OPTIMIZATION FOR A STURM–LIOUVILLE PROBLEM WITH THE SPECTRAL PARAMETER IN THE BOUNDARY CONDITION

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ABSTRACT

We find an optimal mass of a structure described by a Sturm-Liouville (S-L) problem with a spectral parameter in the boundary conditions. While previous work on the subject focused on a somewhat simplified model, we consider a more general S-L problem. We use the calculus of variations approach to determine a set of critical points of a corresponding mass functional, yet these critical points - which we call *predesigns* - do not necessarily themselves represent meaningful solutions. It is natural to expect a mass to be real and positive. To this end, we additionally introduce a set of solvability conditions on the S-L problem data, confirming that these critical points represent meaningful solutions we refer to as *designs*. We further present the analytic continuation of these predesigns in regards to the spectral parameter as well as a discussion of the stability of these (pre)designs. We present a code that allows us to for the given data of the S-L problem check conditions of solvability, plot the design, and calculate the value of the functional that represents the optimal mass.

DEDICATION

I would like to thank my very patient parents and friends.

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LIST OF ABBREVIATIONS

Def, Definition

Thm, Theorem

Lem, Lemma

Ch, Chapter

Sec, Section

IVP, Initial Value Problem

ODE, Ordinary Differential Equation

PDE, Partial Differential Equation

S-L, Sturm-Liouville

WOLOG, Without Loss of Generality

DNE, Does Not Exist

LIST OF SYMBOLS

$p(x)$, the leading coefficient of the differential equation

$q(x)$, $r(x)$, the coefficients of the differential equation

α, β_j, β_j' $(j = 1, 2)$, the parameters in the boundary conditions of a given S-L problem

$\mathcal{D} = \{r(x), \alpha, \beta_j, \beta_j' \ (j = 1, 2)\}$, data set of a given S-L problem

$\mathcal{D}_q = \{q(x), r(x), \alpha, \beta_j, \beta_j' \ (j = 1, 2)\}$, data set of a given S-L problem with a fixed $q(x)$

$\mathcal{P}$ set of positive functions $p(x)$ on the interval $[0, 1]$

$\overline{\mathcal{P}}$, set of functions $p(x)$ which are not positive along all of $[0, 1]$

$\mathcal{Q}$, set of functions $q(x)$ for which $p(x) \in \mathcal{P}$ for a fixed data set $\mathcal{D}$

$\overline{\mathcal{Q}}$, set of functions $q(x)$ for which $p(x) \in \overline{\mathcal{P}}$ for a fixed data set $\mathcal{D}$

$M[p]$, mass functional defined by $\int_0^1 r(x)p(x)dx$

$F[y, p]$, functional accumulating the mass functional and the S-L problem

δF, first variation of $F[y, p]$

$\Lambda_{1,2,3}$, Lagrange multipliers employed in our analysis of δF

$g(x) = \int_0^x \sqrt{r(s)}ds$

$\phi = \cot(\alpha)$

$\psi = \frac{\beta_1 + \lambda_1 \beta_1'}{\beta_2 + \lambda_1 \beta_2'}$

$\zeta = -\frac{\phi + \psi \cosh(2\sqrt{\lambda_1} g(1)) - \int_0^1 q(s) \cosh(2\sqrt{\lambda_1} g(s)) ds}{\psi \sinh(2\sqrt{\lambda_1} g(1)) - \int_0^1 q(s) \sinh(2\sqrt{\lambda_1} g(s)) ds}$

$z = \frac{1}{2} \coth^{-1}(\zeta)$

CHAPTER 1

Introduction

We consider the following regular Sturm–Liouville (S-L) problem with the spectral parameter in the boundary condition

$$(p(x)y'(x))' - q(x)y(x) + \lambda p(x)r(x)y(x) = 0,\ x \in (0,1), \tag{1.0.1}$$

$$\cos(\alpha)y(0) + \sin(\alpha)p(0)y'(0) = 0, \tag{1.0.2}$$

$$-\beta_1 y(1) + \beta_2 p(1)y'(1) = \lambda[\beta_1' y(1) - \beta_2' p(1)y'(1)], \tag{1.0.3}$$

and introduce the "mass" functional

$$M[p] := \int_0^1 p(x)r(x)dx. \tag{1.0.4}$$

Assumption 1.1. We proceed under the following assumptions on the functions $p(x)$, $r(x)$, $q(x)$ and the constants α, β_j, β_j' $(j = 1, 2)$.

a) $p \in C^1[0,1]$, q, $r \in C[0,1]$;

b) p, $r > 0\ \forall x \in [0,1]$;

c) The parameters β_j, $\beta_j' \in \mathbb{R}$ satisfy

$$\delta := \beta_1'\beta_2 - \beta_1\beta_2' > 0; \tag{1.0.5}$$

d) $\alpha \in [0, \pi)$.

Assumption 1.2. Since the given functions and parameters are real-valued, we may assume that all solutions are real-valued.

Assumption 1.1 flows naturally from the fact that our S-L problem is regular [**2**] and will be of great importance in fully describing our solutions y. However, we note that the S-L problem's general theory has been developed under a more general assumption. In particular, condition a) of Assumption 1.1 - that p is a smooth function - may be weakened.

It is known [**2**] that if the conditions a)-c) of Assumption 1.1 are satisfied, then the S-L problem (1.0.1)-(1.0.3) has a discrete spectrum of eigenvalues $\{\lambda_k\}$ with the only point of accumulation $+\infty$. Let λ_1 be the first positive eigenvalue - which we will refer to as the principal eigenvalue for this text. Our goal is to optimize the functional $M[p]$ (1.0.4) subject to the assumption $\lambda = \lambda_1$.

The S-L problem represents a very important tool in the field of Applied Mathematics and is the subject of many studies. In the early 19th century the French mathematicians Charles-François Sturm and Joseph Liouville independently worked on the problem of heat transfer through a metal bar and developed the fundamentals of what we now call *Sturm-Liouville Theory* for regular S-L problems. The general theory of the S-L problem may be found in the book by A. Zettl [**2**], see also [**1**]. Zettl notes that even though the area of research is nearing its third century of study, it is still a fast-growing and rapidly changing field.

Among the panoply of disciplines of Mathematics and Applied Sciences where S-L theory is employed, we mention Acoustic, Elastic, and Electromagnetic Wave Propagation [**14**], Control Theory [**15**], and Optimal Control [**16**]. The Quantum Revolution of the early 20th century - nearly a hundred years into the study of S-L problems - gave us the well-known example of the one-dimensional, time-dependent Schrödinger equation. The following century has given us an additional wealth of research, and indeed too much to mention here.

We restrict ourselves to the discussion of only two directions of research that are not related directly to our studies. First, while we study the linear S-L problem in our work, the theory of nonlinear S-L problems represents a distinct and fast-developing field of study. We only mention three publications: [**18**], [**19**], [**20**]. Further, we study the regular S-L problem. For the singular S-L problem, the structure of the spectrum changes dramatically. For example, the discreteness of the spectrum is no longer guaranteed. We only mention the recent paper [**21**].

Our study is devoted to the regular S-L problem on a finite interval with the spectral parameter in the boundary conditions. It would be impossible to give a complete review of the results in this fast-developing area, so we give only the briefest survey by mentioning the results by Walter [**12**], Fulton [**11**], and Hinton [**10**].

Walter [**12**] proved that the spectrum of an S-L eigenvalue problem operator T consists of an unbounded sequence of real eigenvalues of finite multiplicity with no point of accumulation in $[-\infty, \infty)$. Furthermore, denoting these eigenvalues by $\lambda_1 \leq \lambda_2 \leq \ldots$ and their corresponding (normed and real-valued) eigenfunctions by $u_1, u_2, \ldots$ we have for every function h in the Hilbert space $H := \mathcal{L}^2([a, b]; \mu)$ the expansion formula

$$h = \sum_{k}^{\infty} u_k \int_a^b h(x) u_k(x) d\mu$$

in the sense of strong convergence in H.

Fulton [**11**] showed that the well-known work of E. C. Titchmarsh on eigenfunction expansions associated with second-order DE was readily transferable to regular S-L problems involving a spectral parameter in the boundary condition.

Hinton [**10**] extends the results of the previous two authors to create a uniform convergence theorem for a broad class of self-adjoint operators from the S-L boundary value problem. Then D. Hinton and M. L. McCarthy [**22**] considered a problem complementary to ours wherein the goal is to optimize an eigenvalue while subjecting the coefficients of the operator to certain constraints. The constraint, in particular, is to keep the total mass fixed and attempt to maximize the first eigenvalue which is the same as maximizing the lowest frequency of vibration.

The applications of the S-L problem with a spectral parameter to classical mechanics are numerous, and we only give a brief overview. In [**23**] the authors mention applications to longitudinal oscillations of a rod and to the motion of a body suspended at the endpoint by an inextensible thread. Among other applications are Hele-Shaw flows which model the flooding oil recovery process [**4**]. See also [**24**], for a discussion of how this type of problem relates to the scattering matrix of a system currently undergoing a scattering process - pertaining to the dispersion of particles or radiation.

All aforementioned S-L problems come from classical mechanics. In Quantum Mechanics, the spectral parameter is often the energy of a particle, hence energy-dependent boundary conditions may appear. We mention [**7**] where the authors discuss a model with the boundary condition that contains a logarithmic derivative of the function depending linearly on the energy. Our boundary condition (1.0.3) has this form if $\beta_1 = \beta_2' = 0$.

In 1967 Turner [**5**] discussed the design of an axially vibrating rod supporting a nonstructural mass. Namely, he determined an optimal cross-sectional mass distribution such that a rod of given principal eigenvalue is designed with minimum mass. Turner's technique may also of course be in a sense reversed to determine the optimal cross-sectional mass distribution such that a rod of given total mass is made with the largest principle eigenvalue. The applications of such techniques are complementary: a mass can be optimized to meet a certain given natural frequency, or a given mass may be optimized to yield greatest resistance to resonance. Taylor [**6**] considered the same problem and proved that Turner's design was indeed optimal. Taylor also articulated the duality of these complementary optimization problems employed in [**5**] in a form that assists in generalizing the method.

Belinskiy-Kotval [**3**] extend some of these results further. They use methods from calculus of variations to optimize a mass functional with a given principal eigenvalue - the natural frequency from earlier problems - as a constraint. They found that this mass functional had two critical points which led to optimal masses, yet they did not consider a few key points. First, they did not consider conditions on the resulting critical functions which would guarantee their positivity. In the examples from [**5**] and [**6**], the authors considered mechanical systems where only a positive mass would be expected or indeed physically possible. Second, the authors of [**3**] made the simplifying assumption that $q \equiv 0$. Third, it would seem that they did not find all possible designs. Our work is a continuation of that problem that improves upon earlier work by the inclusion of several factors. First, we refer to "solvability conditions" which guarantee the positivity of a mass. Second, we do not make any simplifying assumptions on q. Third, we find additional designs dependent on $q \not\equiv 0$.

Our overall scheme is as follows: in Chapter 2 we state the main theorem of our work; in Chapter 3 we use the techniques of calculus of variations to determine what we call our pairs of critical functions y and p; Chapter 4 is then dedicated to finding the constant terms C_j found in our critical pairs; we devote Chapter 5 to the solvability conditions on these functions along with a brief discussion of the instability of these designs; Chapter 6 is devoted to the analytic continuation of these functions in regard to $\lambda_1 \leq 0$; in Chapter 7 we discuss various pathological cases involving special boundary conditions; Chapter 8 is dedicated to a comparison of our results with the results of [**3**]; we then give various numerical examples in Chapter 9 including instances where a predesign may slide into a design with slight modification - again pointing to the instability of these predesigns.

CHAPTER 2

Main Theorem

We proceed according to the philosophy of calculus of variations. Our main functional is the "mass" functional $M[p]$ introduced in (1.0.4), though we introduce another functional in Def 2.1 that incorporates the S-L problem (1.0.1)-(1.0.3) as constraints. For this functional, we find the critical points where the term *critical* is as defined in Def 2.2. We further proceed according to the method of Lagrange multipliers [**8**].

