for the density function $h_{N,\boldsymbol{K}}(z)$ for the complex zeros of random polynomials orthogonal on the real line.

Theorem 2.7 *Let the sequence of functions $\{f_j(z)\}_{j=0}^N$ in the definition of the random sum $S_N(z)$ in* (2.1) *be polynomials $p_j(z)$ orthogonal on the real line. For all integers $N > 1$ we have*

$$h_{N,\boldsymbol{K}}(z) = \frac{1}{\pi} \exp\left(- \frac{(K_1^2 + K_2^2) k_{N+1} \operatorname{Im}(z)}{k_N \operatorname{Im}(p_{N+1}(z)\overline{p_N(z)})} \right)$$

$$\times \left\{ \frac{\operatorname{Im}(p'_{N+1}(z)\overline{p'_N(z)})}{\operatorname{Im}(p_{N+1}(z)\overline{p_N(z)})} - \frac{|\overline{p_N(z)}p'_{N+1}(z) - \overline{p_{N+1}(z)}p'_N(z)|^2}{4 \operatorname{Im}(p_{N+1}(z)\overline{p_N(z)})^2} + \frac{1}{4 \operatorname{Im}(z)^2} \right.$$

$$+ \frac{(K_1^2 + K_2^2) k_{N+1}}{k_N} \left(\frac{|\overline{p_N(z)}p'_{N+1}(z) - \overline{p_{N+1}(z)}p'_N(z)|^2 \operatorname{Im}(z)}{8 \operatorname{Im}(p_{N+1}(z)\overline{p_N(z)})^3} \right.$$

$$\left.\left. - \frac{\operatorname{Re}(p_N(z)p'_{N+1}(z) - p_{N+1}(z)\overline{p'_N(z)})}{4 \operatorname{Im}(p_{N+1}(z)\overline{p_N(z)})^2} + \frac{1}{8 \operatorname{Im}(z) \operatorname{Im}(p_{N+1}(z)\overline{p_N(z)})} \right) \right\}.$$

Furthermore, if $\boldsymbol{K}$ is the zero vector, then we have

$$h_{N,\mathbf{0}}(z) = \frac{1}{\pi} \left(\frac{\operatorname{Im}(p'_{N+1}(z)\overline{p'_N(z)})}{\operatorname{Im}(p_{N+1}(z)\overline{p_N(z)})} - \frac{|\overline{p_N(z)}p'_{N+1}(z) - \overline{p_{N+1}(z)}p'_N(z)|^2}{4 \operatorname{Im}(p_{N+1}(z)\overline{p_N(z)})^2} + \frac{1}{4 \operatorname{Im}(z)^2} \right).$$

We remark that, when $\boldsymbol{K}$ is the zero vector in Theorem 2.7, we recover Equation (1.5) of Theorem 1.1 in [55].

Next, the sequence of polynomials $\{\varphi_j(z)\}_{j=0}^\infty$ are orthogonal on the unit circle with respect to a probability Borel measure μ on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ (i.e., the real line modulo 2π) if

$$\int_{\mathbb{T}} \varphi_j(e^{i\theta}) \overline{\varphi_k(e^{i\theta})} \, d\mu(e^{i\theta}) = \delta_{jk},$$

for all $j, k \in \mathbb{N} \cup \{0\}$. As remarked in [55], when μ is restricted to be symmetric with respect to conjugation, the sequence $\{\varphi_j(z)\}_{j=0}^\infty$ will have real coefficients and, hence, be real-valued on the real line. From [49, Theorem 2.2.7], the Christoffel–Darboux formula for this sequence of polynomials is that, for complex variables z and w with $\overline{w}z \neq 1$,

$$\sum_{j=0}^{N} \varphi_j(z)\overline{\varphi_j(w)} = \frac{\overline{\varphi^*_{N+1}(w)}\varphi^*_{N+1}(z) - \overline{\varphi_{N+1}(w)}\varphi_{N+1}(z)}{1-\overline{w}z}, \tag{2.53}$$

where

$$\varphi^*_N = z^N \overline{\varphi_N\left(\frac{1}{\overline{z}}\right)}. \tag{2.54}$$

As before, we find the representations for the kernels $B_{r,N}(z)$ for $0 \leq r \leq 2$. We recall that the polynomials $\varphi_j(z)$ are real-valued on the real line. Thus, we have that $\varphi_j(\overline{z}) = \overline{\varphi_j(z)}$ for $j \geq 0$ and all $z \in \mathbb{C}$. First, from (2.53)

$$B_{0,N}(z) = \sum_{j=0}^{N} \varphi_j(z)\overline{\varphi_j(z)} = \frac{|\varphi^*_{N+1}(z)|^2 - |\varphi_{N+1}(z)|^2}{1-|z|^2}. \tag{2.55}$$

Second, for $B_{1,N}(z)$ we first take the derivative of (2.53) with respect to $\overline{w}$ and use $\overline{\varphi^*_{N+1}(w)} = \varphi^*_{N+1}(\overline{w})$ and $\overline{\varphi_{N+1}(w)} = \varphi_{N+1}(\overline{w})$ to obtain

$$\begin{aligned}\sum_{j=0}^{N} \varphi_j(z)\overline{\varphi'_j(w)} &= \frac{\overline{\varphi^{*\prime}_{N+1}(w)}\varphi^*_{N+1}(z) - \overline{\varphi'_{N+1}(w)}\varphi_{N+1}(z)}{1-\overline{w}z} \\ &\quad + \frac{z(\overline{\varphi^*_{N+1}(w)}\varphi^*_{N+1}(z) - \overline{\varphi_{N+1}(w)}\varphi_{N+1}(z))}{(1-\overline{w}z)^2}.\end{aligned} \tag{2.56}$$

Then putting $w = z$ in (2.56) and applying (2.55) we obtain

$$\begin{aligned}\overline{B_{1,N}(z)} &= \sum_{j=0}^{N} \varphi_j(z)\overline{\varphi'_j(z)} \\ &= \frac{\overline{\varphi^{*\prime}_{N+1}(z)}\varphi^*_{N+1}(z) - \overline{\varphi'_{N+1}(z)}\varphi_{N+1}(z)}{1-|z|^2} + \frac{zB_{0,N}(z)}{1-|z|^2}.\end{aligned} \tag{2.57}$$

Third, for $B_{2,N}(z)$ we first take the derivative of (2.56) with respect to z to obtain

$$\begin{aligned}\sum_{j=0}^{N} \varphi'(z)\overline{\varphi_j'(w)} &= \frac{\overline{\varphi_{N+1}^{*\prime}(w)}\varphi_{N+1}^{*\prime}(z) - \overline{\varphi_{N+1}'(w)}\varphi_{N+1}'(z)}{1-\overline{w}z} \\ &+ \frac{\overline{w}(\overline{\varphi_{N+1}^{*\prime}(w)}\varphi_{N+1}^{*}(z) - \overline{\varphi_{N+1}(w)}\varphi_{N+1}(z))}{(1-\overline{w}z)^2} \\ &+ \frac{z(\overline{\varphi_{N+1}^{*}(w)}\varphi_{N+1}^{*}(z) - \overline{\varphi_{N+1}(w)}\varphi_{N+1}'(z))}{(1-\overline{w}z)^2} \\ &+ \frac{\overline{\varphi_{N+1}^{*}(w)}\varphi_{N+1}^{*}(z) - \overline{\varphi_{N+1}(w)}\varphi_{N+1}(z)}{(1-\overline{w}z)^2} \\ &+ \frac{2\overline{w}z(\overline{\varphi_{N+1}^{*}(w)}\varphi_{N+1}^{*}(z) - \overline{\varphi_{N+1}(w)}\varphi_{N+1}(z))}{(1-\overline{w}z)^3}.\end{aligned} \tag{2.58}$$

Thus, putting $w = z$ in (2.58), applying (2.55) and (2.59), and noting that

$$\frac{\overline{z}(\overline{\varphi_{N+1}^{*\prime}(z)}\varphi_{N+1}^{*}(z) - \overline{\varphi_{N+1}(z)}\varphi_{N+1}(z))}{(1-|z|^2)^2} = \frac{\overline{zB_{1,N}(z)}}{1-|z|^2} - \frac{\overline{z}B_{0,N}(z)}{1-|z|^2}$$

and

$$\frac{z(\overline{\varphi_{N+1}^{*}(z)}\varphi_{N+1}^{*}(z) - \overline{\varphi_{N+1}(z)}\varphi_{N+1}'(z))}{(1-|z|^2)^2} = \frac{zB_{0,N}(z)}{1-|z|^2} - \frac{|z|^2B_{0,N}(z)}{(1-|z|^2)^2},$$

we obtain

$$\begin{aligned}B_{2,N}(z) &= \sum_{j=0}^{N} \varphi_j'(z)\overline{\varphi_j'(z)} \\ &= \frac{|\varphi_{N+1}^{*\prime}(z)|^2 - |\varphi_{N+1}'(z)|^2}{1-|z|^2} + \frac{2\,\mathrm{Re}(zB_{1,N}(z))}{1-|z|^2} + \frac{B_{0,N}(z)}{1-|z|^2}.\end{aligned} \tag{2.59}$$

To facilitate the derivation of the density function $h_{N,\boldsymbol{K}}(z)$ for the complex zeros of random polynomials orthogonal on the unit circle, we note that the formula for the density function $h_{N,\boldsymbol{K}}(z)$ in Theorem 2.1 contains the quotients $(B_{0,N}B_{2,N} - |B_{1,N}|^2)/B_{0,N}^2$ and $|B_{1,N}|^2/B_{0,N}^3$. We treat these quotients in turn.

From (2.55) and (2.59)

$$B_{0,N}(z)B_{2,N}(z) = \frac{|\varphi^{*\prime}_{N+1}(z)\varphi^{*}_{N+1}(z)|^2}{(1-|z|^2)^2} + \frac{|\varphi'_{N+1}(z)\varphi_{N+1}(z)|^2}{(1-|z|^2)^2} - \frac{|\varphi^{*\prime}_{N+1}(z)\varphi_{N+1}(z)|^2}{(1-|z|^2)^2} - \frac{|\varphi^{*}_{N+1}(z)\varphi_{N+1}(z)|^2}{(1-|z|^2)^2} + \frac{2B_{0,N}\operatorname{Re}(zB_{1,N}(z))}{1-|z|^2} + \frac{B_{0,N}(z)^2}{1-|z|^2}. \tag{2.60}$$

From (2.59)

$$\begin{aligned}|B_{1,N}(z)|^2 &= \overline{B_{1,N}(z)}B_{1,N}(z) \\ &= \frac{|\varphi^{*\prime}_{N+1}\varphi^{*}_{N+1}(z)|^2}{(1-|z|^2)^2} + \frac{|\varphi'_{N+1}(z)\varphi_{N+1}(z)|^2}{(1-|z|^2)^2} \\ &\quad - \frac{2\operatorname{Re}(\overline{\varphi^{*\prime}_{N+1}(z)}\varphi^{*}_{N+1}(z)\varphi'_{N+1}(z)\overline{\varphi_{N+1}(z)})}{(1-|z|^2)^2} \\ &\quad + \frac{2B_{0,N}(z)\operatorname{Re}(zB_{1,N}(z))}{1-|z|^2} - \frac{|z|^2B_{0,N}(z)^2}{(1-|z|^2)^2}.\end{aligned} \tag{2.61}$$

Thus, from (2.60) and (2.61)

$$B_{0,N}(z)B_{2,N}(z) - |B_{1,N}(z)|^2 = \frac{(|\varphi^{*}_{N+1}(z)|^2 - |\varphi_{N+1}(z)|^2)^2}{(1-|z|^2)^4} - \frac{|\varphi^{*}_{N+1}(z)\varphi'_{N+1}(z) - \varphi^{*\prime}_{N+1}(z)\varphi_{N+1}(z)|^2}{(1-|z|^2)^2}. \tag{2.62}$$

From (2.55) and (2.61)

$$\begin{aligned}\frac{|B_{1,N}(z)|^2}{B_{0,N}(z)^3} &= \frac{(1-|z|^2)|\overline{\varphi^{*\prime}_{N+1}(z)}\varphi^{*}_{N+1}(z) - \overline{\varphi'_{N+1}(z)}\varphi_{N+1}(z)|^2}{(|\varphi^{*}_{N+1}(z)|^2 - |\varphi_{N+1}(z)|^2)^3} \\ &\quad + \frac{2\operatorname{Re}(z(\varphi^{*\prime}_{N+1}(z)\overline{\varphi^{*}_{N+1}(z)} - \varphi'_{N+1}(z)\overline{\varphi_{N+1}(z)}))}{(|\varphi^{*}_{N+1}(z)|^2 - |\varphi_{N+1}(z)|^2)^2} \\ &\quad + \frac{|z|^2}{(1-|z|^2)(|\varphi^{*}_{N+1}(z)|^2 - |\varphi_{N+1}(z)|^2)}.\end{aligned} \tag{2.63}$$

We deduce from Theorem 2.1, (2.55), (2.62), and (2.63) the following result.

THEOREM 2.8 *Let the sequence of functions $\{f_j(z)\}_{j=0}^N$ in the definition of the random sum $S_N(z)$ in (2.1) be polynomials $\varphi_j(z)$ orthogonal on the unit circle. Let, further, the polynomial $\varphi_N^*(z)$ be given by (2.54). For all integers $N > 1$ we have*

$$h_{N,\boldsymbol{K}}(z) = \frac{1}{\pi} \exp\left\{ -\left(\frac{K_1^2 + K_2^2}{2}\right) \frac{1 - |z|^2}{|\varphi_{N+1}^*(z)|^2 - |\varphi_{N+1}(z)|^2} \right\}$$
$$\times \left\{ \frac{1}{(1-|z|^2)^2} - \frac{|\varphi_{N+1}^*(z)\varphi_{N+1}'(z) - \varphi_{N+1}^{*\prime}(z)\varphi_{N+1}(z)|^2}{(|\varphi_{N+1}^*(z)|^2 - |\varphi_{N+1}(z)|^2)^2} \right.$$
$$+ \left(\frac{K_1^2 + K_2^2}{2}\right) \left(\frac{(1-|z|^2)|\overline{\varphi_{N+1}^{*\prime}(z)}\varphi_{N+1}^*(z) - \overline{\varphi_{N+1}'(z)}\varphi_{N+1}(z)|^2}{(|\varphi_{N+1}^*(z)|^2 - |\varphi_{N+1}(z)|^2)^3} \right.$$
$$+ \frac{2\operatorname{Re}(z(\varphi_{N+1}^{*\prime}(z)\overline{\varphi_{N+1}^*(z)} - \varphi_{N+1}'(z)\overline{\varphi_{N+1}(z)}))}{(|\varphi_{N+1}^*(z)|^2 - |\varphi_{N+1}(z)|^2)^2}$$
$$\left.\left. + \frac{|z|^2}{(1-|z|^2)(|\varphi_{N+1}^*(z)|^2 - |\varphi_{N+1}(z)|^2)} \right) \right\}.$$

Furthermore, if $\boldsymbol{K}$ is the zero vector, then we have

$$h_{N,\mathbf{0}}(z) = \frac{1}{\pi}\left(\frac{1}{(1-|z|^2)^2} - \frac{|\varphi_{N+1}^*(z)\varphi_{N+1}'(z) - \varphi_{N+1}^{*\prime}(z)\varphi_{N+1}(z)|^2}{(|\varphi_{N+1}^*(z)|^2 - |\varphi_{N+1}(z)|^2)^2} \right).$$

Finally, we remark that, when $\boldsymbol{K}$ is the zero vector in Theorem 2.8, we recover Equation (1.6) of Theorem 1.1 in [55].

CHAPTER 3

THE CROSSINGS OF RANDOM SUMS, PART I

An exact formula for the expected number of real zeros of a random polynomial was obtained by Kac [34] under i.i.d., real, standard normal coefficients. For complex coefficients, Dunnage [17, 18] gave some estimates for the number of real zeros. For complex zeros, the expected density of zeros was studied by Shepp and Vanderbei [48] for i.i.d., real, standard normal coefficients and generalized by Ibragimov and Zeitouni [33] for a wider class of distributions of coefficients. Relevant to these investigations is the work of Kostlan [35]. The expected density was dealt with, also, by Hammersley [32], Edelman and Kostlan [20], and Farahmand and Grigorash [28]. Vanderbei [52] generalized the work in [48] to random sums with holomorphic functions that are real-valued on $\mathbb{R}$ as basis functions. Motivated by the studies conducted by Vanderbei [52] and Farahmand [26], the present authors [11] obtained results on the level crossings of these random sums. The chief purpose of the present chapter is to extend certain of these results.