Definition 2.1. We introduce the functional $F[y, p]$ as follows

$$\begin{aligned} F[y,p] :=& M[p] + \int_0^1 \Lambda_1((py')' - qy + \lambda_1 pry)dx \\ &+ \Lambda_2(\cos(\alpha)y(0) + \sin(\alpha)p(0)y'(0)) \\ &+ \Lambda_3([-\beta_1 y(1) + \beta_2 p(1)y'(1)] - \lambda_1[\beta_1' y(1) - \beta_2' p(1)y'(1)]) \end{aligned}$$

with Lagrange multipliers $\Lambda_1(x)$, Λ_2, Λ_3.

Definition 2.2. We call a pair $\{y_j, p_j\}$ critical if $\delta F[y, p] = 0$ where δF refers to the first variation of the functional $F[y, p]$.

Our findings represent a generalization of the result of [**3**], which is based on the engineering papers [**5**] and [**6**]. In [**3**], the authors state that "the calculations for $q \not\equiv 0$ seem to be intractable in the frame of an analytic approach." Here, however, we do solve the problem for $q \not\equiv 0$ analytically. Namely, we prove that there are two pairs of critical points $\{y_1, p_1\}$ and $\{y_2, p_2\}$, find the explicit representation for them, the corresponding "mass" functional $M[p]$, and some conditions of solvability that guarantee the positivity of $M[p]$. The authors of [**3**] do not discuss all the aforementioned issues but we pay much attention to them. According to our Assumption 1.1, the function $p(x) > 0$, $\forall x \in [0, 1]$, though it is not obvious *a priori* that every critical point of the functional satisfies this condition. This issue must be studied in depth. To this end, we consider the parameters α, β_j, β'_j $(j = 1, 2)$ and the function r to be fixed and study the condition of positivity of p by allowing the function q to vary. To this end, to better understand the "shape" of allowable q and p, we introduce several useful definitions.

Definition 2.3. We refer to the set containing the function r and the parameters α, β_j, β'_j $(j = 1, 2)$ as the data set of the problem, $\mathcal{D}$. When a data set is paired with a particular function q, we refer to it as $\mathcal{D}_q$.

It is natural to expect that the condition $p(x) > 0$, $\forall x \in [0, 1]$, on the optimizing functions p is satisfied only for particular data sets $\mathcal{D}_q$.

Definition 2.4. We introduce the complimentary sets of functions

$$\mathcal{P} := \{p | p \in C^1[0, 1],\ p(x) > 0\, \forall x \in [0, 1]\},$$

$$\overline{\mathcal{P}} := \{p | p \in C^1[0, 1],\ p(x) \not> 0\, \forall x \in [0, 1]\}.$$

That is to say, sets of p that meet our criteria for solvability. We further introduce the complimentary sets for a fixed data set $\mathcal{D}$

$$\mathcal{Q} := \{q | q \in C[0, 1],\ p \in \mathcal{P}\},$$

$$\overline{\mathcal{Q}} := \{q | q \in C[0, 1],\ p \in \overline{\mathcal{P}}\}.$$

We will show that the structures of the sets $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ are rather complex and these sets are nonempty for each data set $\mathcal{D}_q$. This matter may not be studied completely through analytical means. That is why we complete our study by developing a computer algorithm that allows, for the given data set $\mathcal{D}_q$ to identify the critical functions p or check that they do (do not) exist in the desired class as well as several numerical examples in Ch 9. Hence, our goal is two-fold: determine critical pairs $\{y_j, p_j\}$ such that $\delta F = 0$ and then formulate conditions on those critical pairs such that p_j may guarantee the positivity of the mass functional $M[p_j]$. We call p_j without such a guarantee of positivity a *predesign*.

Definition 2.5. A function P_k is a predesign if it is a member of a critical pair $\{y_j, p_j\}$, but does not necessarily meet the condition that $P_k \in \mathcal{P}$ so that we may guarantee $M[P_k] > 0$. We will formulate what we refer to as solvability conditions on a given data set to ensure that a predesign is in fact a design.

Our critical points $\{y_j, p_j\}$ without the application of any solvability conditions are given by the following theorem.

Theorem 2.6. Consider the S-L problem (1.1)-(1.3). Let Assumption 1.1 be satisfied and $\lambda_1 > 0$ be the principal eigenvalue. Then two critical points of the functional $F[y,p]$ are given by

$$\begin{cases} y_1 &= \dfrac{1}{\sqrt{\lambda_1 k}} \sinh(\sqrt{\lambda_1} g(x) + C_1), \\ p_1 &= \dfrac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1} g(s) + 2C_1) ds}{2\sqrt{\lambda_1 r(x)} \cosh^2(\sqrt{\lambda_1} g(x) + C_1)} + C_3 \dfrac{\sqrt{r(0)} \cosh^2(C_1)}{\sqrt{r(x)} \cosh^2(\sqrt{\lambda_1} g(x) + C_1)}. \end{cases} \tag{2.0.1}$$

$$\begin{cases} y_2 &= \dfrac{1}{\sqrt{\lambda_1 k}} \cosh(\sqrt{\lambda_1} g(x) + C_2), \\ p_2 &= \dfrac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1} g(s) + 2C_2) ds}{2\sqrt{\lambda_1 r(x)} \sinh^2(\sqrt{\lambda_1} g(x) + C_2)} + C_4 \dfrac{\sqrt{r(0)} \sinh^2(C_2)}{\sqrt{r(x)} \sinh^2(\sqrt{\lambda_1} g(x) + C_2)}. \end{cases} \tag{2.0.2}$$

where C_j, $j \in \{1,2,3,4\}$ are constants determined in Sec 4.1 and Sec 4.2 and $g(x)$ is defined as follows

$$g(x) = \int_0^x \sqrt{r(s)}ds. \tag{2.0.3}$$

LEMMA 2.7. For every design above, and for a fixed data set $\mathcal{D}$, there exist non-empty sets of q such that $p \in \mathcal{P}$ and of q such that $p \in \overline{\mathcal{P}}$.

The proof of Theorem 2.6 takes up the majority of Ch 3 and Ch 4 while the proof of Lemma 2.7 is Ch 5. While the above theorem assumes that the principal eigenvalue λ_1 is positive, we find that an analytic continuation of λ_1 to $\lambda_1 \leq 0$ may also produce valid critical points with accompanying solvability conditions on those predesigns.

THEOREM 2.8. We break down the results of our analytic continuation by sign of λ_1 and region of α. We find for $\lambda_1 = 0$, at $\alpha = 0$

$$\begin{cases} y = g(x), \\ p = \dfrac{1}{\sqrt{r(x)}}[\psi g(1) - \displaystyle\int_x^1 q(s)g(s)ds]. \end{cases}$$

with no other critical points defined at other values for α. For almost all $\lambda_1 < 0$ we find at $\alpha = 0$

$$\begin{aligned} y =& \frac{1}{\sqrt{|\lambda_1|}} \sin(\sqrt{|\lambda_1|}g(x)), \\ p =& \frac{1}{2\sqrt{|\lambda_1|r(x)} \cos^2(\sqrt{|\lambda_1|}g(x))}[\psi \sin(2\sqrt{|\lambda_1|}g(1)) - \int_x^1 q(s) \sin(2\sqrt{|\lambda_1|}g(s))ds]; \end{aligned}$$

at $\alpha = \dfrac{\pi}{2}$

$$\begin{aligned} y =& \frac{1}{\sqrt{|\lambda_1|}} \cos(\sqrt{|\lambda_1|}g(x)), \\ p =& -\frac{\int_0^x q(s) \sin(2\sqrt{|\lambda_1|}g(s))ds}{2\sqrt{|\lambda_1|r(x)} \sin^2(\sqrt{|\lambda_1|}g(x))}; \end{aligned}$$

and at $\alpha \notin \{0, \frac{\pi}{2}\}$

$$\begin{cases} y_1 = \frac{1}{\sqrt{|\lambda_1|}} \sin(\sqrt{|\lambda_1|}g(x) + \frac{1}{2}\cot^{-1}(\zeta')), \\ p_1 = \dfrac{\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s) + \cot^{-1}(\zeta'))ds}{2\sqrt{|\lambda_1|r(x)}\cos^2(\sqrt{|\lambda_1|}g(x) + \frac{1}{2}\cot^{-1}(\zeta'))} - \dfrac{\phi\sin(\cot^{-1}(\zeta'))}{2\sqrt{|\lambda_1|r(x)}\cos^2(\sqrt{|\lambda_1|}g(x) + \frac{1}{2}\cot^{-1}(\zeta'))}. \end{cases}$$

$$\begin{cases} y_2 = \dfrac{1}{\sqrt{|\lambda_1|}} \cos(\sqrt{|\lambda_1|}g(x) + \dfrac{1}{2}\cot^{-1}(\zeta')), \\ p_2 = -\dfrac{\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s) + \cot^{-1}(\zeta'))ds}{2\sqrt{|\lambda_1|r(x)}\sin^2(\sqrt{|\lambda_1|}g(x) + \frac{1}{2}\cot^{-1}(\zeta'))} + \dfrac{\phi\sin(\cot^{-1}(\zeta'))}{2\sqrt{|\lambda_1|r(x)}\sin^2(\sqrt{|\lambda_1|}g(x) + \frac{1}{2}\cot^{-1}(\zeta'))}. \end{cases}$$

Remark 2.9. For our analytic continuation of the functions $\{y, p\}$ in regards to $\lambda_1 < 0$, we must formulate solvability conditions on λ_1 which allow for the critical functions to be defined for a given data set. For example, at $\alpha = 0$ we may require that $\sqrt{|\lambda_1|} < \dfrac{\pi}{2g(1)}$.

Lemma 2.10. For every design above, and for a fixed data set $\mathcal{D}$, there exist non-empty sets of q such that $p \in \mathcal{P}$ and of q such that $p \in \overline{\mathcal{P}}$.

We devote Ch 6 to proof of Theorem 2.8 and Lemma 2.9, as well as a full discussion of Remark 2.9.

Remark 2.11. We discuss the special case $q \equiv 0$ in Ch 8.

CHAPTER 3

Determination of Critical Functions y and Corresponding Designs p

3.1. Determination of y_1, y_2.

Building on the work of [**5**], [**3**] (see also [**6**]) and using conditions (1.0.1)-(1.0.3) upon taking the first variation δ of $F[y,p]$ we have that

$$\delta F = \int_0^1 r\delta p dx + \int_0^1 \Lambda_1[(y'\delta p)' + (p\delta y')' - q\delta y + \lambda_1 r(y\delta p + p\delta y)]dx \tag{3.1.1}$$
$$+ \Lambda_2(\cos(\alpha)\delta y(0) + \sin(\alpha)(y'(0)\delta p(0) + p(0)\delta y'(0)))$$
$$+ \Lambda_3([-\beta_1\delta y(1) + \beta_2(y'(1)\delta p(1) + p(1)\delta y'(1))]$$
$$- \lambda_1[\beta_1'\delta y(1) - \beta_2'(y'(1)\delta p(1) + p(1)\delta y'(1))])$$

where $\Lambda_{1,2,3}$ are Lagrange multipliers. We perform integration by parts on the second integral in (3.1.1),

$$\int_0^1 \Lambda_1[(y'\delta p)' + (p\delta y')']dx = \Lambda_1[y'\delta p + p\delta y']|_0^1 - \int_0^1 \Lambda_1'[y'\delta p + p\delta y']dx. \tag{3.1.2}$$

Upon further breaking down the integral in the right side of the last equality we find

$$\begin{aligned}\int_0^1 \Lambda_1'[y'\delta p + p\delta y']dx &= \int_0^1 [(\Lambda_1'p)'\delta y + \Lambda_1'(p\delta y')] - [(\Lambda_1'p)'\delta y + \Lambda_1'(y'\delta p)]dx \\ &= \int_0^1 (\Lambda_1'p\delta y)'dx - \int_0^1 [(\Lambda_1'p)'\delta y - \Lambda_1'(y'\delta p)]dx \\ &= \Lambda_1'p\delta y|_0^1 - \int_0^1 [(\Lambda_1'p)'\delta y - \Lambda_1'(y'\delta p)]dx \\ \implies (3.1.2) &= \Lambda_1[y'\delta p + p\delta y']|_0^1 - \Lambda_1'p\delta y|_0^1 + \int_0^1 [(\Lambda_1'p)'\delta y - \Lambda_1'(y'\delta p)]dx.\end{aligned}$$

We may then rewrite δF (3.1.1) as

$$\begin{aligned}\delta F = &\int_0^1 r\delta p dx + \int_0^1 \Lambda_1[-q\delta y + \lambda_1 r(y\delta p + p\delta y)]dx + \int_0^1 [(\Lambda_1' p)'\delta y - \Lambda_1'(y'\delta p)]dx \\ &+ (\Lambda_1 y'\delta p)|_0^1 + (\Lambda_1 p\delta y')|_0^1 - (\Lambda_1' p\delta y)|_0^1 \\ &+ \Lambda_2(\cos(\alpha)\delta y(0) + \sin(\alpha)(y'(0)\delta p(0) + p(0)\delta y'(0))) \\ &+ \Lambda_3([-\beta_1\delta y(1) + \beta_2(y'(0)\delta p(1) + p(1)\delta y'(1))] \\ &- \lambda_1[\beta_1'\delta y(1) - \beta_2'(y'(1)\delta p(1) + p(1)\delta y'(1))]).\end{aligned}$$

Upon rearranging the terms of δF above to group δp and δy we have

$$\begin{aligned}\delta F = &\int_0^1 \delta p(r + \Lambda_1\lambda_1 ry - \Lambda_1' y')dx + \int_0^1 \delta y(-\Lambda_1 q + \Lambda_1\lambda_1 rp + (\Lambda_1' p)')dx \\ &+ (\Lambda_1 y'\delta p)|_0^1 + (\Lambda_1 p\delta y')|_0^1 - (\Lambda_1' p\delta y)|_0^1 \\ &+ \Lambda_2(\cos(\alpha)\delta y(0) + \sin(\alpha)(y'(0)\delta p(0) + p(0)\delta y'(0))) \\ &+ \Lambda_3([-\beta_1\delta y(1) + \beta_2(y'(0)\delta p(1) + p(1)\delta y'(1))] \\ &- \lambda_1[\beta_1'\delta y(1) - \beta_2'(y'(1)\delta p(1) + p(1)\delta y'(1))]).\end{aligned} \tag{3.1.3}$$

We call the pair $\{y, p\}$ critical if $\delta F = 0$. To that end, we employ the Fundamental Lemma of the calculus of variations to find

$$\delta p : r + \Lambda_1\lambda_1 ry - \Lambda_1' y' = 0, \tag{3.1.4}$$

$$\delta y : -\Lambda_1 q + \Lambda_1\lambda_1 rp + (\Lambda_1' p)' = 0. \tag{3.1.5}$$

Isolating incidents of independent variations $\delta y(k)$, $\delta y'(k)$, and $\delta p(k)$ ($k \in \{0, 1\}$) in (3.1.3) gives

$$\begin{cases} \delta y(0): & \Lambda_2\cos\alpha - \Lambda_1'(0)p(0) = 0, \\ \delta y'(0): & p(0)(\Lambda_2\sin\alpha + \Lambda_1(0)) = 0, \\ \delta p(0): & y'(0)(\Lambda_2\sin\alpha + \Lambda_1(0)) = 0. \end{cases} \tag{3.1.6}$$

$$
\begin{cases}
\delta y(1): & \Lambda_1' p(1) - \Lambda_3(\beta_1 + \lambda_1 \beta_1') = 0, \\
\delta y'(1): & \Lambda_1 p(1) - \Lambda_3 p(1)(\beta_2 + \lambda_1 \beta_2') = 0, \\
\delta p(1): & \Lambda_1(1) y'(1) - \Lambda_3 y'(1)(\beta_2 + \lambda_1 \beta_2') = 0.
\end{cases} \tag{3.1.7}
$$

From (3.1.6) we derive

$$\Lambda_1'(0)p(0)/\cos\alpha = \Lambda_2 = -\Lambda_1(0)/\sin\alpha.$$

Similarly from (3.1.7) we have

$$\Lambda_1'(1)p(1)/(\beta_1 + \lambda_1\beta_1') = \Lambda_3 = \Lambda_1(1)/(\beta_2 + \lambda_1\beta_2').$$

Thus we have conditions independent of Λ_2 and Λ_3

$$\Lambda_1'(0)p(0)\sin\alpha + \Lambda_1(0)\cos\alpha = 0, \tag{3.1.8}$$

$$-\Lambda_1'(1)p(1)\beta_2 + \Lambda_1(1)\beta_1 = \lambda_1[\beta_2'\Lambda_1'(1)p(1) - \Lambda_1(1)\beta_1']. \tag{3.1.9}$$

Note that the boundary value problem from (3.1.4), (3.1.8), and (3.1.9) matches the boundary value problem from (1.0.1)-(1.0.3). The eigenspace for this type of problem is well known to be 1-dimensional with a principal eigenvalue which has multiplicity 1. Indeed, the boundary condition at the left endpoint (3.1.8) ensures that the eigenspace is one-dimensional. Hence, $\Lambda_1 = ky$ (as in [**3**]) where k is some nontrivial constant. We may assume without loss of generality (WOLOG) that k is positive. Then (3.1.4) becomes

$$(y')^2 - \lambda_1 r y^2 = \pm\, r/k. \tag{3.1.10}$$

To find general solutions, we solve the homogeneous equation

$$(y')^2 - \lambda_1 r y^2 = (y' - \sqrt{\lambda_1 r}y)(y' + \sqrt{\lambda_1 r}y) = 0.$$

Let g be as defined in (2.0.3). Then

$$y' - \sqrt{\lambda_1 r}y = 0 \implies y = Ce^{\sqrt{\lambda_1}g}, \tag{3.1.11a}$$

$$y' + \sqrt{\lambda_1 r}y = 0 \implies y = Ce^{-\sqrt{\lambda_1}g}. \tag{3.1.11b}$$

Assume for the method of undetermined coefficients that $y = A(e^{\sqrt{\lambda_1}g(x)} - e^{-\sqrt{\lambda_1}g(x)})$ is a solution to the particular problem (3.1.10) for some constant A. Since the eigenfunction y is defined up to a factor, we may assume that $A > 0$. Then equation (3.1.10) implies

$$\begin{aligned}\frac{r}{k} =& A^2((e^{\sqrt{\lambda_1}g(x)} - e^{-\sqrt{\lambda_1}g(x)})')^2 - \lambda_1 r(x) A^2 (e^{\sqrt{\lambda_1}g(x)} - e^{-\sqrt{\lambda_1}g(x)})^2 \\ =& A^2\lambda_1 r(x)[e^{2\sqrt{\lambda_1}g(x)} + 2 + e^{-2\sqrt{\lambda_1}g(x)}] - \lambda_1 r(x) A^2 [e^{2\sqrt{\lambda_1}g(x)} - 2 + e^{-2\sqrt{\lambda_1}g(x)}] \\ =& A^2 4\lambda_1 r(x) = \frac{r(x)}{k} \implies A = \frac{1}{2\sqrt{\lambda_1 k}}.\end{aligned}$$

Thus we have first critical y,

$$y_1(x) = \frac{1}{\sqrt{\lambda_1 k}} \sinh(\sqrt{\lambda_1}g(x) + C_1). \tag{3.1.12}$$

Similarly, assume for the method of undetermined coefficients that $A(e^{\sqrt{\lambda_1}g(x)} + e^{-\sqrt{\lambda_1}g(x)})$ is a solution to (3.1.10) for some constant A. Then

$$\begin{aligned}\frac{r}{k} =& A^2((e^{\sqrt{\lambda_1}g(x)} + e^{-\sqrt{\lambda_1}g(x)})')^2 - \lambda_1 r(x) A^2 (e^{\sqrt{\lambda_1}g(x)} - e^{-\sqrt{\lambda_1}g(x)})^2 \\ =& A^2\lambda_1 r(x)[e^{2\sqrt{\lambda_1}g(x)} - 2 + e^{-2\sqrt{\lambda_1}g(x)}] - \lambda_1 r(x) A^2 [e^{2\sqrt{\lambda_1}g(x)} + 2 + e^{-2\sqrt{\lambda_1}g(x)}] \\ =& - A^2 4\lambda_1 r(x) = -\frac{r(x)}{k} \implies A = \frac{1}{2\sqrt{\lambda_1 k}}.\end{aligned}$$

Thus we have second critical y,

$$y_2(x) = \frac{1}{\sqrt{\lambda_1 k}} \cosh(\sqrt{\lambda_1}g(x) + C_2). \tag{3.1.13}$$

A, and therefore y, is defined up to a sign; but we assume WOLOG that these objects are positive because we are interested primarily in p that is not affected by this sign.

3.2. Uniqueness of y_1, y_2.

We have found two solutions of (3.1.10). However, it is possible that these y_1, y_2 are not the only solutions. Certainly, (3.1.10) may be solved explicitly, but we employ the Picard Existence Thm instead [**9**].

To proceed we first make use of the following substitution

$$\begin{cases} t = \sqrt{\lambda_1} g(x), \\ dx = \sqrt{\lambda_1 r(x)} dt, \\ Y(t) := \sqrt{\lambda_1 k} y(x). \end{cases}$$

After employing this substitution, we transform (3.1.10) to

$$Y'^2 - Y^2 = 1 \implies Y' = \pm\sqrt{1+Y^2}. \tag{3.2.1}$$

We examine only the case with plus. The proof for minus is similar. Then we consider the points $(x, Y(x))$ and $(x, Y(x)+\epsilon)$ for some fixed $\epsilon \neq 0$ and any $x \in [0,1]$. We see that the IVP (3.2.1) and therefore (3.1.10) meets the Lipschitz continuity condition. To prove this we note

$$|Y| < \sqrt{1+Y^2},$$

and

$$|Y| + \epsilon < \sqrt{1+(Y+\epsilon)^2}.$$

Then we have that

$$|2Y+\epsilon| \leq |Y| + |Y+\epsilon| \leq \sqrt{1+Y^2} + \sqrt{1+(Y+\epsilon)^2},$$

which implies that

$$\begin{aligned} |2Y\epsilon + \epsilon^2| \leq& |\sqrt{1+Y^2} + \sqrt{1+(Y+\epsilon)^2}| \cdot |\epsilon| \\ \iff |\sqrt{1+Y^2} + \sqrt{1+(Y+\epsilon)^2}| \cdot |\sqrt{1+Y^2} - &\sqrt{1+(Y+\epsilon)^2}| \leq |\sqrt{1+Y^2} + \sqrt{1+(Y+\epsilon)^2}| \cdot |\epsilon| \\ \iff |\sqrt{1+Y^2} - \sqrt{1+(Y+\epsilon)^2}| \leq& |\epsilon|. \end{aligned}$$

This inequality shows that the Lipschitz continuity condition is satisfied. Therefore (3.1.10) has unique solutions. We have therefore determined our only critical functions y_1, y_2 for use in determining critical functions p_1, p_2, respectively.