In what follows, let $\{a_j\}_{j=0}^N$ and $\{b_j\}_{j=0}^N$ be sequences of mutually i.i.d., real, normal random variables defined on the complete probability space $(\Omega, \mathscr{F}, \text{Prob})$ with mean zero and variances $\{\sigma_{a_j}^2\}_{j=0}^N$ and $\{\sigma_{b_j}^2\}_{j=0}^N$. As per usual, Ω is a set with generic elements ω, $\mathscr{F}$ is a σ-field of subsets of Ω and Prob is a probability measure on $\mathscr{F}$. Assume all sub σ-fields contain all sets of measure zero (see [19]). Let $\{f_j\}_{j=0}^N$ be a sequence of holomorphic functions $f_j(z) = u_j(x, y) + iv_j(x, y)$ for $0 \le j \le N$ and $(x, y) \in \mathbb{R}^2$ that are real-valued on $\mathbb{R}$, so that

$\overline{f_j(z)} = f_j(\overline{z})$ for $0 \le j \le N$ and all $z \in \mathbb{C}$. Define

$$S(z) = \sum_{j=0}^{N} (a_j + ib_j) f_j(z). \tag{3.1}$$

It is of interest to study the number of times that S crosses a complex level. If, for each compact subset T of $\mathbb{C}$, $N^S_{\boldsymbol{K}}(T)$ denotes the random number of complex zeros, counted with multiplicity, in T of S in (3.1) that cross the complex level $\boldsymbol{K} = K_1 + iK_2$, where K_1 and K_2 are constants independent of z, then from [11], with probability one, the expected density $h_{\boldsymbol{K}}$ of the complex zeros of

$$S(z) = \boldsymbol{K} \tag{3.2}$$

is given by

$$E(N^S_{\boldsymbol{K}}(T)) = \int_T h_{\boldsymbol{K}}(z)\, dz. \tag{3.3}$$

The explicit derivation of $h_{\boldsymbol{K}}$ constitutes the primary reason for studying the zeros of (3.2). The main device for treating $h_{\boldsymbol{K}}$ throughout $\mathbb{C}$ is the Rice formula. This remarkable result provides a representation for the expected number of zeros of certain random fields. It is reproduced below from [8, Theorem 6.2, pp. 163-164]. (See, also, [2, Theorem 11.2.3, Corollary 11.2.4, pp. 269-271], [6, Theorem 2.1, p. 256], and [7, Theorem 1, p. 3]).

THEOREM 3.1 *Let $Z\colon U \to \mathbb{R}^N$ be a random field, let U be an open subset of $\mathbb{R}^N$ and let $u \in \mathbb{R}^N$ be a fixed point in the codomain. Assume the following conditions are satisfied with probability one:*

(i) Z is normal.

(ii) Almost surely the function $t \rightsquigarrow Z(t)$ is of class C^1.

(iii) For each $t \in U$, $Z(t)$ has a nondegenerate distribution—i.e., $\mathrm{Var}(Z(t)) \succ 0$.

(iv) For each $u \in \mathbb{R}^N$, $\mathrm{Prob}(\exists\, t \in U\colon Z(t) = u, \det(Z'(t)) = 0) = 0$.

If $N_u^Z(B)$ denotes the number of zeros of $Z(t) = u$ that belong to the Borel subset $B \subset U$, then one has

$$E(N_u^Z(B)) = \int_B E(|\det(Z'(t))| \mid Z(t) = u)\, p_{Z(t)}(u)\, dt, \tag{3.4}$$

where $p_{Z(t)}(u)$ is the probability density function of $Z(t)$ at u. If B is compact, then both sides of (3.4) *are finite.*

The function Z in (3.4) is defined on $\mathbb{R}^N$. In our application, we need to find the real and complex zeros of (3.2)—i.e., the real zeros of $\mathrm{Re}(S(x+iy)) = K_1$ and $\mathrm{Im}(S(x+iy)) = K_2$ for $(x, y) \in \mathbb{R}^2$. The conditions (i)–(iv) are easy to check. Formula (3.4) is interesting. It shows that $h_{\mathbf{K}}$, as defined by (3.3), can be expressed through a conditioned mean function of a quadratic form of i.i.d., real, normal random variables conditioned on certain linear combinations.

Theorem 3.2 *Provided all the conditions imposed on S in* (3.1) *and T are satisfied, then for all integers $N > 1$ one has*

$$\begin{aligned}
h_{\boldsymbol{K}} = {} & \frac{1}{2\pi D_0} \exp\left(-\frac{K_1^2 Y_3 + K_2^2 Y_1 - 2K_1K_2Y_2}{2D_0^2}\right) \\
& \times \left\{ D_3 - \frac{|D_1|^2}{D_0}\left(\frac{Y_2 + Y_3}{D_0} - \frac{(K_1Y_3 - K_2Y_2)(K_1(Y_2 + Y_3) - K_2(Y_1 + Y_2))}{D_0^3}\right)\right. \\
& \quad - \frac{|D_2|^2}{D_0}\left(\frac{Y_1 + Y_2}{D_0} - \frac{(K_1Y_2 - K_2Y_1)(K_1(Y_2 + Y_3) - K_2(Y_1 + Y_2))}{D_0^3}\right) \\
& \quad \left. + \frac{|D_1 + iD_2|^2}{D_0}\left(\frac{Y_2}{D_0} - \frac{(K_1Y_3 - K_2Y_2)(K_1Y_2 - K_2Y_1)}{D_0^3}\right)\right\},
\end{aligned}$$

where

$$Y_1(z) = \sum_{j=0}^{N}(\sigma_{a_j}^2 u_j^2 + \sigma_{b_j}^2 v_j^2), \qquad Y_2(z) = \sum_{j=0}^{N}(\sigma_{a_j}^2 - \sigma_{b_j}^2) u_j v_j,$$

$$Y_3(z) = \sum_{j=0}^{N}(\sigma_{b_j}^2 u_j^2 + \sigma_{a_j}^2 v_j^2),$$

and

$$D_0(z) = \sqrt{Y_1(z)Y_3(z) - Y_2^2(z)}, \qquad D_1(z) = \sum_{j=0}^{N} (\sigma_{a_j}^2 u_j - i\sigma_{b_j}^2 v_j)(u_{jx} + iv_{jx}),$$

$$D_2(z) = \sum_{j=0}^{N} (\sigma_{b_j}^2 u_j - i\sigma_{a_j}^2 v_j)(u_{jx} + iv_{jx}), \qquad D_3(z) = \sum_{j=0}^{N} (\sigma_{a_j}^2 + \sigma_{b_j}^2)(u_{jx}^2 + v_{jx}^2),$$

where $u_{jx} = \partial u_j/\partial x$ *and* $v_{jx} = \partial v_j/\partial x$.

In relation to the work in [11], let us first observe that when $\sigma_{a_j}^2 = \sigma_{b_j}^2 = \sigma^2$ for $0 \leq j \leq N$

$$Y_1(z) = Y_3(z) = \sigma^2 B_0(z), \qquad Y_2(z) = 0,$$

and

$$D_0(z) = \sigma^2 B_0(z), \qquad D_1(z) = D_2(z) = \sigma^2 B_1(z), \qquad D_3(z) = 2\sigma^2 B_2(z),$$

where

$$B_0(z) = \sum_{j=0}^{N} |f_j(z)|^2, \qquad B_1(z) = \sum_{j=0}^{N} \overline{f_j(z)} f_j'(z), \qquad B_2(z) = \sum_{j=0}^{N} |f_j'(z)|^2.$$

Then

$$|D_1(z) + iD_2(z)|^2 = |D_1(z)|^2 + |D_2(z)|^2 = 2\sigma^4 |B_1(z)|^2.$$

The following result is obtained by using these substitutions in Theorem 4.1, factoring and simplifying.

THEOREM 3.3 *If $\sigma_{a_j}^2 = \sigma_{b_j}^2 = \sigma^2$ for $0 \leq j \leq N$, then for all integers $N > 1$ one has*

$$h_{\boldsymbol{K}} = \frac{1}{\pi B_0} \exp\left(-\frac{K_1^2 + K_2^2}{2\sigma^2 B_0}\right) \left\{ B_2 - \frac{|B_1|^2}{B_0}\left(1 - \frac{K_1^2 + K_2^2}{2\sigma^2 B_0}\right)\right\}.$$

Then, as a consequence of Theorem 3.3, when σ^2 is set to be one, Theorem 1 in [11] is recovered. Further, if $\boldsymbol{K}$ is the zero vector, Corollary 3 in [11] is recovered, which was proved independently by Yeager [54] and one of the authors [36].

The following result follows, also, from Theorem 4.1.

COROLLARY 3.3.1 *For all vectors $\boldsymbol{K}$ restricted to a circle of radius $K > 0$ and all integers $N > 1$, one has*

$$\begin{aligned} h_{\boldsymbol{K}} = \frac{1}{2\pi D_0} & \exp\left(-\frac{K^2(Y_1 - Y_2 + Y_3)}{2D_0^2}\right) \\ \times & \left\{ D_3 - \frac{|D_1|^2}{D_0}\left(\frac{Y_2+Y_3}{D_0} - \frac{K^2(Y_2-Y_3)(Y_1-Y_2-1)}{D_0^3}\right)\right. \\ & \quad - \frac{|D_2|^2}{D_0}\left(\frac{Y_1+Y_2}{D_0} - \frac{K^2(Y_1-Y_2)(Y_2-Y_3+1)}{D_0^3}\right) \\ & \quad \left. + \frac{|D_1 + iD_2|^2}{D_0}\left(\frac{Y_2}{D_0} - \frac{K^2(Y_1-Y_2)(Y_2-Y_3)}{D_0^3}\right)\right\}. \end{aligned}$$

A special case of Corollary 4.1.1 follows.

COROLLARY 3.3.2 *If $\boldsymbol{K}$ is the zero vector, then for all integers $N > 1$ one has*

$$h_{\boldsymbol{K}} = \frac{D_0^2 D_3 - |D_1|^2(Y_2+Y_3) - |D_2|^2(Y_1+Y_2) + |D_1+iD_2|^2 Y_2}{2\pi D_0^3}.$$

The proof of Theorem 4.1, in the spirit of the method credited to Ibragimov and Zeitouni [33], is presented in Section 4.1. Finally, in relation to the works of Rezakhah and

Shemehsavar [44] and Rezakhah and Soltani [45], an application of Theorem 4.1 entailing a Brownian motion is given in Section 4.2.

3.1 The Density Function for Multivariate Normal Coefficients

The proof of Theorem 4.1 starts with the decomposition

$$S(z) = X_1 + iX_2,$$

where

$$X_1 = \sum_{j=0}^{N}(a_j u_j - b_j v_j), \qquad X_2 = \sum_{j=0}^{N}(a_j v_j + b_j u_j).$$

If the column vector

$$\boldsymbol{X} = (X_1, X_2)'$$

genuinely represents a two-dimensional random field, then, from the Cauchy–Riemann equations, for $z = x + iy$ the Jacobian matrix of $(x, y) \to (X_1, X_2)$ is

$$\nabla \boldsymbol{X} = \begin{pmatrix} \displaystyle\sum_{j=0}^{N}(a_j u_{jx} - b_j v_{jx}) & \displaystyle\sum_{j=0}^{N}(a_j v_{jx} + b_j u_{jx}) \\ \displaystyle\sum_{j=0}^{N}(-a_j v_{jx} - b_j u_{jx}) & \displaystyle\sum_{j=0}^{N}(a_j u_{jx} - b_j v_{jx}) \end{pmatrix}$$

and

$$\begin{aligned} \det(\nabla \boldsymbol{X}) &= \sum_{j=0}^{N}\sum_{k=0}^{N}((a_j a_k + b_j b_k)(u_{jx}u_{kx} + v_{jx}v_{kx}) + (a_j b_k b_j a_k)(v_j u_k - u_j v_k)) \\ &= \sum_{j=0}^{N}(a_j^2 + b_j^2)(u_{jx}^2 + v_{jx}^2) + \sum_{j=0}^{N}\sum_{\substack{k=0 \\ k\neq j}}^{N}((a_j a_k + b_j b_k)(u_{jx}u_{kx} + v_{jx}v_{kx}) \\ &\qquad + (a_j b_k - b_j a_k)(v_{jx}u_{kx} - u_{jx}v_{kx})). \end{aligned} \tag{3.5}$$

It is interesting to note that $\det(\nabla \boldsymbol{X})$ is always nonnegative. Since N is fixed, T contains not more than a finite number of zeros of

$$\boldsymbol{X} = \boldsymbol{K}, \tag{3.6}$$

where

$$\boldsymbol{K} = (K_1, K_2)'. \tag{3.7}$$

Since the set of zeros of (3.6) is of measure zero, assume ∂T does not contain any zeros of (3.6) and T does not contain any such zeros such that $\det(\nabla \boldsymbol{X}) = 0$. Theorem 3.1 applies, and

$$h_{\boldsymbol{K}}(z) = E(\det(\nabla \boldsymbol{X}) \mid \boldsymbol{X} = \boldsymbol{K})\, p_{X_1,X_2}(\boldsymbol{K}'), \tag{3.8}$$

where $p_{x,y}$ denotes the probability density of $\boldsymbol{X}$. By (3.6), and since X_1 and X_2 are linear forms with respect to a_j and b_j for $0 \leq j \leq N$, $h_{\boldsymbol{K}}$ is the conditional mean of a quadratic form with respect to $a_j + ib_j$ for $0 \leq j \leq N$. This form can be calculated in terms of components by means of multivariate analysis.

Based on the assumption that the scalar random variables are independent and normally distributed, the multivariate random vectors

$$\boldsymbol{a} = (a_0, \ldots, a_N)', \qquad \boldsymbol{b} = (b_0, \ldots, b_N)'$$

are such that

$$\mathrm{Cov}(\boldsymbol{a}, \boldsymbol{b} \mid \boldsymbol{Y} = \boldsymbol{K}) = \begin{pmatrix} \boldsymbol{\Sigma}_{\boldsymbol{aa},\boldsymbol{X}} & \boldsymbol{\Sigma}_{\boldsymbol{ab},\boldsymbol{X}} \\ \boldsymbol{\Sigma}_{\boldsymbol{ba},\boldsymbol{X}} & \boldsymbol{\Sigma}_{\boldsymbol{bb},\boldsymbol{X}} \end{pmatrix}. \tag{3.9}$$

The elements can be computed using

$$\boldsymbol{\Sigma}_{\boldsymbol{ab},\boldsymbol{X}} = \boldsymbol{\Sigma}_{\boldsymbol{ab}} - \boldsymbol{\Sigma}_{\boldsymbol{aX}} \boldsymbol{\Sigma}_{\boldsymbol{XX}}^{-1} \boldsymbol{\Sigma}_{\boldsymbol{Xb}} \tag{3.10}$$

and the corresponding expression

$$\boldsymbol{\Sigma}_{\boldsymbol{ab}} = E((\boldsymbol{a} - E(\boldsymbol{a}))(\boldsymbol{b} - E(\boldsymbol{b}))').$$

Since the distribution of a_j and b_j is central for $0 \leq j \leq N$, $E(\boldsymbol{a}) = 0$ and $E(\boldsymbol{b}) = 0$. Then clearly

$$\boldsymbol{\Sigma}_{\boldsymbol{ab}} = E(\boldsymbol{ab}'). \tag{3.11}$$

Thusly, the conditional expected values are expressed in terms of unconditional expected values and covariances.

Then, if $E(X_1) = 0$ and $E(X_2) = 0$, $E(\boldsymbol{X}) = 0$, whence, by (3.11),

$$\boldsymbol{\Sigma}_{\boldsymbol{XX}} = \begin{pmatrix} E(X_1X_1) & E(X_1X_2) \\ E(X_2X_1) & E(X_2X_2) \end{pmatrix} = \begin{pmatrix} Y_1 & Y_2 \\ Y_2 & Y_3 \end{pmatrix}, \tag{3.12}$$

which implies that

$$\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}}) = Y_1Y_3 - Y_2^2 > 0,$$

if X_1 and X_2 are not strictly correlated. Thus,

$$\boldsymbol{\Sigma}_{\boldsymbol{XX}}^{-1} = \frac{1}{Y_1Y_3 - Y_2^2} \begin{pmatrix} Y_3 & -Y_2 \\ -Y_2 & Y_1 \end{pmatrix}. \tag{3.13}$$

Direct evaluation shows that

$$\boldsymbol{\Sigma}_{\boldsymbol{aa}} = E(a_ja_k) = \delta_{jk}\sigma_{a_j}^2 \tag{3.14}$$

and

$$\boldsymbol{\Sigma}_{\boldsymbol{bb}} = E(b_jb_k) = \delta_{jk}\sigma_{b_j}^2 \tag{3.15}$$

for $0 \leq j \leq N$ and $0 \leq k \leq N$, where δ_{jk} denotes the Kronecker delta. Further, notice that

$$\mathbf{\Sigma}_{\boldsymbol{ab}} = E(a_j b_k) = 0 \tag{3.16}$$

and

$$\mathbf{\Sigma}_{\boldsymbol{ba}} = 0. \tag{3.17}$$

Next, since $E(a_j X_1) = \sigma^2_{a_j} u_j$ and $E(a_j X_2) = \sigma^2_{a_j} v_j$ for $0 \leq j \leq N$,

$$\mathbf{\Sigma}_{\boldsymbol{aX}} = (\sigma^2_{a_j} u_j \quad \sigma^2_{a_j} v_j), \tag{3.18}$$

whence

$$\mathbf{\Sigma}_{\boldsymbol{Xa}} = \begin{pmatrix} \sigma^2_{a_k} u_k \\ \\ \sigma^2_{a_k} v_k \end{pmatrix}. \tag{3.19}$$

Analogously, $E(b_j X_1) = -\sigma^2_{b_j} v_j$ and $E(b_j X_2) = \sigma^2_{b_j} u_j$ for $0 \leq j \leq N$. Then

$$\mathbf{\Sigma}_{\boldsymbol{bX}} = (-\sigma^2_{b_j} v_j \quad \sigma^2_{b_j} u_j), \tag{3.20}$$

whence

$$\mathbf{\Sigma}_{\boldsymbol{Xb}} = \begin{pmatrix} -\sigma^2_{b_k} v_k \\ \\ \sigma^2_{b_k} u_k \end{pmatrix} \tag{3.21}$$

for $0 \leq k \leq N$.