3.3. Determination of p_1, p_2.

We return to our S-L problem with boundary conditions given by (1.0.1)-(1.0.3) to solve for corresponding p. Upon rearranging (1.0.1) accordingly, we find that in the situation with unknown p and all other known parameters we have first-order nonhomogeneous ODE given by

$$p'(x) + \frac{y''(x) + \lambda_1 r(x) y(x)}{y'(x)} p(x) = q(x) \frac{y(x)}{y'(x)}. \tag{3.3.1}$$

In this section, we make use of integrating factors s_1, s_2 which are defined as

$$s_1(x) = C_3 \frac{\sqrt{r(0)} \cosh^2(C_1)}{\sqrt{r(x)} \cosh^2(\sqrt{\lambda_1} g(x) + C_1)}, \tag{3.3.2}$$

$$s_2(x) = C_4 \frac{\sqrt{r(0)} \sinh^2(C_2)}{\sqrt{r(x)} \sinh^2(\sqrt{\lambda_1} g(x) + C_2)}. \tag{3.3.3}$$

These are in fact the solutions p_j of [**3**]. Substituting y_1 (3.1.12), into (3.3.1) gives us

$$p_1' + \left[2r(x)\sqrt{\lambda_1} + \frac{\sqrt{\lambda_1} r'(x)}{2\sqrt{r(x)}} \tanh(\sqrt{\lambda_1} g(x) + C_1) \right] p_1 = q(x) \frac{\tanh(\sqrt{\lambda_1} g(x) + C_1)}{\sqrt{\lambda_1 r(x)}}. \tag{3.3.4}$$

After multiplication by integrating factor s_1, we find

$$\left(p_1 \frac{\sqrt{r(x)} \cosh^2(\sqrt{\lambda_1} g(x) + C_1)}{C_3 \sqrt{r(0)} \cosh^2(C_1)} \right)' = q(x) \frac{\sinh(\sqrt{\lambda_1} g(x) + C_1) \cosh(\sqrt{\lambda_1} g(x) + C_1)}{C_3 \sqrt{\lambda_1 r(0)} \cosh^2(C_1)}. \tag{3.3.5}$$

Thus we have that

$$p_1 = s_1 \left(\int q \frac{\sinh(\sqrt{\lambda_1} g(x) + C_1) \cosh(\sqrt{\lambda_1} g(x) + C_1)}{C_3 \sqrt{\lambda_1 r(0)} \cosh^2(C_1)} dx \right).$$

Therefore our first critical pair is

$$\begin{cases} y_1 &= \dfrac{1}{\sqrt{\lambda_1 k}}\sinh(\sqrt{\lambda_1}g(x)+C_1), \\ p_1 &= \dfrac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s)+2C_1)ds}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x)+C_1)} + C_3\dfrac{\sqrt{r(0)}\cosh^2(C_1)}{\sqrt{r(x)}\cosh^2(\sqrt{\lambda_1}g(x)+C_1)}. \end{cases} \tag{3.3.6}$$

Similarly, for y_2 (3.3.1) implies

$$p_2' + \left[2r(x)\sqrt{\lambda_1} + \frac{\sqrt{\lambda_1}r'(x)}{2\sqrt{r(x)}}\coth(\sqrt{\lambda_1}g(x)+C_1)\right]p_2 = q(x)\frac{\coth(\sqrt{\lambda_1}g(x)+C_1)}{\sqrt{\lambda_1 r(x)}}. \tag{3.3.7}$$

Thus we have that

$$p_2 = s_2\left(\int q\frac{\sinh(\sqrt{\lambda_1}g(x)+C_2)\cosh(\sqrt{\lambda_1}g(x)+C_2)}{C_4\sqrt{\lambda_1 r(0)}\sinh^2(C_2)}dx\right).$$

This gives us our second critical pair

$$\begin{cases} y_2 &= \dfrac{1}{\sqrt{\lambda_1 k}}\cosh(\sqrt{\lambda_1}g(x)+C_2), \\ p_2 &= \dfrac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s)+2C_2)ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x)+C_2)} + C_4\dfrac{\sqrt{r(0)}\sinh^2(C_2)}{\sqrt{r(x)}\sinh^2(\sqrt{\lambda_1}g(x)+C_2)}. \end{cases} \tag{3.3.8}$$

Thus we have determined critical p_1 and p_2. Finding constants C_j appears to be a quite nontrivial procedure and we devote Ch 4 to their calculation.

CHAPTER 4

Determination of Constants C_j and of Predesigns $P_k(x)$

This section is devoted to the determination of the constants C_j from (3.3.6) and (3.3.8). Manipulation of condition (1.0.2) and (1.0.3), respectively, gives rise to the following useful constants:

$$\phi := -\frac{p(0)y'(0)}{y(0)} = \cot\alpha, \tag{4.0.1}$$

$$\psi := \frac{p(1)y'(1)}{y(1)} = \frac{\beta_1 + \lambda_1\beta_1'}{\beta_2 + \lambda_1\beta_2'}, \tag{4.0.2}$$

$$\zeta := -\frac{\phi + \psi\cosh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\cosh(2\sqrt{\lambda_1}g(s))ds}{\psi\sinh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds}. \tag{4.0.3}$$

$$z =: \frac{1}{2}\coth^{-1}(\zeta). \tag{4.0.4}$$

Each of these constants has limiting or special cases (such as at $\psi = 0$). Chapter 7 is devoted to a full exploration of those cases. We find that the constant α naturally divides our critical pairs (3.3.6) and (3.3.8) into three cases each, or six cases total, and use those cases to organize them into what we refer to as predesigns - see Def 2.5. Each predesign determined in this way is labeled P_k where k is the case number from the table below. To be considered a design, with an $M[P_k]$, we must first consider the existence and positivity of each predesign. As stated in Lem 2.7 we see regions of existence and nonexistence for each predesign as well as regions of positivity and nonpositivity for a particular data set $\mathcal{D}$. These solvability conditions are explored in Ch 5. Additionally, this section assumes that $\lambda_1 > 0$. We study the analytic continuation of our predesigns P_k for $\lambda_1 \leq 0$ in Ch 6.

Table 1: Summary of Predesigns by Case				
Case No.	$y_j(x), p_j(x)$	α	Predesign	$M[p_j]$
Case 1	$y_1(x), p_1(x)$	$\alpha = 0$	(4.1.2)	(4.3.1)
Case 2	$y_1(x), p_1(x)$	$\alpha = \frac{\pi}{2}$	(4.1.4)	(4.3.2)
Case 3	$y_1(x), p_1(x)$	$\alpha \in (0, \pi)\backslash\{\frac{\pi}{2}\}$	(4.1.10)	(4.3.3)
Case 4	$y_2(x), p_2(x)$	$\alpha = 0$	(4.2.2)	(4.3.1)
Case 5	$y_2(x), p_2(x)$	$\alpha = \frac{\pi}{2}$	(4.2.4)	(4.3.2)
Case 6	$y_2(x), p_2(x)$	$\alpha \in (0, \pi)\backslash\{\frac{\pi}{2}\}$	(4.2.10)	(4.3.4)

Remark 4.1. We note that predesigns given by (4.1.2) and (4.2.2) are identical. The same concidence holds for predesign given by (4.1.4) and (4.2.4). At both $\alpha = 0$ and $\alpha = \frac{\pi}{2}$, we have only one critical point. At $\alpha \neq \{0, \frac{\pi}{2}\}$, we have two critical points represented by (4.1.10) and (4.2.10).

4.1. Critical Point (y_1, p_1).

Case 1: Let $\alpha = 0$. It is immediately apparent upon examining (1.2) that $\alpha = 0$ implies $C_1 = 0$. After algebraic manipulation, boundary condition (1.3) yields

$$C_3 = \frac{1}{2\sqrt{\lambda_1 r(0)}}\left[\psi \sinh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds\right].$$

Then we have predesign at $\alpha = 0$ with arbitrary continuous q such that (y_1, P_1) is given by,

$$y_1(x) = \frac{1}{\sqrt{\lambda_1}}\sinh(\sqrt{\lambda_1}g(x)), \tag{4.1.1}$$

$$P_1(x) = \frac{1}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x))}[\psi \sinh(2\sqrt{\lambda_1}g(1)) - \int_x^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds]. \tag{4.1.2}$$

Case 2: Let $\alpha = \frac{\pi}{2}$. Boundary condition (1.2) yields

$$C_3\sqrt{\frac{r(0)}{k}}\cosh(C_1) = 0.$$

We begin by assuming $\cosh(C_1) = 0$, which implies that $C_{1n} = \frac{2n-1}{2}\pi i$ where $n \in \mathbb{Z}$. If we examine this C_{1n} in context of y_1 we find that

$$y_1(x) = \frac{1}{\sqrt{\lambda_1 k}}\sinh(\sqrt{\lambda_1}g(x) + \frac{2n-1}{2}\pi i)$$

$$=(-1)^{n-1}\frac{i}{\sqrt{\lambda_1 k}}\cosh(\sqrt{\lambda_1}g(x)).$$

We now examine this C_{1n} in context of p_1.

$$\begin{aligned}
p_1(x) = & \frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s)+(2n-1)\pi i)ds}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x)+\frac{2n-1}{2}\pi i)} \\
= & -\frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s))ds}{2\sqrt{\lambda_1 r(x)}(i^2)\sinh^2(\sqrt{\lambda_1}g(x))} \\
= & \frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s))ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x))}.
\end{aligned}$$

If we assume $C_3 = 0$, we have

$$p_1 = \frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s)+2C_1)ds}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x)+C_1)},$$

which attains zero at $x = 0$. This can be circumvented if $\cosh(C_1) = 0$. In any event, we conclude that there is a predesign at $\alpha = \frac{\pi}{2}$ such that (y_1, P_2) takes the form

$$y_1(x) = \frac{1}{\sqrt{\lambda_1}}\cosh(\sqrt{\lambda_1}g(x)), \tag{4.1.3}$$

$$P_2(x) = \frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s))ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x))}. \tag{4.1.4}$$

Case 3: Let $\alpha \notin \{0, \frac{\pi}{2}\}$. Boundary condition (1.0.2) implies that

$$C_3 = -\frac{1}{\sqrt{\lambda_1 r(0)}}\phi\tanh(C_1). \tag{4.1.5}$$

Consider boundary condition (1.0.3) formulated as in (4.0.2) and substitute the pair of critical functions $\{y_1, p_1\}$, then

$$\begin{aligned}
\psi = & \frac{\beta_1+\lambda_1\beta_1'}{\beta_2+\lambda_1\beta_2'} = p_1(1)y_1'(1)/y_1(1) \\
= & \Bigg(\frac{\int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s)+2C_1)ds}{2\sqrt{\lambda_1 r(1)}\cosh^2(\sqrt{\lambda_1}g(1)+C_1)} + C_3\frac{\sqrt{r(0)}\cosh^2(C_1)}{\sqrt{r(1)}\cosh^2(\sqrt{\lambda_1}g(1)+C_1)}\Bigg) \\
& \times \Big(\sqrt{\frac{r(1)}{k}}\cosh(\sqrt{\lambda_1}g(1)+C_1))/(\frac{1}{\sqrt{\lambda_1 k}}\sinh(\sqrt{\lambda_1}g(1)+C_1))
\end{aligned}$$

$$
\begin{aligned}
=&\left(\frac{\int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s)+2C_1)ds}{2\sqrt{\lambda_1}\cosh(\sqrt{\lambda_1}g(1)+C_1)} + C_3\frac{\sqrt{r(0)}\cosh^2(C_1)}{\cosh(\sqrt{\lambda_1}g(1)+C_1)}\right)\\
&\times 1/(\frac{1}{\sqrt{\lambda_1}}\sinh(\sqrt{\lambda_1}g(1)+C_1))\\
=&\frac{\int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s)+2C_1)ds}{\sinh(2\sqrt{\lambda_1}g(1)+2C_1))} + C_3\frac{2\sqrt{\lambda_1 r(0)}\cosh^2(C_1)}{\sinh(2\sqrt{\lambda_1}g(1)+2C_1))}.
\end{aligned}
$$

So we have an alternate formulation for C_3 given by

$$
C_3 = \frac{\psi\sinh(2\sqrt{\lambda_1}g(1)+2C_1)}{2\sqrt{\lambda_1 r(0)}\cosh^2(C_1)} - \frac{\int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s)+2C_1)ds}{2\sqrt{\lambda_1 r(0)}\cosh^2(C_1)}. \tag{4.1.6}
$$

We then equate (4.1.5) and (4.1.6) to solve for C_1,

$$
\begin{aligned}
-\phi\tanh(C_1) =&\frac{1}{2\cosh^2(C_1)}\left[\psi\sinh(2\sqrt{\lambda_1}g(1)+2C_1) - \int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s)+2C_1)ds\right],\\
-\phi\sinh(2C_1) =&\psi[\sinh(2\sqrt{\lambda_1}g(1))\cosh(2C_1)+\cosh(2\sqrt{\lambda_1}g(1))\sinh(2C_1)]\\
&-\int_0^1 q(s)[\sinh(2\sqrt{\lambda_1}g(s))\cosh(2C_1)+\cosh(2\sqrt{\lambda_1}g(s))\sinh(2C_1)]ds,\\
-\phi =&\psi[\sinh(2\sqrt{\lambda_1}g(1))\coth(2C_1)+\cosh(2\sqrt{\lambda_1}g(1))]\\
&-\int_0^1 q(s)[\sinh(2\sqrt{\lambda_1}g(s))\coth(2C_1)+\cosh(2\sqrt{\lambda_1}g(s))]ds,\\
\coth(2C_1) =&-\frac{\phi+\psi\cosh(2\sqrt{\lambda_1 g(1)}) - \int_0^1 q(s)\cosh(2\sqrt{\lambda_1}g(s))ds}{\psi\sinh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds} = \zeta\\
\implies C_1 =&\frac{1}{2}\coth^{-1}(\zeta) = z.
\end{aligned} \tag{4.1.7}
$$

This in turn means for C_3 that

$$
C_3 = -\frac{1}{\sqrt{\lambda_1 r(0)}}\phi\tanh(z). \tag{4.1.8}
$$

We have then for (y_1, P_3) that

$$
y_1(x) =\frac{1}{\sqrt{\lambda_1}}\sinh(\sqrt{\lambda_1}g(x)+z), \tag{4.1.9}
$$

$$
P_3(x) =\frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s)+2z)ds}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x)+z)} - \frac{\phi\sinh(2z)}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x)+z)}. \tag{4.1.10}
$$

4.2. Critical Point (y_2, p_2).

Case 4: Let $\alpha = 0$. Then condition (1.0.2) implies that

$$y_2(0) = \frac{1}{\sqrt{\lambda_1 k}} \cosh(C_2) = 0.$$

We begin with the assumption that $\cosh(C_2) = 0$, which implies that $C_{2n} = \frac{2n-1}{2}\pi i$ where $n \in \mathbb{Z}$. If we examine this C_{2n} in context of $y_2(x)$ we find that

$$\begin{aligned} y_2(x) =& \frac{1}{\sqrt{\lambda_1 k}} \cosh(\sqrt{\lambda_1} g(x) + \frac{2n-1}{2}\pi i) \\ =& (-1)^{n-1} \frac{i}{\sqrt{\lambda_1 k}} \sinh(\sqrt{\lambda_1} g(x)). \end{aligned}$$

We now examine this C_{2n} in context of $p_2(x)$.

$$\begin{aligned} p_2(x) =& \frac{\int_0^x q(s) \sinh(2\sqrt{\lambda_1} g(s) + (2n-1)\pi i) ds}{2\sqrt{\lambda_1 r(x)} \sinh^2(\sqrt{\lambda_1} g(x) + \frac{2n-1}{2}\pi i)} + C_4 \frac{\sqrt{r(0)} \sinh^2(\frac{2n-1}{2}\pi i)}{\sqrt{r(x)} \sinh^2(\sqrt{\lambda_1} g(x) + \frac{2n-1}{2}\pi i)} \\ =& -\frac{\int_0^x q(s) \sinh(2\sqrt{\lambda_1} g(s)) ds}{2\sqrt{\lambda_1 r(x)} (i^2) \cosh^2(\sqrt{\lambda_1} g(x))} + C_4 \frac{\sqrt{r(0)} i^2}{\sqrt{r(x)} i^2 \cosh^2(\sqrt{\lambda_1} g(x))} \\ =& \frac{\int_0^x q(s) \sinh(2\sqrt{\lambda_1} g(s)) ds}{2\sqrt{\lambda_1 r(x)} \cosh^2(\sqrt{\lambda_1} g(x))} + C_4 \frac{\sqrt{r(0)}}{\sqrt{r(x)} \cosh^2(\sqrt{\lambda_1} g(x))}. \end{aligned}$$

Condition (1.0.3) gives us that

$$\begin{aligned} \frac{p_2(1) y_2'(1)}{y_2(1)} =& \frac{\int_0^1 q(s) \sinh(2\sqrt{\lambda_1} g(s)) ds}{\sinh(2\sqrt{\lambda_1} g(1))} + C_4 \frac{\sqrt{\lambda_1 r(0)}}{\sinh(2\sqrt{\lambda_1} g(1))} = \psi \\ \implies C_4 =& \frac{1}{2\sqrt{\lambda_1 r(0)}} \left[\psi \sinh(2\sqrt{\lambda_1} g(1)) - \int_0^1 q(s) \sinh(2\sqrt{\lambda_1} g(s)) ds \right]. \end{aligned}$$

Thus we find a predesign in this case such that (y_2, P_4) is given by

$$y_2(x) = \frac{1}{\sqrt{\lambda_1}} \sinh(\sqrt{\lambda_1} g(x)), \tag{4.2.1}$$

$$P_4(x) = \frac{1}{2\sqrt{\lambda_1 r(x)} \cosh^2(\sqrt{\lambda_1} g(x))} [\psi \sinh(2\sqrt{\lambda_1} g(1)) - \int_x^1 q(s) \sinh(2\sqrt{\lambda_1} g(s)) ds]. \tag{4.2.2}$$

Case 5: Let $\alpha = \frac{\pi}{2}$. Condition (1.0.2) implies

$$p_2(0) y_2'(0) = C_4 \sqrt{\frac{r(0)}{k}} \sinh(C_2) = 0.$$

In the case $C_{2n} = \pi i n$, where $n \in \mathbb{Z}$, p_2 becomes

$$p_2 = \frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s))ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x))}.$$

In the case that $C_4 = 0$, we see that

$$p_2 = \frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s) + 2C_2)ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x) + C_2)}.$$

This function is zero at $x = 0$, and therefore unsuitable as a design unless we require that $C_2 = 0$. Therefore there is a predesign for the critical point (y_2, P_5) at $\alpha = \frac{\pi}{2}$ of the form

$$y_2(x) = \frac{1}{\sqrt{\lambda_1}}\cosh(\sqrt{\lambda_1}g(x)), \tag{4.2.3}$$

$$P_5(x) = \frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s))ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x))}. \tag{4.2.4}$$

Case 6: Let $\alpha \notin \{0, \frac{\pi}{2}\}$. Isolation of the term C_4 in (1.0.2) yields

$$C_4 = -\frac{1}{\sqrt{\lambda_1 r(0)}}\phi\coth(C_2). \tag{4.2.5}$$

Consider boundary condition (1.0.3) formulated as in (4.0.2) and substitute the pair of critical functions $\{y_2, p_2\}$, then

$$\begin{aligned}
\psi =& \frac{\beta_1 + \lambda_1\beta_1'}{\beta_2 + \lambda_1\beta_2'} = p_2(1)y_2'(1)/y_2(1) \\
=& \left(\frac{\int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s) + 2C_2)ds}{2\sqrt{\lambda_1 r(1)}\sinh^2(\sqrt{\lambda_1}g(1) + C_2)} + C_4\frac{\sqrt{r(0)}\sinh^2(C_2)}{\sqrt{r(1)}\sinh^2(\sqrt{\lambda_1}g(1) + C_2)}\right) \\
& \times (\sqrt{\frac{r(1)}{k}}\sinh(\sqrt{\lambda_1}g(1) + C_2))/(\frac{1}{\sqrt{\lambda_1 k}}\cosh(\sqrt{\lambda_1}g(1) + C_2)) \\
=& \left(\frac{\int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s) + 2C_2)ds}{2\sqrt{\lambda_1}\sinh(\sqrt{\lambda_1}g(1) + C_2)} + C_4\frac{\sqrt{r(0)}\sinh^2(C_2)}{\sinh(\sqrt{\lambda_1}g(1) + C_2)}\right) \\
& \times (1)/(\frac{1}{\sqrt{\lambda_1}}\cosh(\sqrt{\lambda_1}g(1) + C_2)) \\
=& \frac{\int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s) + 2C_2)ds}{\sinh(2\sqrt{\lambda_1}g(1) + 2C_2))} + C_4\frac{2\sqrt{\lambda_1 r(0)}\sinh^2(C_2)}{\sinh(2\sqrt{\lambda_1}g(1) + 2C_2))}.
\end{aligned}$$

So we have an alternate formulation for C_4 given by

$$C_4 = \frac{\psi \sinh(2\sqrt{\lambda_1}g(1) + 2C_2)}{2\sqrt{\lambda_1 r(0)}\sinh^2(C_2)} - \frac{\int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s) + 2C_2)ds}{2\sqrt{\lambda_1 r(0)}\sinh^2(C_2)}. \tag{4.2.6}$$

If we equate (4.2.5) and (4.2.6) to solve for C_2 we find

$$\begin{aligned}
-\phi\coth(C_2) =& \frac{1}{2\sinh^2(C_2)}\left[\psi\sinh(2\sqrt{\lambda_1}g(1) + 2C_2) - \int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s) + 2C_2)ds\right], \\
-\phi\sinh(2C_2) =& \psi[\sinh(2\sqrt{\lambda_1}g(1))\cosh(2C_2) + \cosh(2\sqrt{\lambda_1}g(1))\sinh(2C_2)] \\
& - \int_0^1 q(s)[\sinh(2\sqrt{\lambda_1}g(s))\cosh(2C_2) + \cosh(2\sqrt{\lambda_1}g(s))\sinh(2C_2)]ds, \\
-\phi =& \psi[\sinh(2\sqrt{\lambda_1}g(1))\coth(2C_2) + \cosh(2\sqrt{\lambda_1}g(1))] \\
& - \int_0^1 q(s)[\sinh(2\sqrt{\lambda_1}g(s))\coth(2C_2) + \cosh(2\sqrt{\lambda_1}g(s))]ds, \\
\coth(2C_2) =& -\frac{\phi + \psi\cosh(2\sqrt{\lambda_1 g(1)}) - \int_0^1 q(s)\cosh(2\sqrt{\lambda_1}g(s))ds}{\psi\sinh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds} = \zeta \\
\implies C_2 =& \frac{1}{2}\coth^{-1}(\zeta) = z. \qquad (4.2.7)
\end{aligned}$$

This in turn means for C_4 that

$$C_4 = -\frac{1}{\sqrt{\lambda_1 r(0)}}\phi\coth(z). \tag{4.2.8}$$

We therefore have a predesign at $\alpha \notin \{0, \frac{\pi}{2}\}$ such that the critical point (y_2, P_6) is represented by

$$\begin{aligned}
y_2(x) =& \frac{1}{\sqrt{\lambda_1}}\cosh(\sqrt{\lambda_1}g(x) + z), \qquad (4.2.9) \\
P_6(x) =& \frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s) + 2z)ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x) + z)} - \frac{\phi\sinh(2z)}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x) + z))}. \qquad (4.2.10)
\end{aligned}$$

4.3. Determination of $M[P_k]$.

Case 1 and Case 4: We have predesign for P_1 (or P_4) given by (4.1.2) (or (4.2.2)). We determine then for $M[P_1]$ (or $M[P_4]$),

$$\begin{aligned}
M[P_1] = &\int_0^1 r(x)\frac{1}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x))}[\psi\sinh(2\sqrt{\lambda_1}g(1)) - \int_x^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds]dx \\
=&\frac{1}{2\sqrt{\lambda_1}}[\psi\sinh(2\sqrt{\lambda_1}g(1))\int_0^1 \frac{\sqrt{r(x)}}{\cosh^2(\sqrt{\lambda_1}g(x))}dx \\
&-\int_0^1 (\frac{\sqrt{r(x)}}{\cosh^2(\sqrt{\lambda_1}g(x))}\int_x^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds)dx] \\
=&\frac{1}{2\lambda_1}[\psi\sinh(2\sqrt{\lambda_1}g(1))\tanh(\sqrt{\lambda_1}g(1)) \\
&-\int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds\int_0^s \frac{\sqrt{\lambda_1 r(x)}}{\cosh^2(\sqrt{\lambda_1}g(x))}dx] \\
=&\frac{1}{2\lambda_1}[2\psi\sinh^2(\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))\tanh(\sqrt{\lambda_1}g(s))ds] \\
=&\frac{1}{\lambda_1}[\psi\sinh^2(\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\sinh^2(\sqrt{\lambda_1}g(s))ds]. \qquad (4.3.1)
\end{aligned}$$