Then, from (3.10), (3.13), (3.14), (3.18) and (3.19), for the jth row and kth column

$$\mathbf{\Sigma}_{\boldsymbol{aa},\boldsymbol{X}} = \delta_{jk}\sigma^2_{a_j} - \frac{\sigma^2_{a_j}\sigma^2_{a_k}}{Y_1 Y_3 - Y_2^2}(Y_1 v_j v_k - Y_2(u_j v_k + v_j u_k) + Y_3 u_j u_k). \tag{3.22}$$

Likewise, from (3.10), (3.13), (3.15), (3.20) and (3.21)

$$\mathbf{\Sigma}_{\boldsymbol{bb},\boldsymbol{X}} = \delta_{jk}\sigma_{b_j}^2 - \frac{\sigma_{b_j}^2\sigma_{b_k}^2}{Y_1Y_3 - Y_2^2}(Y_1u_ju_k + Y_2(u_jv_k + v_ju_k) + Y_3v_jv_k). \tag{3.23}$$

From (3.10), (3.13), (3.16), (3.18) and (3.21)

$$\mathbf{\Sigma}_{\boldsymbol{ab},\boldsymbol{X}} = -\frac{\sigma_{a_j}^2\sigma_{b_k}^2}{Y_1Y_3 - Y_2^2}(Y_1v_ju_k - Y_2(u_ju_k - v_jv_k) - Y_3u_jv_k). \tag{3.24}$$

From (3.10), (3.13), (3.17), (3.19) and (3.20)

$$\mathbf{\Sigma}_{\boldsymbol{ba},\boldsymbol{X}} = -\frac{\sigma_{b_j}^2\sigma_{a_k}^2}{Y_1Y_3 - Y_2^2}(Y_1u_jv_k - Y_2(u_ju_k - v_jv_k) - Y_3v_ju_k). \tag{3.25}$$

The mean function in (3.8) is then found by applications of

$$E(\boldsymbol{a} \mid \boldsymbol{X} = \boldsymbol{K}) = E(\boldsymbol{a}) + \mathbf{\Sigma}_{\boldsymbol{aX}}\mathbf{\Sigma}_{\boldsymbol{XX}}^{-1}(\boldsymbol{K} - E(\boldsymbol{X})),$$

which, for the aforesaid reasons, reduces to

$$E(\boldsymbol{a} \mid \boldsymbol{X} = \boldsymbol{K}) = \mathbf{\Sigma}_{\boldsymbol{aX}}\mathbf{\Sigma}_{\boldsymbol{XX}}^{-1}\boldsymbol{K}. \tag{3.26}$$

From (3.7), (3.13), (3.18) and (3.26)

$$E(a_j \mid \boldsymbol{X} = \boldsymbol{K}) = \frac{\sigma_{a_j}^2}{Y_1Y_3 - Y_2^2}((K_1Y_3 - K_2Y_2)u_j - (K_1Y_2 - K_2Y_1)v_j). \tag{3.27}$$

From (3.7), (3.13), (3.20) and (3.26)

$$E(b_j \mid \boldsymbol{X} = \boldsymbol{K}) = -\frac{\sigma_{b_j}^2}{Y_1Y_3 - Y_2^2}((K_1Y_3 - K_2Y_2)v_j + (K_1Y_2 - K_2Y_1)u_j). \tag{3.28}$$

Then, from (3.9), (3.22)–(3.25) and (3.27),

$$
\begin{aligned}
E(a_j a_k \mid \boldsymbol{X} = \boldsymbol{K}) &= E(a_j \mid \boldsymbol{X} = \boldsymbol{K})E(a_k \mid \boldsymbol{X} = \boldsymbol{K}) + \mathrm{Cov}(a_j, a_k \mid \boldsymbol{X} = \boldsymbol{K}) \\
&= \frac{\sigma_{a_j}^2 \sigma_{a_k}^2}{(Y_1Y_3 - Y_2^2)^2}((K_1Y_3 - K_2Y_2)^2 u_j u_k + (K_1Y_2 - K_2Y_1)^2 v_j v_k \\
&\quad - (K_1Y_3 - K_2Y_2)(K_1Y_2 - K_2Y_2)(u_j v_k + v_j u_k)) + \delta_{jk}\sigma_{a_j}^2 \\
&\quad - \frac{\sigma_{a_j}^2 \sigma_{a_k}^2}{Y_1Y_3 - Y_2^2}(Y_3 u_j u_k + Y_1 v_j v_k - Y_2(u_j v_k + v_j u_k)).
\end{aligned} \tag{3.29}
$$

From (3.9), (3.22)–(3.25) and (3.28)

$$
\begin{aligned}
E(b_j b_k \mid \boldsymbol{X} = \boldsymbol{K}) &= E(b_j \mid \boldsymbol{X} = \boldsymbol{K})E(b_k \mid \boldsymbol{X} = \boldsymbol{K}) + \mathrm{Cov}(b_j, b_k \mid \boldsymbol{X} = \boldsymbol{K}) \\
&= \frac{\sigma_{b_j}^2 \sigma_{b_k}^2}{(Y_1Y_3 - Y_2^2)^2}((K_1Y_2 - K_2Y_1)^2 u_j u_k + (K_1Y_3 - K_2Y_2)^2 v_j v_k \\
&\quad + (K_1Y_2 - K_2Y_1)(K_1Y_3 - K_2Y_2)(u_j v_k + v_j u_k)) + \delta_{jk}\sigma_{b_j}^2 \\
&\quad - \frac{\sigma_{b_j}^2 \sigma_{b_k}^2}{Y_1Y_3 - Y_2^2}(Y_1 u_j u_k + Y_3 v_j v_k + Y_2(u_j v_k + v_j u_k)).
\end{aligned} \tag{3.30}
$$

From (3.9), (3.22)–(3.25), (3.27) and (3.28)

$$
\begin{aligned}
E(a_j b_k \mid \boldsymbol{X} = \boldsymbol{K}) &= E(a_j \mid \boldsymbol{X} = \boldsymbol{K})E(b_k \mid \boldsymbol{X} = \boldsymbol{K}) + \mathrm{Cov}(a_j, b_k \mid \boldsymbol{X} = \boldsymbol{K}) \\
&= \frac{\sigma_{a_j}^2 \sigma_{b_k}^2}{(Y_1Y_3 - Y_2^2)^2}((K_1Y_2 - K_2Y_1)^2 v_j u_k - (K_1Y_3 - K_2Y_2)^2 u_j v_k \\
&\quad - (K_1Y_3 - K_2Y_2)(K_1Y_2 - K_2Y_1)(u_j u_k - v_j v_k)) \\
&\quad - \frac{\sigma_{a_j}^2 \sigma_{b_k}^2}{Y_1Y_3 - Y_2^2}(Y_1 v_j u_k - Y_2(u_j u_k - v_j v_k) - Y_3 u_j v_k)
\end{aligned} \tag{3.31}
$$

and

$$
\begin{aligned}
E(b_j a_k \mid \boldsymbol{X} = \boldsymbol{K}) &= E(b_j \mid \boldsymbol{X} = \boldsymbol{K}) E(a_k \mid \boldsymbol{X} = \boldsymbol{K}) + \operatorname{Cov}(b_j, a_k \mid \boldsymbol{X} = \boldsymbol{K}) \\
&= \frac{\sigma_{b_j}^2 \sigma_{a_k}^2}{(Y_1 Y_3 - Y_2^2)^2} ((K_1 Y_2 - K_2 Y_1)^2 u_j v_k - (K_1 Y_3 - K_2 Y_2)^2 v_j u_k \\
&\quad - (K_1 Y_3 - K_2 Y_2)(K_1 Y_2 - K_2 Y_1)(u_j u_k - v_j v_k)) \\
&\quad - \frac{\sigma_{b_j}^2 \sigma_{a_k}^2}{Y_1 Y_3 - Y_2^2} (Y_1 u_j v_k - Y_2 (u_j u_k - v_j v_k) - Y_3 v_j u_k).
\end{aligned} \tag{3.32}
$$

Then, from (3.29) and (3.30),

$$
\begin{aligned}
E(a_j a_k + b_j b_k \mid \boldsymbol{X} = \boldsymbol{K}) &= \frac{1}{(Y_1 Y_3 - Y_2^2)^2} ((K_1 Y_3 - K_2 Y_2)^2 (\sigma_{a_j}^2 \sigma_{a_k}^2 u_j u_k \\
&\quad + \sigma_{b_j}^2 \sigma_{b_k}^2 v_j v_k) + (K_1 Y_2 - K_2 Y_1)^2 (\sigma_{a_j}^2 \sigma_{a_k}^2 v_j v_k + \sigma_{b_j}^2 \sigma_{b_k}^2 u_j u_k) \\
&\quad - (K_1 Y_3 - K_2 Y_2)(K_1 Y_2 - K_2 Y_1)(u_j v_k + v_j u_k)(\sigma_{a_j}^2 \sigma_{a_k}^2 - \sigma_{b_j}^2 \sigma_{b_k}^2)) \\
&\quad - \frac{1}{Y_1 Y_3 - Y_2^2} (Y_1 (\sigma_{a_j}^2 \sigma_{a_k}^2 v_j v_k + \sigma_{b_j}^2 \sigma_{b_k}^2 u_j u_k) \\
&\quad + Y_2 (u_j v_k + v_j u_k)(\sigma_{b_j}^2 \sigma_{b_k}^2 - \sigma_{a_j}^2 \sigma_{a_k}^2) + Y_3 (\sigma_{a_j}^2 \sigma_{a_k}^2 u_j u_k + \sigma_{b_j}^2 \sigma_{b_k}^2 v_j v_k)) \\
&\quad + \delta_{jk} (\sigma_{a_j}^2 + \sigma_{b_j}^2).
\end{aligned} \tag{3.33}
$$

From (3.31) and (3.32)

$$
\begin{aligned}
E(a_j b_k - b_j a_k \mid \boldsymbol{X} = \boldsymbol{K}) &= \frac{1}{(Y_1 Y_3 - Y_2^2)^2} ((K_1 Y_2 - K_2 Y_1)^2 (\sigma_{a_j}^2 \sigma_{b_k}^2 v_j u_k \\
&\quad - \sigma_{b_j}^2 \sigma_{a_k}^2 u_j v_k) - (K_1 Y_3 - K_2 Y_2)^2 (\sigma_{a_j}^2 \sigma_{b_k}^2 u_j v_k - \sigma_{b_j}^2 \sigma_{a_k}^2 v_j u_k) \\
&\quad - (K_1 Y_3 - K_2 Y_2)(K_1 Y_2 - K_2 Y_1)(u_j u_k - v_j v_k)(\sigma_{a_j}^2 \sigma_{b_k}^2 - \sigma_{b_j}^2 \sigma_{a_k}^2)) \\
&\quad - \frac{1}{Y_1 Y_3 - Y_2^2} (Y_1 (\sigma_{a_j}^2 \sigma_{b_k}^2 v_j u_k - \sigma_{b_j}^2 \sigma_{a_k}^2 u_j v_k) \\
&\quad + Y_2 (v_j v_k - u_j u_k)(\sigma_{a_j}^2 \sigma_{b_k}^2 - \sigma_{b_j}^2 \sigma_{a_k}^2) - Y_3 (\sigma_{a_j}^2 \sigma_{b_k}^2 u_j v_k - \sigma_{b_j}^2 \sigma_{a_k}^2 v_j u_k)).
\end{aligned} \tag{3.34}
$$

Altogether, in view of (3.5), (3.33) and (3.34), after all the necessary simplifications,

$$\begin{aligned}
E(\det(\nabla \boldsymbol{X}) \mid \boldsymbol{X} = \boldsymbol{K}) &= \sum_{j=0}^{N} (\sigma_{a_j}^2 + \sigma_{b_j}^2)(u_{jx}^2 + v_{jx}^2) \\
&- \left(\frac{Y_1 + Y_2}{Y_1 Y_3 - Y_2^2} - \frac{(K_1 Y_2 - K_2 Y_1)^2 + (K_1 Y_3 - K_2 Y_2)(K_1 Y_2 - K_2 Y_1)}{(Y_1 Y_3 - Y_2^2)^2} \right) \\
&\qquad \times \left| \sum_{j=0}^{N} (\sigma_{b_j}^2 u_j - i\sigma_{a_j}^2 v_j)(u_{jx} + i v_{jx}) \right|^2 \\
&- \left(\frac{Y_2 + Y_3}{Y_1 Y_3 - Y_2^2} - \frac{(K_1 Y_3 - K_2 Y_2)^2 + (K_1 Y_3 - K_2 Y_2)(K_1 Y_2 - K_2 Y_1)}{(Y_1 Y_3 - Y_2^2)^2} \right) \\
&\qquad \times \left| \sum_{j=0}^{N} (\sigma_{a_j}^2 u_j - i\sigma_{b_j}^2 v_j)(u_{jx} + i v_{jx}) \right|^2 \\
&+ \left(\frac{Y_2}{Y_1 Y_3 - Y_2^2} - \frac{(K_1 Y_3 - K_2 Y_2)(K_1 Y_2 - K_2 Y_1)}{(Y_1 Y_3 - Y_2^2)^2} \right) \\
&\qquad \times \left| \sum_{j=0}^{N} ((\sigma_{a_j}^2 u_j - i\sigma_{b_j}^2 v_j) + i(\sigma_{b_j}^2 u_j - i\sigma_{a_j}^2 v_j))(u_{jx} + i v_{jx}) \right|^2 .
\end{aligned} \tag{3.35}$$

Since X_1 and X_2 are random variables distributed according to the normal law, their joint density is

$$p_{X_1,X_2}(\boldsymbol{K}') = \frac{1}{2\pi\sqrt{Y_1 Y_3 - Y_2^2}} \exp\left(-\frac{K_1^2 Y_3 - 2K_1 K_2 Y_2 + K_2^2 Y_1}{2(Y_1 Y_3 - Y_2^2)} \right). \tag{3.36}$$

Finally, in accordance with (3.8), (3.35) and (3.36), the required result is proved.

3.2 A Sequence of Observations of a Brownian Motion

If $\{A_j\}_{j=0}^{\infty}$ and $\{B_j\}_{j=0}^{\infty}$ are sequences of i.i.d., real, normal random variables for which the respective increments $A_j - A_{j-1}$ and $B_j - B_{j-1}$ are independent for $j \geq 0$ and

$A_{-1} = B_{-1} = 0$ by convention, then the increments

$$\Delta_j = (A_j - A_{j-1}) + i(B_j - B_{j-1})$$

are independent, real, normal random variables with mean zero and finite $\mathrm{Var}(\Delta_j)$ such that $A_j + iB_j = \Delta_0 + \cdots + \Delta_j$ for $j \geq 0$. Then $\{A_j + iB_j\}_{j=0}^{\infty}$ can be interpreted as a sequence of successive observations of a Brownian motion. More precisely, $A_j + iB_j = W(t_j)$ for $j \geq 0$, where $t_0 < t_1 < \ldots$ and $\{W(t)\}_{t=0}^{\infty}$ is the standard Brownian motion. It is plain that $\mathrm{Var}(\Delta_j)$ is the distance between the successive times t_{j-1} and t_j for $j \geq 0$. Thus, the sum in (3.2) assumes the form

$$S(z) = \sum_{j=0}^{N}(A_j + iB_j)f_j(z) = \sum_{k=0}^{N} F_k(z)\Delta_k, \tag{3.37}$$

where

$$F_k(z) = \sum_{j=k}^{N} u_j(x, y) + i\sum_{j=k}^{N} v_j(x, y) \tag{3.38}$$

for $0 \leq k \leq N$ and $(x, y) \in \mathbb{R}^2$. In fact, $\{F_k\}_{k=0}^{N}$ is a sequence of holomorphic functions that are real-valued on $\mathbb{R}$. Hence, $\overline{F_k(z)} = F_k(\overline{z})$ for $0 \leq k \leq N$ and all $z \in \mathbb{C}$. Regard that the covariance matrix of Δ_k is given by

$$\Gamma_k = \begin{pmatrix} \sigma_{a_k}^2 & 0 \\ 0 & \sigma_{b_k}^2 \end{pmatrix}$$

for $0 \leq k \leq N$. Then, from Theorem 4.1, the following result is attained.

Theorem 3.4 *Provided all the conditions imposed on S in* (3.37) *and* (3.38) *and T are satisfied, then for all integers $N > 1$ the formula for $h_{\boldsymbol{K}}$ in Theorem 4.1 now holds for*

$$D_0(z) = \sqrt{Y_1(z)Y_3(z) - Y_2^2(z)},$$

where

$$Y_1(z) = \sum_{k=0}^{N} \left(\sigma_{a_k}^2 \left(\sum_{j=k}^{N} u_j \right)^2 + \sigma_{b_k}^2 \left(\sum_{j=k}^{N} v_j \right)^2 \right),$$

$$Y_2(z) = \sum_{k=0}^{N} \left((\sigma_{a_k}^2 - \sigma_{b_k}^2) \left(\sum_{j=k}^{N} u_j \right) \left(\sum_{j=k}^{N} v_j \right) \right),$$

$$Y_3(z) = \sum_{k=0}^{N} \left(\sigma_{b_k}^2 \left(\sum_{j=k}^{N} u_j \right)^2 + \sigma_{a_k}^2 \left(\sum_{j=k}^{N} v_j \right)^2 \right),$$

and

$$D_1(z) = \sum_{k=0}^{N} \left(\sigma_{a_k}^2 \sum_{j=k}^{N} u_j - i\sigma_{b_k}^2 \sum_{j=k}^{N} v_j \right) \left(\sum_{j=k}^{N} u_{jx} + i \sum_{j=k}^{N} v_{jx} \right),$$

$$D_2(z) = \sum_{k=0}^{N} \left(\sigma_{b_k}^2 \sum_{j=k}^{N} u_j - i\sigma_{a_k}^2 \sum_{j=k}^{N} v_j \right) \left(\sum_{j=k}^{N} u_{jx} + i \sum_{j=k}^{N} v_{jx} \right),$$

$$D_3(z) = \sum_{k=0}^{N} (\sigma_{a_k}^2 + \sigma_{b_k}^2) \left(\left(\sum_{j=k}^{N} u_{jx} \right)^2 + \left(\sum_{j=k}^{N} v_{jx} \right)^2 \right).$$

CHAPTER 4

THE CROSSINGS OF RANDOM SUMS, PART II

For real i.i.d. standard normal coefficients, the expected density of real zeros of a random polynomial is given by an exact Kac [34] formula. For complex coefficients, Dunnage [17, 18] gave some estimates for the number of real zeros. For complex zeros, the expected intensity of zeros was studied by Shepp and Vanderbei [48] for real i.i.d. standard normal coefficients and generalized by Ibragimov and Zeitouni [33] for a wider class of distributions of coefficients. The expected intensity was also dealt with by Hammersley [32], Edelman and Kostlan [20], and Farahmand and Grigorash [28]. Then Vanderbei [52] generalized the work in [48] to random sums with holomorphic functions that are real-valued on the real line as the basis functions. Motivated by the studies conducted by Vanderbei [52] and Farahmand [26], the present authors [11, 12] obtained results on the level crossings of these random sums. More precisely, the results were initially based on standard normal coefficients and later extended for coefficients with unrestricted variances. Our interest in this chapter is to answer the basic question about the expected number of complex zeros for coefficients with nonvanishing mean values and unrestricted variances.