Case 2 and Case 5: We have predesign for P_2 (or P_5) given by (4.1.4) (or (4.2.4)). We determine then for $M[P_2]$ (or $M[P_5]$),

$$\begin{aligned}
M[P_2] = &\int_0^1 r(x)\frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s))ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x))}dx \\
=&\frac{1}{2\sqrt{\lambda_1}}\int_0^1 (\frac{\sqrt{r(x)}}{\sinh^2(\sqrt{\lambda_1}g(x))}\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s))ds)dx \\
=&\frac{1}{2\lambda_1}\Big[\int_0^1 \coth(\sqrt{\lambda_1}g(s))q(s)\sinh(2\sqrt{\lambda_1}g(s))ds \\
&-\coth(\sqrt{\lambda_1}g(1))\int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds\Big] \\
=&\frac{1}{2\lambda_1}\Big[2\int_0^1 q(s)\cosh^2(\sqrt{\lambda_1}g(s))ds - \coth(\sqrt{\lambda_1}g(1))\int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds\Big]. \qquad (4.3.2)
\end{aligned}$$

Case 3: We have predesign for P_3 (4.1.10) and determine then for $M[P_3]$,

$$
\begin{aligned}
M[P_3] =& \int_0^1 r(x)\Big[\frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s)+\coth^{-1}(\zeta))ds}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x)+\frac{1}{2}\coth^{-1}(\zeta))} \\
&- \frac{\phi\sinh(\coth^{-1}(\zeta))}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x)+\frac{1}{2}\coth^{-1}(\zeta))}\Big]dx \\
=&\frac{1}{2\sqrt{\lambda_1}}\Big[\int_0^1(\frac{\sqrt{r(x)}}{\cosh^2(\sqrt{\lambda_1}g(x)+\frac{1}{2}\coth^{-1}(\zeta))}\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s)+\coth^{-1}(\zeta))ds)dx \\
&- \phi\sinh(\coth^{-1}(\zeta))\int_0^1\frac{\sqrt{r(x)}}{\cosh^2(\sqrt{\lambda_1}g(x)+\frac{1}{2}\coth^{-1}(\zeta))}dx\Big] \\
=&\frac{1}{2\lambda_1}\Big[(\tanh(\sqrt{\lambda_1}g(x)+\frac{1}{2}\coth^{-1}(\zeta))\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s)+\coth^{-1}(\zeta))ds)|_0^1 \\
&- \int_0^1 \tanh(\sqrt{\lambda_1}g(x)+\frac{1}{2}\coth^{-1}(\zeta))q(x)\sinh(2\sqrt{\lambda_1}g(x)+\coth^{-1}(\zeta))dx \\
&- \phi\sinh(\coth^{-1}(\zeta))\tanh(\sqrt{\lambda_1}g(x)+\frac{1}{2}\coth^{-1}(\zeta))|_0^1\Big] \\
=&\frac{1}{2\lambda_1}\Big[\tanh(\sqrt{\lambda_1}g(1)+z)(\int_0^1 q(x)\sinh(2\sqrt{\lambda_1}g(x)+2z)dx-\phi\sinh(2z)) \\
&- 2(\int_0^1 q(x)\sinh^2(\sqrt{\lambda_1}g(x)+z)dx-\phi\sinh^2(z))\Big].
\end{aligned} \tag{4.3.3}
$$

Case 6: We have predesign for P_6 (4.2.10) and determine then for $M[P_6]$,

$$
\begin{aligned}
M[P_6] =& \int_0^1 r(x)\Big[\frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s)+\coth^{-1}(\zeta))ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x)+\frac{1}{2}\coth^{-1}(\zeta))} \\
&- \frac{\phi\sinh(\coth^{-1}(\zeta))}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x)+\frac{1}{2}\coth^{-1}(\zeta))}\Big]dx \\
=&\frac{1}{2\sqrt{\lambda_1}}\Big[\int_0^1(\frac{\sqrt{r(x)}}{\sinh^2(\sqrt{\lambda_1}g(x)+\frac{1}{2}\coth^{-1}(\zeta))}\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s)+\coth^{-1}(\zeta))ds)dx \\
&- \phi\sinh(\coth^{-1}(\zeta))\int_0^1\frac{\sqrt{r(x)}}{\sinh^2(\sqrt{\lambda_1}g(x)+\frac{1}{2}\coth^{-1}(\zeta))}dx\Big] \\
=&\frac{1}{2\lambda_1}[-\coth(\sqrt{\lambda_1}g(x)+\frac{1}{2}\coth^{-1}(\zeta))\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s)+\coth^{-1}(\zeta))ds]_0^1 \\
&+ \frac{1}{2\lambda_1}[2\int_0^1 q(s)\cosh^2(\sqrt{\lambda_1}g(s)+\frac{1}{2}\coth^{-1}(\zeta))ds] \\
&+ \frac{1}{2\lambda_1}[\phi\sinh(\coth^{-1}(\zeta))\coth(\sqrt{\lambda_1}g(x)+\frac{1}{2}\coth^{-1}(\zeta))]_0^1
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{2\lambda_1}[-\coth(\sqrt{\lambda_1}g(1)+\frac{1}{2}\coth^{-1}(\zeta))\int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s)+\coth^{-1}(\zeta))ds]\\
&+\frac{1}{2\lambda_1}[2\int_0^1 q(s)\cosh^2(\sqrt{\lambda_1}g(s)+\frac{1}{2}\coth^{-1}(\zeta))ds]\\
&+\frac{1}{2\lambda_1}[\phi\sinh(\coth^{-1}(\zeta))\coth(\sqrt{\lambda_1}g(1)+\frac{1}{2}\coth^{-1}(\zeta))-2\phi\cosh^2(\frac{1}{2}\coth^{-1}(\zeta))]\\
=&\frac{1}{2\lambda_1}\Bigg[\coth(\sqrt{\lambda_1}g(1)+z)(\phi\sinh(2z)-\int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s)+2z)ds) \qquad (4.3.4)\\
&+2(\int_0^1 q(s)\cosh^2(\sqrt{\lambda_1}g(s)+z)ds-\phi\cosh^2(z))\Bigg]
\end{aligned}
$$

4.4. Comparison of $M[P_3]$ and $M[P_6]$.

Our goal in this study is to determine the minimum mass of a specified structure with a given data set, however for the case when $\alpha \notin \{0, \frac{\pi}{2}\}$ we have two critical points. In this section, we determine the necessary conditions for one critical point or the other to be a point of minimum by studying the function

$$
f(z) := M[P_3] - M[P_6], \qquad (4.4.1)
$$

where the masses are as defined in (4.3.3) and (4.3.4), respectively, and we fix the functions q and r and vary the data z (4.0.4). We observe that both $M[P_3]$ and $M[P_6]$ contain the factor λ_1^{-1}, so we may not include it without changing the overall result.

$$
\begin{aligned}
f(z) =&\frac{1}{2}\tanh(\sqrt{\lambda_1}g(1)+z)(\int_0^1 q(x)\sinh(2\sqrt{\lambda_1}g(x)+2z)dx-\phi\sinh(2z))\\
&-\int_0^1 q(x)\sinh^2(\sqrt{\lambda_1}g(x)+z)dx+\phi\sinh^2(z)\\
&+\frac{1}{2}\coth(\sqrt{\lambda_1}g(1)+z)(\int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s)+2z)ds-\phi\sinh(2z))\\
&-\int_0^1 q(x)\cosh^2(\sqrt{\lambda_1}g(x)+z)dx+\phi\cosh^2(z)\\
=&\frac{1}{2}(\tanh(\sqrt{\lambda_1}g(1)+z)+\coth(\sqrt{\lambda_1}g(1)+z))
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \Big(\int_0^1 q(x)\sinh(2\sqrt{\lambda_1}g(x)+2z)dx - \phi\sinh(2z)\Big)\\
&- \int_0^1 q(x)(\sinh^2(\sqrt{\lambda_1}g(x)+z)+\cosh^2(\sqrt{\lambda_1}g(x)+z))dx\\
&+ \phi(\cosh^2(z)+\sinh^2(z))\\
=&\coth(2\sqrt{\lambda_1}g(1)+2z)\Big(\int_0^1 q(x)\sinh(2\sqrt{\lambda_1}g(x)+2z)dx - \phi\sinh(2z)\Big) \qquad (4.4.2)\\
&- \int_0^1 q(x)\cosh(2\sqrt{\lambda_1}g(x)+2z)dx + \phi\cosh(2z).
\end{aligned}
$$

Here the way seems impassable without some additional assumptions on q, and we proceed by way of the simplifying assumption that $q \equiv 0$. We begin by determining the conditions under which $M[P_3] = M[P_6]$ or $f(z) = 0$.

$$
\begin{aligned}
f(z) =&0\\
\implies 0 =& -\coth(2\sqrt{\lambda_1}g(1)+2z)\phi\sinh(2z) + \phi\cosh(2z),\\
\phi\cosh(2z) =&\coth(2\sqrt{\lambda_1}g(1)+2z)\phi\sinh(2z).
\end{aligned}
$$

At $\alpha \notin \{0, \frac{\pi}{2}\}$, $\phi \neq 0$. Then we have that

$$
\coth(2z) = \coth(2\sqrt{\lambda_1}g(1)+2z). \qquad (4.4.3)
$$

As the function coth is monotonic on every subinterval of the domain and $\sqrt{\lambda_1}g(1)$ is nonzero, the above equality is not possible. Hence we have no real data sets $\mathcal{D}$ which can produce this result. We conclude then that one mass must be inferior to the other. It is clear that the function $f(z)$ under our simplifying assumption has a vertical asymptote at $z = -\sqrt{\lambda_1}g(1)$. If we begin by assuming that $\phi > 0$, then we have

$$
\begin{aligned}
\lim_{z\to\sqrt{\lambda_1}g(1)^+} f(z) =&\infty, \quad \lim_{z\to\infty} f(z) = 0;\\
\lim_{z\to\sqrt{\lambda_1}g(1)^-} f(z) =& -\infty, \quad \lim_{z\to-\infty} f(z) = 0.
\end{aligned}
$$

As (4.4.3) shows us that $f(z) \neq 0$, it is clear that $f > 0, \forall z > -\sqrt{\lambda_1}g(1)$, and $f < 0, \forall z < -\sqrt{\lambda_1}g(1)$. If we return to the definition of z and ζ we see that

$$\begin{aligned}
f(z) > 0 &\implies z > -\sqrt{\lambda_1}g(1) \\
&\implies \zeta > -\coth(2\sqrt{\lambda_1}g(1)), \\
-\frac{\phi + \psi\cosh(\sqrt{\lambda_1}g(1))}{\psi\sinh(\sqrt{\lambda_1}g(1))} &> -\coth(\sqrt{\lambda_1}g(1)), \\
\frac{\phi + \psi\cosh(\sqrt{\lambda_1}g(1))}{\psi} - \frac{\psi\cosh(\sqrt{\lambda_1}g(1))}{\psi} &< 0, \\
\frac{\phi}{\psi} &< 0 \\
\implies \psi &< 0.
\end{aligned}$$

Similarly,

$$M[P_6] > M[P_3] \implies \psi < 0.$$

These inequalities are reversed for $\phi < 0$, so we may construct the following table of minimum masses.

Table 2: Minimum Mass by sign of ϕ, ψ		
	$\phi > 0$	$\phi < 0$
$\psi > 0$	$M[P_3]$	$M[P_6]$
$\psi < 0$	$M[P_6]$	$M[P_3]$

We conclude that even for $q \equiv 0$ we may not guarantee *a-priori* that the design P_3 or P_6 is the point of a minimum. Instead, each design may give a minimum, but not both. The same holds for an arbitrary q. We further unravel this problem by studying the behavior of the function

$$h := \left|\frac{P_3}{P_6}\right|.$$

When we simplify this function using our known, explicit formulae for P_3 and P_6 we find

$$h(x) = \left| \frac{\frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s)+2z)ds}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x)+z)} - \frac{\phi\sinh(2z)}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x)+z)}}{\frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s)+2z)ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x)+z)} - \frac{\phi\sinh(2z)}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x)+z)}} \right|$$

$$= \frac{\sinh^2(\sqrt{\lambda_1}g(x)+z)}{\cosh^2(\sqrt{\lambda_1}g(x)+z)} < 1.$$

We conclude that since the denominator P_6 is larger than the numerator P_3, then $M[P_3]$ will have the smaller mass - as long as the solvability conditions on P_3 are met. As we will discuss in Ch 5, the solvability conditions on P_3 are a precursor to the solvability conditions on P_6.