Let

$$S(z) = \sum_{j=0}^{N} (a_j + ib_j) f_j(z),$$

where a_j and b_j are mutually independent and identically distributed, real, random variables such that $a_j \sim \mathcal{N}(\mu_{a_j}, \sigma_{a_j}^2)$ and $b_j \sim \mathcal{N}(\mu_{b_j}, \sigma_{b_j}^2)$ and $f_j(z) = u_j(x, y) + iv_j(x, y)$ are holomorphic functions that are real-valued on the real line for $0 \leq j \leq N$. By the Schwarz reflection principle, $\overline{f_j(z)} = f_j(\overline{z})$ for $0 \le j \le N$ and all $z \in \mathbb{C}$. Let us assume that T is a manifold in the complex plane. Assume that T is compact and the boundary of T does not contain points for which $S(z) = \boldsymbol{K}$, where $\boldsymbol{K} = K_1 + iK_2$ with K_1 and K_2 being constants independent of z. Assume, further, that the origin does not belong to T.

Denote

$$X_1 = \sum_{j=0}^{N}(a_j u_j - b_j v_j), \quad X_2 = \sum_{j=0}^{N}(a_j v_j + b_j u_j),$$

as the real and imaginary parts of S and

$$\boldsymbol{X} = (X_1, X_2)'.$$

Then, from the Cauchy–Riemann equations, for $z = x + iy$ the Jacobian of the random transformation $(x, y) \to (X_1, X_2)$ is

$$\nabla \boldsymbol{X} = \begin{pmatrix} \sum_{j=0}^{N}(a_j u_{jx} - b_j v_{jx}) & \sum_{j=0}^{N}(a_j v_{jx} + b_j u_{jx}) \\ \sum_{j=0}^{N}(-a_j v_{jx} - b_j u_{jx}) & \sum_{j=0}^{N}(a_j u_{jx} - b_j v_{jx}) \end{pmatrix},$$

where $u_{jx} = \partial u_j/\partial x$ and $v_{jx} = \partial v_j/\partial x$.

Assume that there are no points in T for which both equalities $S(z) = \boldsymbol{K}$ and $\det(\nabla \boldsymbol{X}) = 0$ take place. Since N is fixed, T contains not more than a finite number of zeros of $\boldsymbol{X} = \boldsymbol{K}$, where $\boldsymbol{K} = (K_1, K_2)'$. If $N^S_{\boldsymbol{K}}(T)$ denotes the random number of complex zeros, counted with multiplicity, in T of S that cross the complex level $\boldsymbol{K}$, then according to Rice formula

$$E(N^S_{\boldsymbol{K}}(T)) = \int_T h_{\boldsymbol{K}}(z)\, dz, \tag{4.1}$$

where

$$h_{\boldsymbol{K}}(z) = E(|\det(\nabla \boldsymbol{X})| \mid \boldsymbol{X} = \boldsymbol{K})\, p_{X_1,X_2}(\boldsymbol{K}'). \tag{4.2}$$

Here, $p_{x,y}$ is used to denote the probability density of the random vector $\boldsymbol{X}$. (See [2, Theorem 11.2.3, Corollary 11.2.4, pp. 269-271] and [6,7].) The explicit derivation of $h_{\boldsymbol{K}}$ constitutes the primary reason for studying the zeros of $S(z) = \boldsymbol{K}$.

For the sake of brevity, let

$$Y_1(z) = \sum_{j=0}^{N}(\sigma_{a_j}^2 u_j^2 + \sigma_{b_j}^2 v_j^2) - \left(\sum_{j=0}^{N}(\mu_{a_j} u_j - \mu_{b_j} v_j)\right)^2,$$

$$Y_2(z) = \sum_{j=0}^{N}(\sigma_{a_j}^2 - \sigma_{b_j}^2) u_j v_j - \left(\sum_{j=0}^{N}(\mu_{a_j} u_j - \mu_{b_j} v_j)\right)\left(\sum_{j=0}^{N}(\mu_{a_j} v_j + \mu_{b_j} u_j)\right),$$

$$Y_3(z) = \sum_{j=0}^{N}(\sigma_{a_j}^2 v_j^2 + \sigma_{b_j}^2 u_j^2) - \left(\sum_{j=0}^{N}(\mu_{a_j} v_j + \mu_{b_j} u_j)\right)^2.$$

Then define

$$D_0(z) = \sqrt{Y_1(z) Y_3(z) - Y_2(z)^2}$$

and

$$M(z) = \sum_{j=0}^{N} E(a_j + i b_j) f_j'(z), \qquad D_1(z) = \sum_{j=0}^{N}(A_{j,1}(z) - i B_{j,1}(z)) f_j'(z),$$

$$D_2(z) = \sum_{j=0}^{N}(B_{j,2}(z) - i A_{j,2}(z)) f_j'(z), \quad D_3(z) = \sum_{j=0}^{N}(\sigma_{a_j}^2 + \sigma_{b_j}^2)|f_j'(z)|^2,$$

where

$$A_{j,1}(z) = \sigma_{a_j}^2 u_j - \mu_{a_j} E(X_1), \quad A_{j,2}(z) = \sigma_{a_j}^2 v_j - \mu_{a_j} E(X_2),$$

$$B_{j,1}(z) = \sigma_{b_j}^2 v_j + \mu_{b_j} E(X_1), \quad B_{j,2}(z) = \sigma_{b_j}^2 u_j - \mu_{b_j} E(X_2),$$

for $0 \leq j \leq N$. The following theorem is proved in Section 4.1.

THEOREM 4.1 *For all integers $N > 1$ one has*

$$
\begin{aligned}
&h_{\boldsymbol{K}} \\
&= \frac{1}{2\pi D_0} \exp\left(-\frac{(K_1 - E(X_1))^2 Y_3 + (K_2 - E(X_2))^2 Y_1 - 2(K_1 - E(X_1))(K_2 - E(X_2))Y_2}{2D_0^2}\right) \\
&\times \left\{ D_3 - \frac{|D_1|^2}{D_0^2}\left(Y_3 - \frac{((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)^2}{D_0^2}\right)\right. \\
&-\frac{|D_2|^2}{D_0^2}\left(Y_1 - \frac{((K_1 - E(X_1))Y_2 - (K_2 - E(X_2))Y_1)^2}{D_0^2}\right) \\
&+\left(\frac{|D_1 + iD_2|^2 - |D_1|^2 - |D_2|^2}{D_0^2}\right) \\
&\times \left(Y_2 - \frac{((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)((K_1 - E(X_1))Y_2 - (K_2 - E(X_2))Y_1)}{D_0^2}\right) \\
&-\left(\frac{|M + D_1|^2 - |M|^2 - |D_1|^2}{D_0^2}\right)((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2) \\
&\left.+\left(\frac{|M + iD_2|^2 - |M|^2 - |D_2|^2}{D_0^2}\right)((K_1 - E(X_1))Y_2 - (K_2 - E(X_2))Y_1)\right\}.
\end{aligned}
$$

Several consequences of Theorem 4.1 are of special interest. They are used to recover the key results from our work in [11] and [12]. These consequences are derived in Section 4.2.

4.1 A Generalized Density Function for Multivariate Normal Coefficients

In order to utilize (4.1) and (4.2), first note that for values of X_1 and X_2

$$
\begin{aligned}
\det(\nabla \boldsymbol{X}) &= \sum_{j=0}^{N}\sum_{k=0}^{N}((a_j a_k + b_j b_k)(u_{jx}u_{kx} + v_{jx}v_{kx}) + (a_j b_k - b_j a_k)(v_j u_k - u_j v_k)) \\
&= \sum_{j=0}^{N}(a_j^2 + b_j^2)(u_{jx}^2 + v_{jx}^2) + \sum_{j=0}^{N}\sum_{\substack{k=0 \\ k\neq j}}^{N}((a_j a_k + b_j b_k)(u_{jx}u_{kx} + v_{jx}v_{kx})
\end{aligned}
$$

$$+ (a_j b_k - b_j a_k)(v_{jx} u_{kx} - u_{jx} v_{kx})).$$

Thus, the evaluation of $h_{\boldsymbol{K}}$ leads to the computation of the expected value of a quadratic form $\det(\nabla \boldsymbol{X})$ of i.i.d. random variables, conditioned on two linear combinations. We define

$$\boldsymbol{\Sigma}_{\boldsymbol{ab}} = E((\boldsymbol{a} - E(\boldsymbol{a}))(\boldsymbol{b} - E(\boldsymbol{b}))')$$

for the generalized nonconditional covariance matrix of the two vectors $\boldsymbol{a} = (a_0, \ldots, a_N)'$ and $\boldsymbol{b} = (b_0, \ldots, b_N)'$, and

$$\boldsymbol{\Sigma}_{\boldsymbol{ab},\boldsymbol{X}} = \boldsymbol{\Sigma}_{\boldsymbol{ab}} - \boldsymbol{\Sigma}_{\boldsymbol{aX}} \boldsymbol{\Sigma}_{\boldsymbol{XX}}^{-1} \boldsymbol{\Sigma}_{\boldsymbol{Xb}}. \tag{4.3}$$

Based on the assumption that all the scalar random variables involved are i.i.d., by standard multivariate analysis,

$$\operatorname{Cov}(\boldsymbol{a}, \boldsymbol{b} \mid \boldsymbol{Y} = \boldsymbol{K}) = \begin{pmatrix} \boldsymbol{\Sigma}_{\boldsymbol{aa},\boldsymbol{X}} & \boldsymbol{\Sigma}_{\boldsymbol{ab},\boldsymbol{X}} \\ \boldsymbol{\Sigma}_{\boldsymbol{ba},\boldsymbol{X}} & \boldsymbol{\Sigma}_{\boldsymbol{bb},\boldsymbol{X}} \end{pmatrix}$$

and

$$E(\boldsymbol{a} \mid \boldsymbol{X} = \boldsymbol{K}) = E(\boldsymbol{a}) + \boldsymbol{\Sigma}_{\boldsymbol{aX}} \boldsymbol{\Sigma}_{\boldsymbol{XX}}^{-1} (\boldsymbol{K} - E(\boldsymbol{X})). \tag{4.4}$$

The formula for $E(\boldsymbol{b} \mid \boldsymbol{X} = \boldsymbol{K})$ is analogous.

Direct computation leads to the equalities

$$E(X_1) = \sum_{j=0}^{N} (\mu_{a_j} u_j - \mu_{b_j} v_j), \quad E(X_2) = \sum_{j=0}^{N} (\mu_{a_j} v_j + \mu_{b_j} u_j).$$

Hence,

$$\mathbf{\Sigma}_{\boldsymbol{XX}} = \begin{pmatrix} E(X_1^2) - (E(X_1))^2 & E(X_1X_2) - E(X_1)E(X_2) \\ E(X_1X_2) - E(X_1)E(X_2) & E(X_2^2) - (E(X_2))^2 \end{pmatrix}.$$

Now, since

$$E(X_1^2) = \sum_{j=0}^{N}(\sigma_{a_j}^2 u_j^2 + \sigma_{b_j}^2 v_j^2), \quad E(X_2^2) = \sum_{j=0}^{N}(\sigma_{a_j}^2 v_j^2 + \sigma_{b_j}^2 u_j^2),$$

and

$$E(X_1X_2) = \sum_{j=0}^{N}(\sigma_{a_j}^2 - \sigma_{b_j}^2)u_j v_j,$$

it follows that

$$\mathbf{\Sigma}_{\boldsymbol{XX}} = \begin{pmatrix} Y_1 & Y_2 \\ Y_2 & Y_3 \end{pmatrix}.$$

Thus,

$$\det(\mathbf{\Sigma}_{\boldsymbol{XX}}) = Y_1Y_3 - Y_2^2 > 0,$$

if X_1 and X_2 are not strictly correlated. Hence,

$$\mathbf{\Sigma}_{\boldsymbol{XX}}^{-1} = \frac{1}{Y_1Y_3 - Y_2^2}\begin{pmatrix} Y_3 & -Y_2 \\ -Y_2 & Y_1 \end{pmatrix}.$$

Expanding our definitions, we obtain $\mathbf{\Sigma}_{\boldsymbol{aa},\boldsymbol{X}}, \mathbf{\Sigma}_{\boldsymbol{bb},\boldsymbol{X}}, \mathbf{\Sigma}_{\boldsymbol{ab},\boldsymbol{X}}$, and $\mathbf{\Sigma}_{\boldsymbol{ba},\boldsymbol{X}}$ as follows. For the jth row and kth column

$$\mathbf{\Sigma}_{\boldsymbol{aa}} = E(a_j a_k) - E(a_j)E(a_k) = \delta_{jk}\sigma_{a_j}^2 - \mu_{a_j}\mu_{a_k},$$

$$\mathbf{\Sigma}_{\boldsymbol{bb}} = E(b_j b_k) - E(b_j)E(b_k) = \delta_{jk}\sigma_{b_j}^2 - \mu_{b_j}\mu_{b_k},$$

$$\mathbf{\Sigma}_{\boldsymbol{ab}} = E(a_j a_k) - E(a_j)E(b_k) = -\mu_{a_j}\mu_{b_k},$$

$$\mathbf{\Sigma}_{\boldsymbol{ba}} = \mathbf{\Sigma}'_{\boldsymbol{ab}} = -\mu_{b_j}\mu_{a_k},$$

where δ_{jk} denotes the Kronecker delta. Since

$$\begin{aligned}
E((a_j - E(a_j))(X_1 - E(X_1))) &= E\left(\sum_{k=0}^{N}(a_j a_k u_k - a_j b_k v_k)\right) - \mu_{a_j}E(X_1) \\
&= A_{j,1}, \\
E((a_j - E(a_j))(X_2 - E(X_2))) &= E\left(\sum_{k=0}^{N}(a_j a_k v_k + a_j b_k u_k)\right) - \mu_{a_j}E(X_2) \\
&= A_{j,2}, \\
E((b_j - E(b_j))(X_1 - E(X_1))) &= E\left(\sum_{k=0}^{N}(b_j a_k u_k - b_j b_k v_k)\right) - \mu_{b_j}E(X_1) \\
&= -B_{j,1}, \\
E((b_j - E(b_j))(X_2 - E(X_2))) &= E\left(\sum_{k=0}^{N}(b_j a_k v_k + b_j b_k u_k)\right) - \mu_{b_j}E(X_2) \\
&= B_{j,2},
\end{aligned}$$

it follows that

$$\mathbf{\Sigma}_{\boldsymbol{aX}} = (A_{j,1} \quad A_{j,2}), \quad \mathbf{\Sigma}_{\boldsymbol{aX}} = (-B_{j,1} \quad B_{j,2})$$

for $0 \leq j \leq N$. Using (4.3), simple algebra then leads to

$$\begin{aligned}
\mathbf{\Sigma}_{\boldsymbol{aa},\boldsymbol{X}} &= \delta_{jk}\sigma^2_{a_j} - \mu_{a_j}\mu_{a_k} \\
&\quad - (A_{j,1}A_{k,1}Y_3 + A_{j,2}A_{k,2}Y_1 - (A_{j,1}A_{k,2} + A_{j,2}A_{k,1})Y_2)/(Y_1Y_3 - Y_2^2), \\
\mathbf{\Sigma}_{\boldsymbol{bb},\boldsymbol{X}} &= \delta_{jk}\sigma^2_{b_j} - \mu_{b_j}\mu_{b_k} \\
&\quad - (B_{j,1}B_{k,1}Y_3 + B_{j,1}B_{k,2}Y_1 + (B_{j,1}B_{k,2} + B_{j,2}B_{k,1})Y_2)/(Y_1Y_3 - Y_2^2), \\
\mathbf{\Sigma}_{\boldsymbol{ab},\boldsymbol{X}} &= -\mu_{a_j}\mu_{b_k}
\end{aligned}$$

$$+ (A_{j,1}B_{k,1}Y_3 - A_{j,2}B_{k,2}Y_1 + (A_{j,1}B_{k,2} - A_{j,2}B_{k,1})Y_2)/(Y_1Y_3 - Y_2^2),$$

$$\boldsymbol{\Sigma}_{\boldsymbol{ba},\boldsymbol{X}} = -\mu_{b_j}\mu_{a_k}$$
$$+ (B_{j,1}A_{k,1}Y_3 - B_{j,2}A_{k,2}Y_1 + (B_{j,2}A_{k,1} - B_{j,1}A_{k,1})Y_2)/(Y_1Y_3 - Y_2^2).$$

Using these in (4.4), we obtain

$$E(\boldsymbol{a} \mid \boldsymbol{X} = \boldsymbol{K}) = \mu_{a_j} - (((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)A_{j,1}$$
$$+ ((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2)A_{j,2})/(Y_1Y_3 - Y_2^2),$$

$$E(\boldsymbol{b} \mid \boldsymbol{X} = \boldsymbol{K}) = \mu_{b_j} + (((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)B_{j,1}$$
$$- ((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2)B_{j,2})/(Y_1Y_3 - Y_2^2).$$

Thus, for the jth row and kth column

$$E(a_ja_k \mid \boldsymbol{X} = \boldsymbol{K}) = E(a_j \mid \boldsymbol{X} = \boldsymbol{K})E(a_k \mid \boldsymbol{X} = \boldsymbol{K}) + \mathrm{Cov}(a_ja_k \mid \boldsymbol{X} = \boldsymbol{K})$$
$$= \delta_{jk}\sigma_{a_j}^2 - (A_{j,1}A_{k,1}Y_3 + A_{j,2}A_{k,2}Y_1 - (A_{j,1}A_{k,2} + A_{j,2}A_{k,1})Y_2)/(Y_1Y_3 - Y_2^2)$$
$$- (((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)(\mu_{a_j}A_{k,1} + \mu_{a_k}A_{j,1})$$
$$+ ((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2)(\mu_{a_j}A_{k,2} + \mu_{a_k}A_{j,2})/(Y_1Y_3 - Y_2^2)$$
$$+ ((K_1 - E(X_1))Y_3 - (K_2 - E(K_2))Y_2)((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2)$$
$$\times (A_{j,1}A_{k,2} + A_{j,2}A_{k,1})/(Y_1Y_3 - Y_2^2)^2$$
$$+ (((K_1 + E(X_1))Y_3 - (K_2 - E(X_2))Y_2)^2A_{j,1}A_{k,1}$$
$$+ ((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2^2A_{j,2}A_{k,2})/(Y_1Y_3 - Y_2^2)^2,$$

$$E(b_jb_k \mid \boldsymbol{X} = \boldsymbol{K}) = E(b_j \mid \boldsymbol{X} = \boldsymbol{K})E(b_k \mid \boldsymbol{X} = \boldsymbol{K}) + \mathrm{Cov}(b_jb_k \mid \boldsymbol{X} = \boldsymbol{K})$$
$$= \delta_{jk}\sigma_{b_j}^2 - (B_{j,1}B_{k,1}Y_3 + B_{j,2}B_{k,2}Y_1 + (B_{j,1}B_{k,2} + B_{j,2}B_{k,1})Y_2)/(Y_1Y_3 - Y_2^2)$$

$$+\,((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)(\mu_{b_j} B_{k,1} + \mu_{b_k} B_{j,1})$$

$$-\,((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2)(\mu_{b_j} B_{k,2} + \mu_{b_k} B_{j,2})/(Y_1 Y_3 - Y_2^2)$$

$$-\,((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2)$$

$$\times\,(B_{j,1} B_{k,2} + B_{j,2} B_{k,1})/(Y_1 Y_3 - Y_2^2)^2$$

$$+\,(((K_1 - E(X_1))Y_3 - (K_2 - E(X_2)Y_2)^2 B_{j,1} B_{k,1}$$

$$+\,((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2)^2) B_{j,2} B_{k,2})/(Y_1 Y_3 - Y_2^2)^2,$$

$$E(a_j b_k \mid \boldsymbol{X} = \boldsymbol{K}) = E(a_j \mid \boldsymbol{X} = \boldsymbol{K}) E(b_k \mid \boldsymbol{X} = \boldsymbol{K}) + \mathrm{Cov}(a_j b_k \mid \boldsymbol{X} = \boldsymbol{K})$$

$$= (A_{j,1} B_{k,1} Y_3 - A_{j,2} B_{k,2} Y_1 + (A_{j,1} B_{k,2} - A_{j,2} B_{k,1}) Y_2)/(Y_1 Y_3 - Y_2^2)$$

$$-\,(((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)(\mu_{b_k} A_{j,1} - \mu_{a_j} B_{k,1})$$

$$+\,((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2)(\mu_{a_j} B_{k,2} + \mu_{b_k} A_{j,2}))/(Y_1 Y_3 - Y_2^2)$$

$$+\,((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2)$$

$$\times\,(A_{j,1} B_{k,2} - A_{j,2} B_{k,1})/(Y_1 Y_3 - Y_2^2)^2$$

$$+\,(((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)^2 A_{j,1} B_{k,1}$$

$$+\,((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2)^2 A_{j,2} B_{k,2})/(Y_1 Y_3 - Y_2^2)^2,$$

$$E(b_j a_k \mid \boldsymbol{X} = \boldsymbol{K}) = E(b_j \mid \boldsymbol{X} = \boldsymbol{K}) E(a_k \mid \boldsymbol{X} = \boldsymbol{K}) + \mathrm{Cov}(b_j a_k \mid \boldsymbol{X} = \boldsymbol{K})$$

$$= (B_{j,1} A_{k,1} Y_3 - B_{j,2} A_{k,2} Y_1 + (B_{j,2} A_{k,1} - B_{j,1} A_{k,2}) Y_2)/(Y_1 Y_3 - Y_2^2)$$

$$-\,(((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)(\mu_{b_j} A_{k,1} - \mu_{a_k} B_{j,1})$$

$$+\,((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2)(\mu_{b_j} A_{k,2} + \mu_{a_k} B_{j,2}))/(Y_1 Y_3 - Y_2^2)$$

$$+\,((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2)$$

$$\times\,(B_{j,2} A_{k,1} - B_{j,1} A_{k,2})/(Y_1 Y_3 - Y_2^2)^2$$

$$-\,(((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)^2 B_{j,1} A_{k,1}$$

$$- ((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2)^2 B_{j,2}A_{k,2})/(Y_1Y_3 - Y_2^2)^2.$$

Having obtained the four expectations above, we can now derive the required expectations for computing the value of $E(|\det(\nabla \boldsymbol{X})| \mid \boldsymbol{X} = \boldsymbol{K})$. Thus, for $j \neq k$

$$\begin{aligned}
&E(a_j a_k + b_j b_k \mid \boldsymbol{X} = \boldsymbol{K}) \\
&\quad = (((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)^2 (A_{j,1}A_{k,1} + B_{j,1}B_{k,1}) \\
&\qquad + ((K_1 - E(X_1))Y_1 - (K_2 - E(X_2))Y_2)^2 (A_{j,2}A_{k,2} + B_{j,2}B_{k,2}))/(Y_1Y_3 - Y_2^2) \\
&\qquad - (((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)(\mu_{a_j}A_{k,1} + \mu_{a_k}A_{j,1} - \mu_{b_j}B_{k,1} - \mu_{b_k}B_{j,1}) \\
&\qquad + ((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2) \\
&\qquad \times (\mu_{a_j}A_{k,2} + \mu_{a_k}A_{j,2} + \mu_{b_j}B_{k,2} + \mu_{b_k}B_{j,2}))/(Y_1Y_3 - Y_2^2) \\
&\qquad + (((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2) \\
&\qquad \times (A_{j,1}A_{k,2} + A_{j,2}A_{k,1} - B_{j,1}B_{k,2} - B_{j,2}B_{k,1}))/(Y_1Y_3 - Y_2^2)^2 \\
&\qquad + (((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)^2 (A_{j,1}A_{k,1} + B_{j,1}B_{k,1}) \\
&\qquad + ((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2)^2 (A_{j,2}A_{k,2} + B_{j,2}B_{k,2}))/(Y_1Y_3 - Y_2^2)^2,
\end{aligned}$$

$$\begin{aligned}
&E(a_j b_k - b_j a_k \mid \boldsymbol{X} = \boldsymbol{K}) \\
&\quad = -((B_{j,1}A_{k,1} - A_{j,1}B_{k,1})Y_3 + (A_{j,2}B_{k,2} - B_{j,2}A_{k,2})Y_1 \\
&\qquad - (A_{j,1}B_{k,2} + B_{j,1}A_{k,2} - A_{j,2}B_{k,1} - B_{j,2}A_{k,1})Y_2)/(Y_1Y_3 - Y_2^2) \\
&\qquad + (((K_1 - E(X_1))Y_3 - (K_2 E(X_2))Y_2)(\mu_{a_j}B_{k,1} + \mu_{b_j}A_{k,1} - \mu_{a_k}B_{j,1} - \mu_{b_k}A_{j,1}) \\
&\qquad + ((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2) \\
&\qquad \times (\mu_{b_j}A_{k,2} + \mu_{a_k}B_{j,2} - \mu_{b_k}A_{j,2} - \mu_{a_j}B_{k,2}))/(Y_1Y_3 - Y_2^2) \\
&\qquad + (((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2) \\
&\qquad \times (A_{j,1}B_{k,2} + B_{j,1}A_{k,2} - A_{j,2}B_{k,1} - B_{j,2}A_{k,1}))/(Y_1Y_3 - Y_2^2)^2
\end{aligned}$$

$$+ (((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)^2 (B_{j,1} A_{k,1} - A_{j,1} B_{k,1})$$

$$+ ((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2)^2 (A_{j,2} B_{k,1} - B_{j,2} A_{k,2}))/(Y_1 Y_3 - Y_2^2)^2,$$

$$\begin{aligned}
&E(a_j^2 + b_j^2 \mid \boldsymbol{X} = \boldsymbol{K}) \\
&\quad = (\sigma_{a_j}^2 + \sigma_{b_j}^2) - ((A_{j,1}^2 + B_{j,1}^2)Y_3 + ((A_{j,2}^2 + B_{j,2}^2)Y_1 \\
&\qquad - 2(A_{j,1} A_{j,2} - B_{j,1} B_{j,2})Y_2)/(Y_1 Y_3 - Y_2^2) \\
&\qquad - 2[((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_1)(\mu_{aj} A_{j,1} - \mu_{b_j} B_{j,1}) \\
&\qquad + ((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2)(\mu_{a_j} A_{j,2} + \mu_{b_j} B_{j,2})]/(Y_1 Y_3 - Y_2^2) \\
&\qquad + 2(((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2) \\
&\qquad \times (A_{j,1} A_{j,2} - B_{j,1} B_{j,2}))/(Y_1 Y_3 - Y_2^2)^2 \\
&\qquad + (((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)^2 (A_{j,1}^2 + B_{j,1}^2) \\
&\qquad + ((K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2)^2 (A_{j,2}^2 + B_{j,2}^2))/(Y_1 Y_3 - Y_2^2)^2.
\end{aligned}$$

After all the necessary simplifications, since $\det(\nabla \boldsymbol{X})$ is nonnegative,

$$\begin{aligned}
E(\det(\nabla \boldsymbol{X}) \mid \boldsymbol{X} = \boldsymbol{K}) = \sum_{j=0}^{N} &(\sigma_{a_j}^2 + \sigma_{b_j}^2)(u_{jx}^2 + v_{jx}^2) \\
&- (I_1 S_1 - I_2 S_2 - I_3 S_3 - I_4 S_4 - I_5 S_5)/(Y_1 Y_3 - Y_2^2),
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= Y_3 - ((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)^2/(Y_1 Y_3 - Y_2^2), \\
I_2 &= Y_1 - ((K_1 - E(X_1))Y_2 - (K_2 - E(X_2))Y_1)^2/(Y_1 Y_3 - Y_2^2), \\
I_3 &= Y_2 + ((K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2)((K_2 - E(X_2))Y_1 \\
&\qquad - (K_1 - E(X_1))Y_2)/(Y_1 Y_3 - Y_2^2), \\
I_4 &= (K_1 - E(X_1))Y_3 - (K_2 - E(X_2))Y_2,
\end{aligned}$$

$$I_5 = (K_2 - E(X_2))Y_1 - (K_1 - E(X_1))Y_2,$$

and

$$S_1 = \sum_{j=0}^{N}\sum_{k=0}^{N}((A_{j,1}A_{k,1} + B_{j,1}B_{k,1})(u_{jx}u_{kx} + v_{jx}v_{kx})$$
$$+ (B_{j,1}A_{k,1} - A_{j,1}B_{k,1})(v_{jx}u_{kx} - u_{jx}v_{kx})),$$
$$S_2 = \sum_{j=0}^{N}\sum_{k=0}^{N}((A_{j,2}A_{k,2} + B_{j,2}B_{k,2})(u_{jx}u_{kx} + v_{jx}v_{kx})$$
$$+ (A_{j,2}B_{k,2} - B_{j,2}A_{k,2})(v_{jx}u_{kx} - u_{jx}v_{kx})),$$
$$S_3 = \sum_{j=0}^{N}\sum_{k=0}^{N}((A_{j,1}A_{k,2} + A_{j,2}A_{k,1} - B_{j,1}B_{k,2} - B_{j,2}B_{k,1})(u_{jx}u_{kx} + v_{jx}v_{kx})$$
$$+ (A_{j,1}B_{k,2} + B_{j,1}A_{k,2} - A_{j,2}B_{k,1} - B_{j,2}A_{k,2})(v_{jx}u_{kx} - u_{jx}v_{kx})),$$
$$S_4 = \sum_{j=0}^{N}\sum_{k=0}^{N}((\mu_{a_j}A_{k,1} + \mu_{a_k}A_{j,1} - \mu_{b_j}B_{k,1} - \mu_{b_k}B_{j,1})(u_{jx}u_{kx} + v_{jx}v_{kx})$$
$$+ (\mu_{b_k}A_{j,2} + \mu_{a_j}B_{k,2} - \mu_{b_j}A_{k,2} - \mu_{a_k}B_{j,2})(v_{jx}u_{kx} - u_{jx}v_{kx})),$$
$$S_5 = \sum_{j=0}^{N}\sum_{k=0}^{N}((\mu_{a_j}A_{k,2} + \mu_{a_k}A_{j,2} + \mu_{b_j}B_{k,2} + \mu_{b_k}B_{j,2})(u_{jx}u_{kx} + v_{jx}v_{kx})$$
$$+ (\mu_{b_k}A_{j,2} + \mu_{a_j}B_{k,2} - \mu_{b_j}A_{k,2} - \mu_{a_k}B_{j,2})(v_{jx}u_{kx} - u_{jx}v_{kx})).$$

On noting that

$$S_1 = |D_1|^2, \quad S_2 = |D_2|^2,$$
$$S_3 = |D_1 + iD_2|^2 - |D_1| - |D_2|^2,$$
$$S_4 = |M + D_1|^2 - |M|^2 - |D_1|^2,$$
$$S_5 = |M + iD_2|^2 - |M|^2 - |D_2|^2,$$

after regrouping the terms, with a little algebra we can write

$$E(\det(\nabla \boldsymbol{X}) \mid \boldsymbol{X} = \boldsymbol{K}) = D_3^* - \frac{1}{Y_1Y_3^* - (Y_2^*)^2}(|D_1^*|^2 I_1 - |D_2^*|^2 I_2$$
$$+ (|D_1^* + iD_2^*|^2 - |D_1^*|^2 - |D_2^*|^2) I_3 - (|M + D_1^*|^2 - |M|^2 - |D_1^*|^2) I_4$$
$$- (|M + iD_2^*|^2 - |M|^2 - |D_2^*|^2) I_5).$$

Now, the joint density of two random normal variables X_1 and X_2 is given by

$$p_{X_1X_2}(K_1, K_2) = \frac{1}{2\pi\sqrt{Y_1Y_3 - Y_2^2}} \exp\left(-\frac{1}{2}(\boldsymbol{K} - E(\boldsymbol{X}))' \boldsymbol{\Sigma}_{\boldsymbol{XX}}^{-1} (\boldsymbol{K} - E(\boldsymbol{X}))\right)$$
$$= \frac{1}{2\pi\sqrt{Y_1Y_3 - Y_2^2}}$$
$$\times \exp\left(-\frac{(K_1 - E(X_1))^2 Y_3 + (K_2 - E(X_2))^2 Y_1 - 2(K_1 - E(X_1))(K_2 - E(X_2)) Y_2}{2(Y_1Y_3 - Y_2^2)}\right).$$

This completes the proof of the theorem.

4.2 Some Ramifications

Theorem 4.1 has several interesting consequences. First, if $\mu_{a_j} = \mu_{b_j} = \mu$ for $0 \leq j \leq N$ in Theorem 4.1, then $h_{\boldsymbol{K}}$ holds with some modifications to the auxiliary functions, namely,

$$E(X_1) = \mu \sum_{j=0}^{N} (u_j - v_j), \quad E(X_2) = \mu \sum_{j=0}^{N} (u_j + v_j),$$
$$Y_1(z) = \sum_{j=0}^{N} (\sigma_{a_j}^2 u_j^2 + \sigma_{b_j}^2 v_j^2) - \mu^2 \left(\sum_{j=0}^{N} (u_j - v_j)\right)^2,$$
$$Y_2(z) = \sum_{j=0}^{N} (\sigma_{a_j}^2 - \sigma_{b_j}^2) u_j v_j - \mu^2 \left(\sum_{j=0}^{N} (u_j - v_j)\right)\left(\sum_{j=0}^{N} (u_j + v_j)\right),$$
$$Y_3(z) = \sum_{j=0}^{N} (\sigma_{a_j}^2 v_j^2 + \sigma_{b_j}^2 u_j^2) - \mu^2 \left(\sum_{j=0}^{N} (u_j + v_j)\right)^2,$$

and

$$A_{j,1}(z) = \sigma_{a_j}^2 u_j - \mu^2 \sum_{k=0}^{N} (u_k - v_k), \quad A_{j,2}(z) = \sigma_{a_j}^2 v_j - \mu^2 \sum_{k=0}^{N} (u_k + v_k),$$
$$B_{j,1}(z) = \sigma_{b_j}^2 v_j + \mu^2 \sum_{k=0}^{N} (u_k - v_k), \quad B_{j,2}(z) = \sigma_{b_j}^2 u_j - \mu^2 \sum_{k=0}^{N} (u_k + v_k),$$

for $0 \le j \le N$. Observe that M is changed implicitly by the change in $E(a_j + ib_j)$.