This concludes our proof of Thm 2.6.

CHAPTER 5

Solvability Conditions on Predesign P_k

Recall Def 2.4 of the sets $\mathcal{Q}$ and $\overline{\mathcal{Q}}$, namely the set of functions q such that the corresponding p is (is not) positive at every point $x \in [0,1]$. We introduce the following notation for two additional sets of real-valued functions q:

$$\{Q_{M^+} \,|\, q(x) \geq M > 0,\ \forall x \in [0,1]\},\ \{Q_{M^-} \,|\, q(x) \leq -M < 0,\ \forall x \in [0,1]\}.$$

The constants M and $-M$ will be different on different occasions. Their existence but not the exact values will be important. We devote this chapter to the proof of Lem 2.7.

5.1. Conditions on q such that p is defined $\forall x \in [0,1]$.

We note that in some cases the constants C_j - and therefore y_k, p_k - depend on the constant ζ for existence. Therefore we construct in this section conditions on $q(x)$ such that we may find a suitable ζ. For this section, we refer to the numerator and denominator of ζ separately by

$$\zeta := -\frac{\phi + \psi\cosh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\cosh(2\sqrt{\lambda_1}g(s))ds}{\psi\sinh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds} := -\frac{\zeta_1}{\zeta_2}. \tag{5.1.1}$$

Note that as $\zeta_2 \to 0$, $\zeta \to \pm\infty$ and

$$\lim_{\zeta\to\infty} C_1 = 0 \implies \lim_{\zeta\to\infty} C_3 = 0,$$

$$\lim_{\zeta\to\infty} C_2 = 0 \implies \lim_{\zeta\to\infty} C_4 = \text{undef.}$$

For this section, we exclude the case where $\zeta_1 = \zeta_2 = 0$. A full discussion of this possibility is included in Ch 7. As the constant ζ does not appear in P_1 or P_2, we concern ourselves with P_3 and P_6.

5.1.1. $P_3(x)$**:** Let y_1, p_1 as described in (3.3.6) with C_1, C_3 as described in (4.1.7) and (4.1.8), respectively. We note that for very large ζ, $\coth^{-1}(\zeta) \to 0$. Thus as $\zeta \to \infty$, we have

$$P_3(x) = \frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s))ds}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x))} = 0, \text{ at } x = 0.$$

However, we require that $p > 0$ for the entire domain. This results in no suitable design for P_1. Our first condition, therefore, is that ζ_2 must be nonzero. Second, the $\coth^{-1}\zeta$ function is defined for $|\zeta| > 1$.

These two requirements give rise to the following four subcases:

Table 3: Summary of Cases (P_3)		
	$\zeta_2 > 0$	$\zeta_2 < 0$
$\zeta > 1$	(5.1.2)	(5.1.3)
$\zeta < -1$	(5.1.4)	(5.1.5)

a. Assuming $\zeta_2 > 0$, $\zeta > 1 \implies$

$$-\frac{\phi + \psi\cosh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\cosh(2\sqrt{\lambda_1}g(s))ds}{\psi\sinh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds} > 1$$

$$\implies \int_0^1 q(s)\exp(2\sqrt{\lambda_1}g(s))ds > \phi + \psi\exp(2\sqrt{\lambda_1}g(1)) \tag{5.1.2}$$

which is true at least if $q \in Q_M^+$ and false if at least $q(x) \in Q_M^-$ for large values of M.

b. Similarly, assuming $\zeta_2 > 0$ and $\zeta < -1$,

$$\implies \int_0^1 q(s)\exp(-2\sqrt{\lambda_1}g(s))ds < \phi + \psi\exp(-2\sqrt{\lambda_1}g(1)) \tag{5.1.3}$$

which is true at least if $q \in Q_M^-$ and false if at least $q \in Q_M^+$ for large values of M.

c. If we assume $\zeta_2 < 0$ and $\zeta > 1$,

$$-\frac{\phi + \psi\cosh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\cosh(2\sqrt{\lambda_1}g(s))ds}{\psi\sinh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds} > 1$$

$$\implies \int_0^1 q(s)\exp(2\sqrt{\lambda_1}g(s))ds < \phi + \psi\exp(2\sqrt{\lambda_1}g(1)) \tag{5.1.4}$$

which is true at least if $q \in Q_M^-$ and false if at least $q \in Q_M^+$ for large values of M.

d. Similarly, assuming $\zeta_2 < 0$ and $\zeta < -1$,

$$\implies \int_0^1 q(s)\exp(-2\sqrt{\lambda_1}g(s))ds > \phi + \psi\exp(-2\sqrt{\lambda_1}g(1)) \tag{5.1.5}$$

which is true at least if $q \in Q_M^+$ and false if at least $q \in Q_M^-$ for large values of M.

Therefore, in all these four sub-cases (see Table 2) it is possible to find q such that a suitable design p_1 exists or not. Hence, both $\mathcal{Q}$ and $\overline{\mathcal{Q}} \neq \emptyset$.

5.1.2. $P_6(x)$**:** Let y_2, p_2 as described in (3.3.8) with C_2, C_4 as described in (4.2.7) and (4.2.8), respectively. The denominator of P_6 contains the term

$$\sinh(\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta)).$$

To ensure that P_6 is defined along the entire domain we require

$$\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta) \neq 0,\ x \in [0,1].$$

To guarantee this outcome, we require one of the following to be true,

$$\coth^{-1}(\zeta) > 0,\ \text{i.e. } \zeta > 1; \tag{5.1.6}$$

$$\coth^{-1}(\zeta) < -2\sqrt{\lambda_1}g(1). \tag{5.1.7}$$

Similar to P_3 above, we must also consider the domain of $\coth^{-1}(\zeta)$. This gives rise to four subcases given by

Table 4: Summary of Cases (P_6)		
	$\zeta_2 > 0$	$\zeta_2 < 0$
(5.6)	(5.1.2)	(5.1.3)
(5.7)	(5.1.8)	(5.1.9)

We note that condition (5.1.6) gives rise to conditions (5.1.2) and (5.1.3) as in **Case 3**, and as we know, both $\mathcal{Q}$ and $\overline{\mathcal{Q}} \neq \emptyset$. However, if we assume condition (5.1.7) and $\zeta_2 > 0$ we find

$$\coth(-2\sqrt{\lambda_1}g(1)) > -\frac{\phi + \psi\cosh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\cosh(2\sqrt{\lambda_1}g(s))ds}{\psi\sinh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds}$$

$$\Longrightarrow \int_0^1 q(s)[\cosh(2\sqrt{\lambda_1}g(1)) - \cosh(2\sqrt{\lambda_1}g(s))]ds < \phi \tag{5.1.8}$$

which is true at least if $q(x) \in Q_M^-$ and false at least if $q(x) \in Q_M^+$ for large positive values of M. Similarly, if we assume (5.1.7) and $\zeta_2 < 0$,

$$\Longrightarrow \int_0^1 q(s)[\cosh(2\sqrt{\lambda_1}g(s)) - \cosh(2\sqrt{\lambda_1}g(1))]ds > \phi \tag{5.1.9}$$

which is true at least if $q(x) \in Q_M^+$ and false at least if $q(x) \in Q_M^-$ for large positive values of M.

Therefore, in all these four sub-cases (Table 4) it is possible to find $q(x)$ such that a suitable design P_6 exists or not. Hence, $\mathcal{Q}$, $\overline{\mathcal{Q}} \neq \emptyset$.

5.2. Conditions on q to ensure positivity of p.

This subsection is devoted to the derivation of conditions on q such that a positive design p exists as well as a few examples of some satisfactory q. First, we assume that $\psi > 0$. Then, we discuss the opposite case.

5.2.1. P_1**:** Let P_1 be as in (4.1.2),

$$P_1 = \frac{1}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x))}[\psi\sinh(2\sqrt{\lambda_1}g(1)) - \int_x^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds].$$

The denominator, $2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x))$, is positive everywhere on the domain, so we focus solely on the numerator to determine the sign of p_1. We find

$$\begin{aligned} \psi \sinh(2\sqrt{\lambda_1}g(1)) - \int_x^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds > 0 \\ \implies \int_x^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds < \psi\sinh(2\sqrt{\lambda_1}g(1)),\ \forall x \in [0,1]. \end{aligned} \tag{5.2.1}$$

We present a few examples and non-examples to show that $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ are nonempty.

a) For $q(x) := q_a(x) \equiv 0$, (5.2.1) holds true, i.e. $q_a \in \mathcal{Q}$.

b) Let $q(x) := q_b(x) = 2\psi\sqrt{\lambda_1 r(x)}$, where $\psi > 0$. Then

$$\begin{aligned} \psi\sinh(2\sqrt{\lambda_1}g(1)) >& \psi \int_x^1 2\sqrt{\lambda_1 r(s)}\sinh(2\sqrt{\lambda_1}g(s))ds \\ \implies \psi\sinh(2\sqrt{\lambda_1}g(1)) >& \psi\cosh(2\sqrt{\lambda_1}g(1)) - \psi\cosh(2\sqrt{\lambda_1}g(x)) \\ \implies -\psi\exp(-2\sqrt{\lambda_1}g(1)) >& -\psi\cosh(2\sqrt{\lambda_1}g(x)) \\ \implies \exp(-2\sqrt{\lambda_1}g(1)) <& \cosh(2\sqrt{\lambda_1}g(x)), \end{aligned}$$

which is true $\forall x \in [0,1]$, so we conclude that our choice of q is valid, i.e. $q_b \in \mathcal{Q}$.

c) Similarly, if we change the sign of $q_b(x)$ and let $q_c(x) = 2\psi\sqrt{\lambda_1 r(x)}$ where $\psi < 0$ we find

$$\begin{aligned} \psi\sinh(2\sqrt{\lambda_1}g(1)) >& -\psi \int_x^1 2\sqrt{\lambda_1 r(s)}\sinh(2\sqrt{\lambda_1}g(s))ds \\ \implies \psi\sinh(2\sqrt{\lambda_1}g(1)) >& -\psi\cosh(2\sqrt{\lambda_1}g(1)) + \psi\cosh(2\sqrt{\lambda_1}g(x)) \\ \implies \psi\exp(2\sqrt{\lambda_1}g(1)) >& \psi\cosh(2\sqrt{\lambda_1}g(x)) \\ \implies \exp(2\sqrt{\lambda_1}g(1)) >& \cosh(2\sqrt{\lambda_1}g(x)), \end{aligned}$$

which is true $\forall x \in [0,1]$, so we conclude that our choice of q is valid, i.e. $q_c \in \mathcal{Q}$.

d) Finally, if we assume $\psi < 0$, then neither b) nor c) hold, i.e. $q_b, q_c \in \overline{\mathcal{Q}}$.

Hence, both $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ are nonempty for P_1.

Lemma 5.1. The sets $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ do not intersect, yet contain infinitely close elements.

We give a brief proof of the lemma as it pertains specifically to $P_1(x)$. The intersection of these sets is empty - obvious from Def 2.4. We now demonstrate an example of infinitely close elements. Let $\psi_{1,2} > 0$. We have seen see that $q(x) := 2\psi_1\sqrt{\lambda_1 r(x)} \in \mathcal{Q}$ and $\overline{q}(x) := -2\psi_2\sqrt{\lambda_1 r(x)} \in \overline{\mathcal{Q}}$ in our examples above (see Subsection 5.2.1). We observe that $||q - \overline{q}||_{C[0,1]} \leq 2\sqrt{\lambda_1}\max_{[0,1]}\sqrt{r(x)}|\psi_1 + \psi_2|$, which is small enough if $|\psi_1 + \psi_2|$ is small enough. This explicitly demonstrates that the sets $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ contain infinitely close elements, even though the sets do not intersect.