Further, if $\mu_{a_j} = \mu_{b_j} = 0$ for $0 \le j \le N$, then $E(X_1) = E(X_2) = 0$. Thus, $A_{j,1} = \sigma_{a_j}^2 u_j$, $A_{j,2} = \sigma_{a_j}^2 v_j$, $B_{j,1} = \sigma_{b_j}^2 v_j$, $B_{j,2} = \sigma_{b_j}^2 u_j$, and $M = 0$. Consequently, Theorem 2 in [12] is recovered, which can be summarized as follows.

COROLLARY 4.1.1 *If $\mu_{a_j} = \mu_{b_j} = 0$ for $0 \le j \le N$, then for all integers $N > 1$ one has*

$$\begin{aligned} h_{\boldsymbol{K}} = {} & \frac{1}{2\pi D_0} \exp\left(-\frac{K_1^2 Y_3 + K_2^2 Y_1 - 2K_1 K_2 Y_2}{2D_0^2} \right) \\ & \times \left\{ D_3 - \frac{|D_1|^2}{D_0^2} \left(Y_3 - \frac{(K_1 Y_3 - K_2 Y_2)^2}{D_0^2} \right) - \frac{|D_2|^2}{D_0^2} \left(Y_1 - \frac{(K_1 Y_2 - K_2 Y_1)^2}{D_0^2} \right) \right. \\ & \left. + \left(\frac{|D_1 + iD_2|^2 - |D_1|^2 - |D_2|^2}{D_0^2} \right) \left(Y_2 - \frac{(K_1 Y_3 - K_2 Y_2)(K_1 Y_2 - K_2 Y_1)}{D_0^2} \right) \right\}, \end{aligned}$$

where

$$Y_1(z) = \sum_{j=0}^{N} (\sigma_{a_j}^2 u_j^2 + \sigma_{b_j}^2 v_j^2), \quad Y_2(z) = \sum_{j=0}^{N} (\sigma_{a_j}^2 - \sigma_{b_j}^2) u_j v_j, \quad Y_3(z) = \sum_{j=0}^{N} (\sigma_{a_j}^2 v_j^2 + \sigma_{b_j}^2 u_j^2),$$

and

$$D_0(z) = \sqrt{Y_1(z) Y_3(z) - Y_2(z)^2}, \qquad D_1(z) = \sum_{j=0}^{N} (\sigma_{a_j}^2 u_j - i\sigma_{b_j}^2 v_j) f_j'(z),$$
$$D_2(z) = \sum_{j=0}^{N} (\sigma_{b_j}^2 u_j - i\sigma_{a_j}^2 v_j) f_j'(z), \quad D_3(z) = \sum_{j=0}^{N} (\sigma_{a_j}^2 + \sigma_{b_j}^2) |f_j'(z)|^2.$$

Second, if $\sigma_{a_j}^2 = \sigma_{b_j}^2 = \sigma^2$ for $0 \le j \le N$ in Theorem 4.1, then $Y_2(z) = -E(X_1)E(X_2)$ and

$$Y_1(z) = \sigma^2 \sum_{j=0}^{N} |f_j(z)|^2 - (E(X_1))^2, \quad Y_3(z) = \sigma^2 \sum_{j=0}^{N} |f_j(z)|^2 - (E(X_2))^2.$$

Observe that

$$A_{j,1}(z) - iB_{j,1}(z) = \sigma^2 \overline{f_j(z)} - E(X_1)E(a_j + ib_j),$$
$$B_{j,2}(z) - iA_{j,2}(z) = \sigma^2 \overline{f_j(z)} + iE(X_2)E(a_j + ib_j).$$

Altogether, the formula for $h_{\boldsymbol{K}}$ in Theorem 4.1 holds with

$$D_0(z) = \sqrt{\left(\sigma^2 \sum_{j=0}^{N} |f_j(z)|^2\right)^2 - \sigma^2 \sum_{j=0}^{N} |f_j(z)|^2 \left|\sum_{j=0}^{N} f_j(z)E(a_j + ib_j)\right|^2},$$
$$D_1(z) = \sigma^2 \sum_{j=0}^{N} \overline{f_j(z)} f_j'(z) + E(X_1) \sum_{j=0}^{N} f_j'(z)E(a_j + ib_j),$$
$$D_2(z) = \sigma^2 \sum_{j=0}^{N} \overline{f_j(z)} f_j'(z) + iE(X_2) \sum_{j=0}^{N} f_j'(z)E(a_j + ib_j),$$
$$D_3(z) = 2\sigma^2 \sum_{j=0}^{N} |f_j(z)|^2.$$

The form of M remains unchanged. Further, if $\mu_{a_j} = \mu_{b_j} = \mu$ for $0 \le j \le N$,

$$E(X_1) = \mu \sum_{j=0}^{N} (u_j - v_j), \quad E(X_2) = \mu \sum_{j=0}^{N} (u_j + v_j).$$

Then the formula for $h_{\boldsymbol{K}}$ in Theorem 4.1 now holds with

$$D_0(z) = \sqrt{\left(\sigma^2 \sum_{j=0}^{N} |f_j(z)|^2\right)^2 - 2\mu^2\sigma^2 \sum_{j=0}^{N} |f_j(z)|^2 \left|\sum_{j=0}^{N} f_j(z)\right|^2},$$
$$D_1(z) = \sigma^2 \sum_{j=0}^{N} \overline{f_j(z)} f_j'(z) + \mu \sum_{j=0}^{N} (u_j - v_j) \sum_{j=0}^{N} f_j'(z)E(a_j + ib_j),$$

$$D_2(z) = \sigma^2 \sum_{j=0}^{N} \overline{f_j(z)} f_j'(z) + i\mu \sum_{j=0}^{N} (u_j + v_j) \sum_{j=0}^{N} f_j'(z) E(a_j + ib_j),$$
$$D_3(z) = 2\sigma^2 \sum_{j=0}^{N} |f_j(z)|^2.$$

Third, in Corollary 4.1.1 if the random variables a_j and b_j are i.i.d. with mean 0 and variance 1 for $0 \le j \le N$, then

$$Y_1(z) = Y_3(z) = \sum_{j=0}^{N} |f_j(z)|^2, \quad Y_2(z) = 0,$$

so that

$$D_0(z) = \sum_{j=0}^{N} |f_j(z)|^2.$$

From

$$A_{j,1}(z) - iB_{j,1}(z) = \overline{f_j(z)}, \quad B_{j,2}(z) - iA_{j,2}(z) = \overline{f_j(z)},$$

it follows that

$$D_1(z) = D_2(z) = \sum_{j=0}^{N} \overline{f_j(z)} f_j'(z), \quad D_3(z) = 2 \sum_{j=0}^{N} |f_j'(z)|^2.$$

Then immediate by Corollary 4.1.1 is the following consequence, which recovers Theorem 1 in [11].

Corollary 4.1.2 *In the notation of Theorem 1 in [11], let*

$$B_0(z) = \sum_{j=0}^{N} |f_j(z)|^2, \quad B_1(z) = \sum_{j=0}^{N} \overline{f_j(z)} f_j'(z), \quad B_2(z) = \sum_{j=0}^{N} |f_j'(z)|^2.$$

If $\sigma_{a_j}^2 = \sigma_{b_j}^2 = 1$ *for* $0 \le j \le N$*, then for all integers* $N > 1$ *one has*

$$h_{\boldsymbol{K}} = \frac{1}{B_0} \exp\left(-\frac{K_1^2 + K_2^2}{2B_0}\right) \left\{ B_2 - \frac{|B_0|^2}{B_0} \left(1 - \frac{K_1^2 + K_2^2}{2B_0}\right) \right\}.$$

Fourth, and finally, the following follows from Theorem 4.1.

Corollary 4.1.3 *For any vector* $\boldsymbol{K}$ *restricted to a circle of radius* $K > 0$ *and all integers* $N > 1$*, one has*

$$\begin{aligned} h_{\boldsymbol{K}} = {} & \frac{1}{2\pi D_0} \exp\left(-\frac{(K - E(X_1))^2 Y_3 + (K - E(X_2))^2 Y_1 - 2(K - E(X_1))(K - E(X_2))Y_2}{2D_0^2}\right) \\ & \times \left\{ D_3 - \frac{|D_1|^2}{D_0^2} \left(Y_3 - \frac{((K - E(X_1))Y_3 - (K - E(X_2))Y_2)^2}{D_0^2}\right) \right. \\ & - \frac{|D_2|^2}{D_0^2} \left(Y_1 - \frac{((K - E(X_1))Y_2 - (K - E(X_2))Y_1)^2}{D_0^2}\right) \\ & + \left(\frac{|D_1 + iD_2|^2 - |D_1|^2 - |D_2|^2}{D_0^2}\right) \\ & \times \left(Y_2 - \frac{((K - E(X_1))Y_3 - (K - E(X_2))Y_2)((K - E(X_1))Y_2 - (K - E(X_2))Y_1)}{D_0^2}\right) \\ & - \left(\frac{|M + D_1|^2 - |M|^2 - |D_1|^2}{D_0^2}\right) ((K - E(X_1))Y_3 - (K - E(X_2))Y_2) \\ & \left. + \left(\frac{|M + iD_2|^2 - |M|^2 - |D_2|^2}{D_0^2}\right) ((K - E(X_1))Y_2 - (K - E(X_2))Y_1) \right\}. \end{aligned}$$

Then immediate by Corollary 4.1.3 is the following result.

Corollary 4.1.4 *If* $\boldsymbol{K}$ *is the zero vector, then for all integers* $N > 1$ *one has*

$$h_{\boldsymbol{K}} = \frac{1}{2\pi D_0} \exp\left(-\frac{(E(X_1))^2 Y_3 + (E(X_2))^2 Y_1 - 2E(X_1)E(X_2)Y_2}{2D_0^2}\right)$$

$$
\begin{aligned}
&\times \left\{ D_3 - \frac{|D_1|^2}{D_0^2} \left(Y_3 - \frac{(E(X_1)Y_3 - E(X_2)Y_2)^2}{D_0^2} \right) \right. \\
&- \frac{|D_2|^2}{D_0^2} \left(Y_1 - \frac{(E(X_1)Y_2 - E(X_2)Y_1)^2}{D_0^2} \right) + \left(\frac{|D_1 + iD_2|^2 - |D_1|^2 - |D_2|^2}{D_0^2} \right) \\
&\times \left(Y_2 - \frac{(E(X_1)Y_3 - E(X_2)Y_2)(E(X_1)Y_2 - E(X_2)Y_1)}{D_0^2} \right) \\
&- \left(\frac{|M + D_1|^2 - |M|^2 - |D_1|^2}{D_0^2} \right) (E(X_1)Y_3 - E(X_2)Y_2) \\
&\left. + \left(\frac{|M + iD_2|^2 - |M|^2 - |D_2|^2}{D_0^2} \right) (E(X_1)Y_2 - E(X_2)Y_1) \right\}.
\end{aligned}
$$

CHAPTER 5

THE CROSSINGS OF GAUSSIAN TRIGONOMETRIC POLYNOMIALS

The classical random trigonometric polynomial $\sum_{j=0}^{N} a_j \cos j\theta$ with identically distributed standard Gaussian coefficients $\{a_j\}_{j=0}^{N}$ was studied by Dunnage [16]. He showed the number of real zeros in the interval $(0, 2\pi)$, other than a set of measure 0, is $2n/\sqrt{3}$ plus an error term that is at most $O(n^{11/13}(\log n)^{3/13})$. Farahmand [22, 24, 25] later computed the expected number of real zeros of the equation $\sum_{j=0}^{N} a_j \cos j\theta = K$. He showed that this asymptotic formula remains valid. This result for different assumptions on the distribution of the coefficients was also obtained by Sambandham and Renganathan [47], Farahmand [23] and others. A study involving coefficients with different means and variances was studied by Farahmand and Sambandham [30]. It shows an interesting result for the expected number of level crossings in the interval $(0, 2\pi)$. Based on these works, the aim of the present chapter is to study the complex zeros of a random trigonometric polynomial of a different form.

Let $\{a_j\}_{j=0}^{N}$, $\{b_j\}_{j=0}^{N}$, $\{c_j\}_{j=0}^{N}$, and $\{d_j\}_{j=0}^{N}$ be sequences of real i.i.d. standard Gaussian random variables. Let $\eta_j = a_j + ib_j$ and $\gamma_j = c_j + id_j$ for $0 \le j \le N$. Further, let z be the complex variable $x + iy$. We consider the random trigonometric polynomial

$$D(z) = \sum_{j=0}^{N} (\eta_j \cos jz + \gamma_j \sin jz).$$

It is of interest to study the number of times that D crosses a complex level. If, for each compact subset T of $\mathbb{C}$, $N_{\boldsymbol{K}}^{D}(T)$ denotes the random number of complex zeros, counted with multiplicity, in T of D that cross the complex level $\boldsymbol{K} = K_1 + iK_2$, where K_1 and K_2

are constants independent of z, then, with probability one, the expected density $h_{\boldsymbol{K}}$ of the complex zeros of

$$D(z) = \boldsymbol{K}$$

is given by

$$E(N_{\boldsymbol{K}}^D(T)) = \int_T h_{\boldsymbol{K}}(z)\, dz.$$

The explicit derivation of $h_{\boldsymbol{K}}$ constitutes the primary reason for studying the zeros of $D(z) = \boldsymbol{K}$. The main device for treating $h_{\boldsymbol{K}}$ throughout $\mathbb{C}$ is the Rice formula, which provides a representation for the expected number of zeros of certain random fields. The following theorem is proved in Section 5.1.

THEOREM 5.1 *Provided all the conditions imposed on D in $D(z) = \boldsymbol{K}$ and T are satisfied, then for all integers $N > 1$ one has*

$$h_{\boldsymbol{K}} = \exp\left(-(K_1^2 + K_2^2)\Big/ 2\sum_{j=0}^{N} \cosh 2jy\right)\left(\pi \sum_{j=0}^{N} \cosh 2jy\right)^{-1}$$
$$\times \left(\sum_{j=0}^{N} j^2 \cosh 2jy - \frac{\left(\sum_{j=0}^{N} j \sinh 2jy\right)^2}{\sum_{j=0}^{N} \cosh 2jy}\left(1 - (K_1^2 + K_2^2)\Big/ 2\sum_{j=0}^{N} \cosh 2jy\right)\right).$$

It is of special interest to study the behavior of $h_{\boldsymbol{K}}$ for large values of N. In Section 5.2 we prove the following corollary.

COROLLARY 5.1.1 *One has*

$$\lim_{N\to\infty} \frac{1}{N^2} h_{\boldsymbol{K}} = 0.$$

5.1 The Derivation of the Density Function

If

$$\boldsymbol{X} = (X_1, X_2)'$$

denotes a two-dimensional random field in $\mathbb{C}$, where

$$X_1 = \mathrm{Re}(D(z)) = \sum_{j=0}^{N}((a_j \cos jx + c_j \sin jx)\cosh jy + (b_j \sin jx - d_j \cos jx)\sinh jy),$$

$$X_2 = \mathrm{Im}(D(z)) = \sum_{j=0}^{N}((b_j \cos jx + d_j \sin jx)\cosh jy - (a_j \sin jx - c_j \cos jx)\sinh jy).$$

then the Jacobian matrix of the random transformation $(x, y) \to (X_1, X_2)$ is given by

$$\nabla \boldsymbol{X} = \begin{pmatrix} \dfrac{\partial X_1}{\partial x} & \dfrac{\partial X_2}{\partial x} \\ \dfrac{\partial X_1}{\partial y} & \dfrac{\partial X_2}{\partial y} \end{pmatrix}.$$

Assume that there are no points in T for which both equalities $S(z) = \boldsymbol{K}$ and $\det(\nabla \boldsymbol{X}) = 0$ take place. Since N is fixed, T contains not more than a finite number of zeros of $\boldsymbol{X} = \boldsymbol{K}$, where $\boldsymbol{K} = (K_1, K_2)'$. Then, by [2, Theorem 11.2.3, Corollary 11.2.4, pp. 269-271], the expected density $h_{\boldsymbol{K}}$ can be expressed through the conditioned expected value

$$h_{\boldsymbol{K}}(z) = E(|\det(\nabla \boldsymbol{X})| \mid \boldsymbol{X} = \boldsymbol{K})\, p_{X_1,X_2}(\boldsymbol{K}'),$$

where $p_{X_1,X_2}(K_1, K_2)$ denotes the probability density of the random vector $\boldsymbol{X}$ at the point indicated by $\boldsymbol{K}'$. On noting that

$$\frac{\partial X_1}{\partial x} = \sum_{j=0}^{N} j((c_j \cos jx - a_j \sin jx)\cosh jy + (b_j \cos jx + d_j \sin jx)\sinh jy),$$

$$\frac{\partial X_2}{\partial x} = -\sum_{j=0}^{N} j((b_j \sin jx - d_j \cos jx)\cosh jy + (a_j \cos jx + c_j \sin jx)\sinh jy),$$

$$\frac{\partial X_1}{\partial y} = \sum_{j=0}^{N} j(a_j \cos jx + c_j \sin jx) \sinh jy + (b_j \sin jx - d_j \cos jx) \cosh jy),$$
$$\frac{\partial X_2}{\partial y} = \sum_{j=0}^{N} j((b_j \cos jx + d_j \sin jx) \sinh jy + (c_j \cos jx - a_j \sin jx) \cosh jy),$$

with a little algebra we have

$$\begin{aligned}
\det(\nabla \boldsymbol{X}) = & \left(\sum_{j=0}^{N} j((c_j \cos jx - a_j \sin jx) \cosh y + (b_j \cos jx + d_j \sin jx) \sinh jy \right)^2 \\
& + \left(\sum_{j=0}^{N} j((a_j \cos jx + c_j \sin x) \sinh jy + (b_j \sin jx - d_j \cos jx) \cosh jy) \right)^2 \\
= & \frac{1}{2} \sum_{j=0}^{N} \sum_{k=0}^{N} jk((c_j c_k + d_j d_k)(\cos jx \cos k\overline{z} + \cos j\overline{z} \cos kz) \\
& + (a_j a_k + b_j b_k)(\sin jz \sin k\overline{z} + \cos j\overline{z} \sin kz) \\
& - (d_j b_k + c_j a_k)(\cos jz \sin k\overline{z} + \cos j\overline{z} \sin kz) \\
& - (b_j d_k + a_j c_k)(\sin jz \cos k\overline{z} + \sin j\overline{z} \cos kz) \\
& + i(c_j b_k - d_j a_k)(\cos jz \sin k\overline{z} - \cos j\overline{z} \sin kz) \\
& - i(b_j c_k - a_j d_k)(\sin jz \cos k\overline{z} - \cos j\overline{z} \cos kz) \\
& - i(c_j d_k - d_j c_k)(\cos jz \cos k\overline{z} - \cos j\overline{z} \cos kz) \\
& - i(a_j b_k - b_j a_k)(\sin jz \sin k\overline{z} - \sin j\overline{z} \sin kz)).
\end{aligned}$$

It is clear that $\det(\nabla \boldsymbol{X})$ is always positive.

We now calculate the covariance matrices

$$\mathrm{Cov}(\boldsymbol{a}, \boldsymbol{b} \mid \boldsymbol{Y} = \boldsymbol{K}), \qquad \mathrm{Cov}(\boldsymbol{c}, \boldsymbol{d} \mid \boldsymbol{Y} = \boldsymbol{K}), \qquad \mathrm{Cov}(\boldsymbol{a}, \boldsymbol{c} \mid \boldsymbol{Y} = \boldsymbol{K}),$$
$$\mathrm{Cov}(\boldsymbol{b}, \boldsymbol{d} \mid \boldsymbol{Y} = \boldsymbol{K}), \qquad \mathrm{Cov}(\boldsymbol{c}, \boldsymbol{b} \mid \boldsymbol{Y} = \boldsymbol{K}), \qquad \mathrm{Cov}(\boldsymbol{a}, \boldsymbol{d} \mid \boldsymbol{Y} = \boldsymbol{K}),$$

where

$$\boldsymbol{a} = (a_0, \ldots, a_N)', \qquad \boldsymbol{b} = (b_0, \ldots, b_N)',$$

$$\boldsymbol{c} = (c_0, \ldots, c_N)', \qquad \boldsymbol{d} = (d_0, \ldots, d_N)',$$

and

$$\mathrm{Cov}(\boldsymbol{a}, \boldsymbol{b} \mid \boldsymbol{X} = \boldsymbol{K}) = \begin{pmatrix} \boldsymbol{\Sigma}_{\boldsymbol{aa},\boldsymbol{X}} & \boldsymbol{\Sigma}_{\boldsymbol{ab},\boldsymbol{X}} \\ \boldsymbol{\Sigma}_{\boldsymbol{ba},\boldsymbol{X}} & \boldsymbol{\Sigma}_{\boldsymbol{bb},\boldsymbol{X}} \end{pmatrix}$$

with

$$\boldsymbol{\Sigma}_{\boldsymbol{ab},\boldsymbol{X}} = \boldsymbol{\Sigma}_{\boldsymbol{ab}} - \boldsymbol{\Sigma}_{\boldsymbol{aX}} \boldsymbol{\Sigma}_{\boldsymbol{XX}}^{-1} \boldsymbol{\Sigma}_{\boldsymbol{Xb}}$$

and

$$\boldsymbol{\Sigma}_{\boldsymbol{ab}} = E((\boldsymbol{a} - E(\boldsymbol{a}))(\boldsymbol{b} - E(\boldsymbol{b}))').$$

We first find $\boldsymbol{\Sigma}_{\boldsymbol{XX}}^{-1}$. Since $E(X_1) = E(X_2) = 0$, direct computation leads to

$$\boldsymbol{\Sigma}_{\boldsymbol{XX}} = \begin{pmatrix} E(X_1X_1) & E(X_1X_2) \\ E(X_2X_1) & E(X_2X_2) \end{pmatrix} = \begin{pmatrix} \displaystyle\sum_{j=0}^{N} \cosh 2jy & 0 \\ 0 & \displaystyle\sum_{j=0}^{N} \cosh 2jy \end{pmatrix},$$

from which follows

$$\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}}) = \left(\sum_{j=0}^{N} \cosh 2jy \right)^2.$$

Thus,

$$\boldsymbol{\Sigma}_{\boldsymbol{XX}}^{-1} = \frac{1}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $0 \leq j \leq N$ and $0 \leq k \leq N$

$$\mathbf{\Sigma}_{\boldsymbol{aa}} = E((\boldsymbol{a} - E(\boldsymbol{a}))(\boldsymbol{a} - E(\boldsymbol{a}))') = E(\boldsymbol{aa}') = E(a_j a_k) = (\delta_{jk}),$$

where δ_{jk} denotes the Kronecker delta. Similarly,

$$\mathbf{\Sigma}_{\boldsymbol{bb}} = \mathbf{\Sigma}_{\boldsymbol{cc}} = \mathbf{\Sigma}_{\boldsymbol{dd}} = (\delta_{jk}).$$

Further,

$$\mathbf{\Sigma}_{\boldsymbol{ab}} = \mathbf{\Sigma}_{\boldsymbol{ba}} = 0, \quad \mathbf{\Sigma}_{\boldsymbol{cd}} = \mathbf{\Sigma}_{\boldsymbol{dc}} = 0, \quad \mathbf{\Sigma}_{\boldsymbol{ac}} = \mathbf{\Sigma}_{\boldsymbol{ca}} = 0,$$
$$\mathbf{\Sigma}_{\boldsymbol{bd}} = \mathbf{\Sigma}_{\boldsymbol{db}} = 0, \quad \mathbf{\Sigma}_{\boldsymbol{ad}} = \mathbf{\Sigma}_{\boldsymbol{da}} = 0.$$

Now, for $0 \leq j \leq N$

$$\begin{aligned}
E(a_j X_1) &= \cos jx \cosh jy, & E(a_j X_2) &= -\sin jx \sinh jy, \\
E(b_j X_1) &= \sin jy \sinh jy, & E(b_j X_2) &= \cos jx \cosh jy, \\
E(c_j X_1) &= \sin jx \cosh jy, & E(c_j X_2) &= \cos jx \sinh jy, \\
E(d_j X_1) &= -\cos jx \sinh jy, & E(d_j X_2) &= \sin jx \cosh jy.
\end{aligned}$$

Thus, expanding our definitions, we obtain for $0 \leq j \leq N$

$$\begin{aligned}
\mathbf{\Sigma}_{\boldsymbol{aX}} &= (\cos jx \cosh jy \quad -\sin jx \sinh jy), \\
\mathbf{\Sigma}_{\boldsymbol{bX}} &= (\sin jy \sinh jy \quad \cos jy \cosh jy), \\
\mathbf{\Sigma}_{\boldsymbol{cX}} &= (\sin jx \cosh jy \quad \cos jx \sinh jy), \\
\mathbf{\Sigma}_{\boldsymbol{dX}} &= (-\cos jx \sinh jy \quad \sin jx \cosh jy).
\end{aligned}$$

Simple algebra leads to for $0 \le j \le N$ and $0 \le k \le N$

$$\begin{aligned}
\boldsymbol{\Sigma}_{\boldsymbol{aa},\boldsymbol{X}} = \boldsymbol{\Sigma}_{\boldsymbol{bb},\boldsymbol{X}} &= \delta_{jk} - \frac{\cos jz \cos k\overline{z} + \cos j\overline{z} \cos kz}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}}, \\
\boldsymbol{\Sigma}_{\boldsymbol{cc},\boldsymbol{X}} = \boldsymbol{\Sigma}_{\boldsymbol{dd},\boldsymbol{X}} &= \delta_{jk} - \frac{\sin jz \sin k\overline{z} + \sin j\overline{z} \sin kz}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}}, \\
\boldsymbol{\Sigma}_{\boldsymbol{ba},\boldsymbol{X}} = \boldsymbol{\Sigma}'_{\boldsymbol{ab},\boldsymbol{X}} &= -\frac{i(\cos jz \cos k\overline{z} - \cos j\overline{z} \cos kz)}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}}, \\
\boldsymbol{\Sigma}_{\boldsymbol{dc},\boldsymbol{X}} = \boldsymbol{\Sigma}'_{\boldsymbol{cd},\boldsymbol{X}} &= -\frac{i(\sin jz \sin k\overline{z} - \sin j\overline{z} \sin kz)}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}}, \\
\boldsymbol{\Sigma}_{\boldsymbol{ac},\boldsymbol{X}} = \boldsymbol{\Sigma}_{\boldsymbol{bd},\boldsymbol{X}} &= -\frac{\cos jz \sin k\overline{z} + \cos j\overline{z} \sin kz}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}}, \\
\boldsymbol{\Sigma}_{\boldsymbol{ca},\boldsymbol{X}} = \boldsymbol{\Sigma}_{\boldsymbol{db},\boldsymbol{X}} &= -\frac{\sin jz \cos k\overline{z} + \sin j\overline{z} \cos kz}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}}, \\
\boldsymbol{\Sigma}_{\boldsymbol{cb},\boldsymbol{X}} = \boldsymbol{\Sigma}'_{\boldsymbol{bc},\boldsymbol{X}} &= \frac{i(\sin jz \cos k\overline{z} - \sin j\overline{z} \cos kz)}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}}, \\
\boldsymbol{\Sigma}_{\boldsymbol{da},\boldsymbol{X}} = \boldsymbol{\Sigma}'_{\boldsymbol{ad},\boldsymbol{X}} &= -\frac{i(\sin jz \cos k\overline{z} - \sin j\overline{z} \cos kz)}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}}.
\end{aligned}$$

Then the conditional expectations follow from

$$E(\boldsymbol{a} \mid \boldsymbol{X} = \boldsymbol{K}) = E(\boldsymbol{a}) + \boldsymbol{\Sigma}_{\boldsymbol{aX}} \boldsymbol{\Sigma}_{\boldsymbol{XX}}^{-1} (\boldsymbol{K} - E(\boldsymbol{X})).$$

Thus, for $0 \le j \le N$

$$\begin{aligned}
E(\boldsymbol{a} \mid \boldsymbol{X} = \boldsymbol{K}) &= \frac{-K_1 \cos jx \cosh jy + K_2 \sin jx \sinh jy}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}}, \\
E(\boldsymbol{b} \mid \boldsymbol{X} = \boldsymbol{K}) &= \frac{-K_1 \sin jx \sinh jy - K_2 \cos jx \cosh jy}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}}, \\
E(\boldsymbol{c} \mid \boldsymbol{X} = \boldsymbol{K}) &= \frac{-K_1 \sin jx \cosh jy - K_2 \cos jx \sinh jy}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}}, \\
E(\boldsymbol{d} \mid \boldsymbol{X} = \boldsymbol{K}) &= \frac{K_1 \cos jx \sinh jy - K_2 \sin jx \cosh jy}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}}.
\end{aligned}$$

Next, for $0 \leq j \leq N$ and $0 \leq k \leq N$

$$\begin{aligned}
E(a_j a_k \mid \boldsymbol{X} = \boldsymbol{K}) &= E(a_j \mid \boldsymbol{X} = \boldsymbol{K}) E(a_k \mid \boldsymbol{X} = \boldsymbol{K}) + \mathrm{Cov}(a_j a_k \mid \boldsymbol{X} = \boldsymbol{K}) \\
&= \frac{K_1^2 \cos jx \cosh jy \cos kx \cosh ky + K_2^2 \sin jx \sinh jy \sin kx \sinh ky}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad - \frac{K_1 K_2 (\cos jx \cosh jy \sin kx \sinh ky + \sin jx \sinh jy \cos kx \cosh ky)}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad + \delta_{jk} - \frac{\cos jz \cos k\overline{z} + \cos j\overline{z} \cos kz}{2\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}},
\end{aligned}$$

$$\begin{aligned}
E(b_j b_k \mid \boldsymbol{X} = \boldsymbol{K}) &= E(b_j \mid \boldsymbol{X} = \boldsymbol{K}) E(b_k \mid \boldsymbol{X} = \boldsymbol{K}) + \mathrm{Cov}(b_j b_k \mid \boldsymbol{X} = \boldsymbol{K}) \\
&= \frac{K_1^2 \sin jx \sinh jy \sin kx \sinh ky + K_2^2 \cos jx \cosh jy \cos kx \cosh ky}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad + \frac{K_1 K_2 (\sin jx \sinh jy \cos kx \cosh ky + \cos jx \cosh jy \sin kx \sinh ky)}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad + \delta_{jk} - \frac{\cos jz \cos k\overline{z} + \cos j\overline{z} \cos kz}{2\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}},
\end{aligned}$$

$$\begin{aligned}
E(c_j c_k \mid \boldsymbol{X} = \boldsymbol{K}) &= E(c_j \mid \boldsymbol{X} = \boldsymbol{K}) E(c_k \mid \boldsymbol{X} = \boldsymbol{K}) + \mathrm{Cov}(c_j c_k \mid \boldsymbol{X} = \boldsymbol{K}) \\
&= \frac{K_1^2 \sin jx \cosh jy \sin kx \cosh ky + K_2^2 \cos jx \sinh jy \cos kx \sinh ky}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad + \frac{K_1 K_2 (\sin jx \cosh jy \cos kx \sinh ky + \cos jx \sinh jy \sin kx \cosh ky)}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad + \delta_{jk} - \frac{\sin jz \sin k\overline{z} + \sin j\overline{z} \sin kz}{2\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}},
\end{aligned}$$

$$\begin{aligned}
E(d_j d_k \mid \boldsymbol{X} = \boldsymbol{K}) &= E(d_j \mid \boldsymbol{X} = \boldsymbol{K}) E(d_k \mid \boldsymbol{X} = \boldsymbol{K}) + \mathrm{Cov}(d_j d_k \mid \boldsymbol{X} = \boldsymbol{K}) \\
&= \frac{K_1^2 \cos jx \sinh jy \cos kx \sinh ky + K_2^2 \sin jx \cosh jy \sin kx \cosh ky}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad - \frac{K_1 K_2 (\cos jx \sinh jy \sin kx \cosh ky + \sin jx \cosh jy \cos kx \sinh ky)}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad + \delta_{jk} - \frac{\sin jz \sin k\overline{z} + \sin j\overline{z} \sin kz}{2\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}},
\end{aligned}$$

$$\begin{aligned}
E(a_j b_k \mid \boldsymbol{X}=\boldsymbol{K}) &= E(a_j \mid \boldsymbol{X}=\boldsymbol{K})E(b_k \mid \boldsymbol{X}=\boldsymbol{K}) + \mathrm{Cov}(a_j b_k \mid \boldsymbol{X}=\boldsymbol{K}) \\
&= \frac{K_1^2 \cos jx \cosh jy \sin kx \sinh ky - K_2^2 \sin jx \sinh jy \cos kx \cosh ky}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad + \frac{K_1 K_2(\cos jx \cosh jy \cos kx \cosh ky - \sin jx \sinh jy \sin kx \sinh ky)}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad + \frac{i(\cos jz \cos k\overline{z} - \cos j\overline{z} \cos kz)}{2\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}},
\end{aligned}$$

$$\begin{aligned}
E(b_j a_k \mid \boldsymbol{X}=\boldsymbol{K}) &= E(b_j \mid \boldsymbol{X}=\boldsymbol{K})E(a_k \mid \boldsymbol{X}=\boldsymbol{K}) + \mathrm{Cov}(b_j a_k \mid \boldsymbol{X}=\boldsymbol{K}) \\
&= \frac{K_1^2 \sin jx \sinh jy \cos kx \cosh ky - K_2^2 \cos jx \cosh jy \sin kx \sinh ky}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad + \frac{K_1 K_2(\cos jx \cosh jy \cos kx \cosh ky - \sin jx \sinh jy \sin kx \sinh ky)}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad - \frac{i(\cos jz \cos k\overline{z} - \cos j\overline{z} \cos kz)}{2\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}},
\end{aligned}$$

$$\begin{aligned}
E(c_j d_k \mid \boldsymbol{X}=\boldsymbol{K}) &= E(c_j \mid \boldsymbol{X}=\boldsymbol{K})E(d_k \mid \boldsymbol{X}=\boldsymbol{K}) + \mathrm{Cov}(c_j d_k \mid \boldsymbol{X}=\boldsymbol{K}) \\
&= \frac{-K_1^2 \sin jx \cosh jy \cos kx \sinh ky + K_2^2 \cos jx \sinh jy \sin kx \cosh ky}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad + \frac{K_1 K_2(\sin jx \cosh jy \sin kx \cosh ky - cos jx \sinh jy \cos kx \sinh ky)}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad + \frac{i(\sin jz \sin k\overline{z} - \sin j\overline{z} \sin kz)}{2\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}},
\end{aligned}$$

$$\begin{aligned}
E(d_j c_k \mid \boldsymbol{X}=\boldsymbol{K}) &= E(d_j \mid \boldsymbol{X}=\boldsymbol{K})E(c_k \mid \boldsymbol{X}=\boldsymbol{K}) + \mathrm{Cov}(d_j c_k \mid \boldsymbol{X}=\boldsymbol{K}) \\
&= \frac{-K_1^2 \cos jx \sinh jy \sin kx \cosh ky + K_2^2 \sin jx \cos jy \cos kx \sinh ky}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad - \frac{K_1 K_2(\cos jx \sinh jy \cos kx \sinh ky - \sin jx \cosh ky \sin kx \cosh ky)}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad - \frac{i(\sin jz \sin k\overline{z} - \sin j\overline{z} \sin kz)}{2\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}},
\end{aligned}$$

$$
\begin{aligned}
E(a_j c_k \mid \boldsymbol{X} = \boldsymbol{K}) &= E(a_j \mid \boldsymbol{X} = \boldsymbol{K}) E(c_k \mid \boldsymbol{X} = \boldsymbol{K}) + \operatorname{Cov}(a_j c_k \mid \boldsymbol{X} = \boldsymbol{K}) \\
&= \frac{K_1^2 \cos jx \cosh jy \sin kx \cosh ky - K_2^2 \sin jx \sinh jy \cos kx \sinh ky}{\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})} \\
&\quad - \frac{K_1 K_2 (\cos jx \cosh jy \cos kx \sinh ky - \sin jx \sinh jy \sin kx \cosh ky)}{\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})} \\
&\quad - \frac{\cos jz \sin k\overline{z} + \cos j\overline{z} \sin kz}{2\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})}},
\end{aligned}
$$

$$
\begin{aligned}
E(c_j a_k \mid \boldsymbol{X} = \boldsymbol{K}) &= E(c_j \mid \boldsymbol{X} = \boldsymbol{K}) E(a_k \mid \boldsymbol{X} = \boldsymbol{K}) + \operatorname{Cov}(c_j a_k \mid \boldsymbol{X} = \boldsymbol{K}) \\
&= \frac{K_1^2 \sin jx \sinh jy \cos kx \sinh ky + K_2^2 \cos jx \cosh jy \sin kx \cosh ky}{\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})} \\
&\quad + \frac{K_1 K_2 (\sin jx \sinh jy \sin kx \cosh ky - \cos jx \cosh jy \cos kx \sinh ky)}{\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})} \\
&\quad - \frac{\sin jz \cos k\overline{z} + \sin j\overline{z} \cos kz}{2\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})}},
\end{aligned}
$$

$$
\begin{aligned}
E(b_j d_k \mid \boldsymbol{X} = \boldsymbol{K}) &= E(b_j \mid \boldsymbol{X} = \boldsymbol{K}) E(d_k \mid \boldsymbol{X} = \boldsymbol{K}) + \operatorname{Cov}(b_j d_k \mid \boldsymbol{X} = \boldsymbol{K}) \\
&= \frac{-K_1^2 \sin jx \sinh jy \cos kx \sinh ky + K_2^2 \cos jx \cosh jy \sin kx \cosh ky}{\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})} \\
&\quad + \frac{K_1 K_2 (\sin jx \sinh jy \sin kx \cosh ky - \cos jx \cosh jy \cos kx \sinh ky)}{\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})} \\
&\quad - \frac{\cos jz \sin k\overline{z} + \cos j\overline{z} \sin kz}{2\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})}},
\end{aligned}
$$

$$
\begin{aligned}
E(d_j b_k \mid \boldsymbol{X} = \boldsymbol{K}) &= E(d_j \mid \boldsymbol{X} = \boldsymbol{K}) E(b_k \mid \boldsymbol{X} = \boldsymbol{K}) + \operatorname{Cov}(d_j b_k \mid \boldsymbol{X} = \boldsymbol{K}) \\
&= \frac{-K_1^2 \cos jx \sinh jy \sin kx \sinh ky + K_2^2 \sin jx \cosh jy \cos kx \cosh ky}{\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})} \\
&\quad - \frac{K_1 K_2 (\cos jx \sinh jy \cos kx \cosh ky - sinjx \cosh jy \sin kx \sinh ky)}{\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})} \\
&\quad - \frac{\sin jz \cos k\overline{z} + \sin j\overline{z} \cos kz}{2\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})}},
\end{aligned}
$$

$$
\begin{aligned}
E(b_j c_k \mid \boldsymbol{X} = \boldsymbol{K}) &= E(b_j \mid \boldsymbol{X} = \boldsymbol{K}) E(c_k \mid \boldsymbol{X} = \boldsymbol{K}) + \mathrm{Cov}(b_j c_k \mid \boldsymbol{X} = \boldsymbol{K}) \\
&= \frac{K_1^2 \sin jx \sinh jy \sin kx \cosh ky + K_2^2 \cos jx \cosh jy \cos kx \sinh ky}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad + \frac{K_1 K_2 (\sin jx \sinh jy \cos kx \sinh ky + \cos jx \cosh jy \sin kx \cosh ky)}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad - \frac{i(\cos jz \sin k\overline{z} - \cos j\overline{z} \sin kz)}{2\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}},
\end{aligned}
$$

$$
\begin{aligned}
E(c_j b_k \mid \boldsymbol{X} = \boldsymbol{K}) &= E(c_j \mid \boldsymbol{X} = \boldsymbol{K}) E(b_k \mid \boldsymbol{X} = \boldsymbol{K}) + \mathrm{Cov}(c_j b_k \mid \boldsymbol{X} = \boldsymbol{K}) \\
&= \frac{K_1^2 \sin jx \cosh jy \sin kx \sinh ky + K_2^2 \cos jx \sinh jy \cos kx \cosh ky}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad + \frac{K_1 K_2 (\sin jx \cosh jy \cos kx \cosh ky + \cos jx \sin jy \sin kx \sinh ky)}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad + \frac{i(\sin jz \cos k\overline{z} - \sin j\overline{z} \cos kz)}{2\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}},
\end{aligned}
$$

$$
\begin{aligned}
E(a_j d_k \mid \boldsymbol{X} = \boldsymbol{K}) &= E(a_j \mid \boldsymbol{X} = \boldsymbol{K}) E(d_k \mid \boldsymbol{X} = \boldsymbol{K}) + \mathrm{Cov}(a_j d_k \mid \boldsymbol{X} = \boldsymbol{K}) \\
&= \frac{-K_1^2 \cos jx \cosh jy \cos kx \sinh ky - K_2^2 \sin jx \sinh jy \sin kx \ \ cosh ky}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad + \frac{K_1 K_2 (\cos jx \cosh jy \sin kx \cosh ky + \sin jx \sinh jy \cos kx \sinh ky)}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad + \frac{i(\cos jz \sin k\overline{z} - \cos j\overline{z} \sin kz)}{2\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}},
\end{aligned}
$$

$$
\begin{aligned}
E(d_j a_k \mid \boldsymbol{X} = \boldsymbol{K}) &= E(d_j \mid \boldsymbol{X} = \boldsymbol{K}) E(a_k \mid \boldsymbol{X} = \boldsymbol{K}) + \mathrm{Cov}(d_j a_k \mid \boldsymbol{X} = \boldsymbol{K}) \\
&= \frac{-K_1^2 \cos jx \sinh jy \cos kx \cosh ky - K_2^2 \sin jx \cosh ju \sin kx \sinh ky}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad + \frac{K_1 K_2 (\cos jx \sinh jy \sin kx \sinh ky + \sin jx \cosh jy \cos kx \cosh ky)}{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} \\
&\quad - \frac{i(\sin jz \cos k\overline{z} - \sin j\overline{z} \cos kz)}{2\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}}.
\end{aligned}
$$

We combine these in appropriate sums in order to compute the value of $E(\det(\nabla \boldsymbol{X}) \mid \boldsymbol{X} = \boldsymbol{K})$. Thus, for $0 \leq j \leq N$ and $0 \leq k \leq N$

$$
\begin{aligned}
E(a_j a_k + b_j b_k \mid \boldsymbol{X} = \boldsymbol{K}) = &\frac{(K_1^2 + K_2^2)(\cos jz \cos k\overline{z} + \cos jz \cos kz)}{2\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})} \\
&- \frac{\cos jz \cos k\overline{z} + \cos j\overline{z} \cos kz}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})}} + 2\delta_{jk},
\end{aligned}
$$

$$
\begin{aligned}
E(c_j c_k + d_j d_k \mid \boldsymbol{X} = \boldsymbol{K}) = &\frac{(K_1^2 + K_2^2)(\sin jz \sin k\overline{z} + \sin j\overline{z} \sin kz)}{2\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})} \\
&- \frac{\sin jz \sin k\overline{z} + \sin j\overline{z} \sin kz}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})}} + 2\delta_{jk},
\end{aligned}
$$

$$
\begin{aligned}
E(d_j b_k + c_j a_k \mid \boldsymbol{X} = \boldsymbol{K}) = &\frac{(K_1^2 + K_2^2)(\sin jz \cos k\overline{z} + \sin j\overline{z} \cos kz)}{2\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})} \\
&- \frac{\sin jz \cos k\overline{z} + \sin j\overline{z} \cos kz}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})}},
\end{aligned}
$$

$$
\begin{aligned}
E(b_j d_k + a_j c_k \mid \boldsymbol{X} = \boldsymbol{K}) = &\frac{(K_1^2 + K_2^2)(\cos jz \sin k\overline{z} + \cos j\overline{z} \sin kz)}{2\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})} \\
&- \frac{\cos jz \sin k\overline{z} + \cos j\overline{z} \sin kz}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})}},
\end{aligned}
$$

$$
\begin{aligned}
E(c_j b_k - d_j a_k \mid \boldsymbol{X} = \boldsymbol{K}) = &-\frac{i(K_1^2 + K_2^2)(\sin jz \cos k\overline{z} - \sin j\overline{z} \cos kz)}{2\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})} \\
&+ \frac{i(\sin jz \cos k\overline{z} - \sin j\overline{z} \cos kz)}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})}},
\end{aligned}
$$

$$
\begin{aligned}
E(b_j c_k - a_j d_k \mid \boldsymbol{X} = \boldsymbol{K}) = &\frac{i(K_1^2 + K_2^2)(\cos jz \sin k\overline{z} - \cos j\overline{z} \sin kz)}{2\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})} \\
&- \frac{i(\cos jz \sin k\overline{z} - \cos j\overline{z} \sin kz)}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}})}},
\end{aligned}
$$

$$E(c_j d_k - d_j c_k \mid \boldsymbol{X} = \boldsymbol{K}) = -\frac{i(K_1^2 + K_2^2)(\sin jz \sin k\overline{z} - \sin j\overline{z} \sin kz)}{2\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} + \frac{i(\sin jz \sin k\overline{z} - \sin j\overline{z} \sin kz)}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}},$$

$$E(a_j b_k - b_j a_k \mid \boldsymbol{X} = \boldsymbol{K}) = -\frac{i(K_1^2 + K_2^2)(\cos jz \sin k\overline{z} - \cos j\overline{z} \sin kz)}{2\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})} + \frac{i(\cos jz \cos k\overline{z} - \cos j\overline{z} \cos kz)}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}},$$

After all the necessary simplifications, we have

$$\begin{aligned}
&E(\det(\nabla \boldsymbol{X}) \mid \boldsymbol{X} = \boldsymbol{K}) \\
&= \sum_{j=0}^{N}\sum_{k=0}^{N} jk \Bigg\{ \delta_{jk}(\cos jz \cos k\overline{z} + \cos j\overline{z} \cos kz) \\
&\quad + \delta_{jk}(\sin jz \sin k\overline{z} + \sin j\overline{z} \sin kz) - \left(\frac{1}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}} - \frac{K_2^2 + K_2^2}{2\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}\right) \\
&\quad \times \Bigg((\cos jz \cos k\overline{z} + \cos j\overline{z} \cos kz)(\sin jz \sin k\overline{z} + \sin j\overline{z} \sin kz) \\
&\qquad - (\cos jz \sin k\overline{z} + \cos j\overline{z} \sin kz)(\sin jz \cos k\overline{z} + \sin j\overline{z} \cos kz) \\
&\qquad + (\cos jz \sin k\overline{z} - \cos j\overline{z} \sin kz)(\sin jz \cos k\overline{z} - \sin j\overline{z} \cos kz) \\
&\qquad - (\cos jz \cos k\overline{z} - \cos j\overline{z} \cos kz)(\sin jz \sin k\overline{z} - \sin j\overline{z} \sin kz) \Bigg) \Bigg\} \\
&= 2\sum_{j=0}^{N} j^2 \cosh^2 jy - \left(\frac{1}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}} - \frac{K_2^2 + K_2^2}{2\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}\right) \\
&\qquad \times \sum_{j=0}^{N}\sum_{k=0}^{N} 2jk \sinh 2jy \sinh 2ky
\end{aligned}$$

$$= 2\sum_{j=0}^{N} j^2 \cosh^2 jy - \frac{2\left(\sum_{j=0}^{N} \sinh 2jy\right)^2}{\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}} \left(1 - \frac{K_1 + K_2^2}{2\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}}\right).$$

Since X_1 and X_2 are random variables distributed according to the normal law, their joint density is

$$\begin{aligned} p_{X_1X_2}(K_1, K_2) &= \frac{1}{2\pi\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}} \exp\left(-\frac{1}{2}(\boldsymbol{K} - E(\boldsymbol{X}))'\boldsymbol{\Sigma}_{\boldsymbol{XX}}^{-1}(\boldsymbol{K} - E(\boldsymbol{X}))\right) \\ &= \frac{1}{2\pi\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}} \exp\left(-\frac{K_1^2 + K_2^2}{2\sqrt{\det(\boldsymbol{\Sigma}_{\boldsymbol{XX}})}}\right). \end{aligned}$$

Combining this with the above expression for $E(\det(\nabla \boldsymbol{X}) \mid \boldsymbol{X} = \boldsymbol{K})$, the required result is obtained.

5.2 The Asymptotic Behavior

In this section we prove the corollary. For brevity's sake, let us write

$$h_{\boldsymbol{K}} = \frac{1}{\pi B_0} \exp\left(-\frac{K_1^2 + K_2^2}{2B_0}\right)\left(B_2 - \frac{B_1^2}{B_0}\left(1 - \frac{K_1^2 + K_2^2}{2B_0}\right)\right),$$

where

$$B_0 = \sum_{j=0}^{N} \cosh 2jy, \qquad B_1 = \sum_{j=0}^{N} j \sinh 2jy, \qquad B_2 = \sum_{j=0}^{N} j^2 \cosh 2jy.$$

We shall make use of the identity

$$B_0 = \operatorname{csch} y \sinh(N+1)y \cosh Ny.$$

We have

$$B_1 = \frac{1}{2}(N \operatorname{csch} y \sinh(N+1)y \sinh Ny$$
$$+ (N+1)\operatorname{csch}(N+1)y \cosh(N+1)y \cosh Ny - (\coth y)B_0)$$

and

$$B_2 = \frac{1}{4}((N^2 + (N+1)^2 - 1)B_0 - 4(\coth y)B_1$$
$$+ 2N(N+1)\operatorname{csch} y \cosh(N+1)y \sinh Ny).$$

Then

$$\lim_{N\to\infty} B_0 = \infty.$$

Further,

$$\begin{aligned}\lim_{N\to\infty} \frac{B_1}{NB_0} &= \lim_{N\to\infty} \frac{1}{2}\left(\tanh Ny + \frac{N+1}{N}\coth(N+1)y - \frac{1}{N}\coth y\right)\\ &= \begin{cases} 1 & \text{if } y > 0, \\ -1 & \text{if } y < 0, \end{cases}\end{aligned}$$

and

$$\begin{aligned}\lim_{N\to\infty} \frac{B_2}{N^2B_0} &= \lim_{N\to\infty} \frac{1}{4}\left(2 + \frac{2}{N} - \frac{4}{N}(\coth y)\left(\frac{B_1}{NB_0}\right) + \frac{2(N+1)}{N}\right)\\ &= 1.\end{aligned}$$

Altogether,

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N^2}h_{\boldsymbol{K}} &= \lim_{N\to\infty} \frac{1}{\pi}\exp\left(-\frac{K_1^2+K_2^2}{2B_0}\right)\left(\frac{B_2}{N^2B_0} - \left(\frac{B_1}{NB_0}\right)^2\left(1 - \frac{K_1^2+K_2^2}{B_0}\right)\right)\\ &= 0.\end{aligned}$$

CHAPTER 6

CONCLUSION

The aim of this dissertation was to study the distribution of the complex zeros and level crossings of random sums with holomorphic functions that are real-valued on the real line as the basis functions. Our main results were obtained through refinements of existing methods and techniques from random fields first pioneered by Rice [46] and applied by Ibragimov and Zeitouni [33].

In Chapter 2 we computed an explicit formula for the expected density of the complex zeros and level crossings of a family of random sums constructed from sequences of i.i.d. random complex standard Gaussian variables and sequences of given holomorphic functions as basis functions. We applied this result to several practical choices of basis function that include random Weyl polynomials, random root-binomial polynomials, and random truncated Fourier sine and cosine series. We obtained limiting behavior for the density function and plots of the density function and empirical distributions of the zeros with these chosen basis functions. Further, we considered random sums whose basis functions are polynomials orthogonal on the real line and orthogonal on the unit circle.

In Chapter 3 we generalized the main result from Chapter 2 for random variables with zero mean and general variances. We applied this more general density function to random sums constructed from a sequence of successive observations of a Brownian motion and obtained an explicit density function for the complex zeros of random sums constructed in this manner.

In Chapter 4 we considered the distrubtion of the zeros of random sums with coefficients of nonvanishing means and general variances. The main result generalizes results from Chapters 2 and 3.

Finally, in Chapter 5 we applied the Rice formula to a certain random trigonometric polynomial. We obtained an explicit density function for the complex zeros and investigated the asymptotic behavior of the density function.

There are several directions for future work. First, a modification of the method used could be applied to the case of random sums constructed from real standard Gaussian random variables. This would provide an alternate proof of the results in [48, 52], as well as similar results on complex level crossings. Second, a modification of the method used in [36,48,52,54,55], which utilizes Cauchy's argument principle and the Cholesky decomposition of a covariance matrix, to obtain the expected density of the level crossings would provide a second proof of the main result in Chapter 2. Third, a continuation of the work in Chapter 5 would involve the derivation of a central limit theorem.

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