Remark 5.2. This result demonstrates a level of instability in our design with respect to $q(x)$. We will remark further on the stability of our designs in our section on numerical examples.

5.2.2. P_2**:** Let P_2 be as in (4.1.4),

$$P_2(x) = \frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s))ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x))}.$$

Here we have a potential discontinuity at $x = 0$. We proceed via L'Hopital's Rule.

$$\begin{aligned}\lim_{x\to 0} p_1 &= \lim_{x\to 0}\frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s))ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x))}\\ &= \lim_{x\to 0}\frac{1}{2\sqrt{\lambda_1 r(x)}}\lim_{x\to 0}\frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s))ds}{\sinh^2(\sqrt{\lambda_1}g(x))}\\ &= \frac{1}{2\sqrt{\lambda_1 r(0)}}\lim_{x\to 0}\frac{q(x)\sinh(2\sqrt{\lambda_1}g(x))}{\sqrt{\lambda_1 r(x)}\sinh(2\sqrt{\lambda_1}g(x))}\\ &= \frac{q(0)}{2\lambda_1 r(0)}.\end{aligned}$$

Thus we have a positive solution which is continuous at $x = 0$ if at least $q(x) > 0$ along $[0, 1]$ and an unsuitable solution if at least $q(x) < 0$ along $[0, 1]$.

5.2.3. P_3**:** Let P_3 be as in (4.1.10),

$$P_3 = \frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s) + \coth^{-1}(\zeta))ds}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta))} - \frac{\phi\sinh(\coth^{-1}(\zeta))}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta))}.$$

The denominator is positive everywhere on the domain, so we may focus on the numerator. We formulate the condition

$$\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s) + \coth^{-1}(\zeta))ds - \phi\sinh(\coth^{-1}(\zeta)) > 0, \ \forall x \in [0, 1].$$

At $x = 0$, this implies $\phi\sinh(\coth^{-1}(\zeta)) < 0$. For $\alpha \in (0, \frac{\pi}{2})$, $\phi > 0$ so we require

$$\sinh(\coth^{-1}(\zeta)) < 0$$

$$\implies \coth^{-1}(\zeta) < 0$$

$$\implies \zeta < -1$$

Here we arrive at conditions (5.1.2) and (5.1.4) depending on the sign of ζ_2. The inequality is reversed for $\alpha \in (\frac{\pi}{2}, \pi)$, and we find conditions (5.1.3) or (5.1.5) depending on the sign of ζ_2. We again conclude that both sets Ω and $\overline{\Omega}$ are nonempty.

5.2.4. P_6**:** Let P_6 be as in 4.2.10,

$$P_6 = \frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s) + \coth^{-1}(\zeta))ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta))} - \frac{\phi\sinh(\coth^{-1}(\zeta))}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta))}.$$

We note that the denominator is positive - Sec 4.1 details the restrictions we require on $q(x)$ such that this is the case - so we require

$$\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s) + \coth^{-1}(\zeta))ds - \phi\sinh(\coth^{-1}(\zeta)) > 0, \ \forall x \in [0, 1].$$

This implies a similar situation as in Case 3 above. For $\alpha \in (0, \frac{\pi}{2})$, we require condition (5.1.3) and (5.1.5) depending on the sign of ζ_2. The inequality is reversed for $\alpha \in (\frac{\pi}{2}, \pi)$, and we find conditions (5.1.2) or (5.1.4) depending on the sign of ζ_2. Again, both sets $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ are nonempty.

Remark 5.3. This completes the proof of Lem 2.7 and explanation of Tables 1, 2, and 3.

CHAPTER 6

Analytic Continuation of P_k in Regard to λ_1

Thm 2.6 assumes that $\lambda_1 > 0$. Here we construct an analytic continuation of the previously found designs toward $\lambda_1 \leq 0$. We again are specifically interested in the designs $p \in \mathcal{P}$ and prove that the sets $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ are not empty - namely we show that there exists some $q(x)$ such that a design $p(x)$ exists for a given data set $\mathcal{D}$. We first consider the case where $\lambda_1 = 0$ in the limiting sense, taking $\lambda_1 \to 0$; then the case where $\lambda_1 < 0$. Note $\sqrt{\lambda_1} = i\sqrt{|\lambda_1|}$.

6.1. Case: $\lambda_1 = 0$.

6.1.1. P_1**:** If we take C_3 as in the predesign $P_1(x)$ (4.1.2) above, we find that

$$\begin{aligned}\lim_{\lambda_1\to 0} C_3 =& \frac{1}{\sqrt{r(0)}}\Big[\lim_{\lambda_1\to 0}\frac{\psi\sinh(2\sqrt{\lambda_1}g(1))}{2\sqrt{\lambda_1}} - \lim_{\lambda_1\to 0}\int_0^1\frac{q(s)\sinh(2\sqrt{\lambda_1}g(s))ds}{2\sqrt{\lambda_1}}\Big]\\ =& \frac{1}{\sqrt{r(0)}}[\psi g(1) - \int_0^1 q(s)g(s)ds]. \end{aligned} \tag{6.1.1}$$

We see then that for P_1 as $\lambda_1 \to 0$

$$\begin{aligned}\lim_{\lambda_1\to 0} P_1(x) =& \lim_{\lambda_1\to 0}\frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s))ds}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x))} + \lim_{\lambda_1\to 0} C_3\frac{\sqrt{r(0)}}{\sqrt{r(x)}\cosh^2(\sqrt{\lambda_1}g(x))}\\ =& \frac{1}{\sqrt{r(x)}}[\psi g(1) - \int_x^1 q(s)g(s)ds]. \end{aligned} \tag{6.1.2}$$

Thus we have a solution at $\lambda = 0$. To ensure the positivity of this design we check the condition

$$\psi g(1) > \int_x^1 q(s)g(s)ds,\ \forall x \in [0,1]. \tag{6.1.3}$$

At $x = 1$, the above implies $\psi g(1) > 0$ which is only true for $\psi > 0$. If we assume that $\psi > 0$, then (6.1.3) holds for at least $q(x) \in Q_M^-$ and is false for at least $q(x) \in Q_M^+$. After integration we see that $M[P_1]$ for $\lambda_1 = 0$ computes to

$$\begin{aligned} M[P_1] &= \int_0^1 \sqrt{r(x)}[\psi g(1) - \int_x^1 q(s)g(s)ds]dx \\ &= \psi g^2(1) - \int_0^1 q(x)g(x)dx \int_0^x \sqrt{r(s)}ds \\ &= \psi g^2(1) - \int_0^1 q(x)g^2(x)dx. \end{aligned} \tag{6.1.4}$$

6.1.2. P_2**:** We see no suitable design in this case in general. $P_2(x)$ (4.1.4) is undefined under these conditions.

6.1.3. P_3 **and** P_6**:** The functions (4.1.10) and (4.2.10) are undefined under $\lambda_1 = 0$ and we see no suitable design.

6.2. Case: $\lambda_1 < 0$

6.2.1. P_1**:** If we take C_3 as in the scheme (4.1.2) above, we find that

$$\begin{aligned} C_3 &= \frac{1}{2i\sqrt{|\lambda_1|r(0)}}[\psi \sinh(2i\sqrt{|\lambda_1|}g(1)) - \int_0^1 q(s)\sinh(2i\sqrt{|\lambda_1|}g(s))ds] \\ &= \frac{1}{2i\sqrt{|\lambda_1|r(0)}}[i\psi \sin(2\sqrt{|\lambda_1|}g(1)) - i\int_0^1 q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds] \\ &= \frac{1}{2\sqrt{|\lambda_1|r(0)}}[\psi \sin(2\sqrt{|\lambda_1|}g(1)) - \int_0^1 q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds]. \end{aligned}$$

We have that C_3 is real-valued. For (4.1.2) we have that $C_1 = 0$, so we have for $P_1(x)$

$$\begin{aligned} P_1 &= \frac{\int_0^x q(s)\sinh(2i\sqrt{|\lambda_1|}g(s))ds}{2i\sqrt{|\lambda_1|r(x)}\cosh^2(i\sqrt{|\lambda_1|}g(x))} + C_3\frac{\sqrt{r(0)}\cosh^2(C_1)}{\sqrt{r(x)}\cosh^2(i\sqrt{|\lambda|}g(x))} \\ &= \frac{i\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds}{2i\sqrt{|\lambda_1|r(x)}\cos^2(\sqrt{|\lambda_1|}g(x))} + C_3\frac{\sqrt{r(0)}}{\sqrt{r(x)}\cos^2(\sqrt{|\lambda_1|}g(x))} \\ &= \frac{1}{2\sqrt{|\lambda_1|r(x)}\cos^2(\sqrt{|\lambda_1|}g(x))}[\psi \sin(2\sqrt{|\lambda_1|}g(1)) - \int_x^1 q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds]. \end{aligned} \tag{6.2.1}$$

Thus (y_1, P_1) are real-valued even for negative λ_1. However, this presents a new challenge. The denominator of $P_1(x)$ contains $\cos(\sqrt{|\lambda_1|}g(x))$, and if the argument of this function is equal to $\frac{\pi}{2}+\pi n$ the function $P_1(x)$ is undefined. Thus we may require the following condition for the solvability of the system,

$$\sqrt{|\lambda_1|} < \frac{\pi}{2g(1)}, \tag{6.2.2}$$

so that the solution allows the analytic continuation on the segment $x \in [0,1]$. Alternately, we investigate the possibility that for some $x_n \in [0,1]$,

$$\psi \sin(2\sqrt{|\lambda_1|}g(1)) - \int_{x_n}^{1} q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds = 0, \text{ and } \sqrt{|\lambda_1|} = \frac{2n-1}{2g(x_n)}\pi. \tag{6.2.3}$$

Then the limit as $x \to x_n$ of $P_1(x)$ may be finite and $P_1(x)$ may in fact be defined at x_n by continuity. We proceed via L'Hopital's Rule.

$$\begin{aligned}
\lim_{x\to x_n} P_1(x) = &\lim_{x\to x_n} \frac{1}{2\sqrt{|\lambda_1|r(x)}\cos^2(\sqrt{|\lambda_1|}g(x))}\Big[\psi\sin(2\sqrt{|\lambda_1|}g(1)) \\
&- \int_x^1 q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds\Big] \\
= &\lim_{x\to x_n}\frac{1}{2\sqrt{|\lambda_1|r(x)}}\lim_{x\to x_n}\frac{1}{\cos^2(\sqrt{|\lambda_1|}g(x))}\Big[\psi\sin(2\sqrt{|\lambda_1|}g(1)) \\
&- \int_x^1 q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds\Big] \\
=&\frac{1}{2\sqrt{|\lambda_1|r(x_n)}}\lim_{x\to x_n}\frac{q(x)\sin(2\sqrt{|\lambda_1|}g(x))}{-\sqrt{|\lambda_1|r(x)}\sin(2\sqrt{|\lambda_1|}g(x))} \\
=&\frac{q(x_n)}{-2|\lambda_1|r(x_n)}.
\end{aligned} \tag{6.2.4}$$

It would appear then that we have continuity at all $x_n \in [0,1]$ such that (6.2.3) holds true. As for positivity of the design, we have that the denominator of (6.2.1) is positive everywhere inside the region described by (6.2.2). We focus then on the numerator and formulate the positivity condition

$$\psi \sin(2\sqrt{|\lambda_1|}g(1)) - \int_x^1 q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds > 0,\ \forall x \in [0,1]. \tag{6.2.5}$$

Using the definition of $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ given in Def 2.4, we present here a few examples to show that both sets are nonempty. First, we consider for these examples that $\psi > 0$ and then the case $\psi < 0$. We consider the case $\psi = 0$ as one of the exceptional cases and leave its discussion for Ch 7. We restrict our consideration by considering only λ_1 in the range described by (6.2.2).

a) For $q(x) := q_a(x) = 0$, (6.2.5) holds true, i.e. $q_a \in \mathcal{Q}$.

b) Let $q(x) := q_b(x) \in Q_M^-$. As $\sin(2\sqrt{|\lambda_1|}g(x)) > 0$ for $x \in [0,1]$ and (6.2.2), we have that (6.2.5) holds true. i.e. $q_b \in \mathcal{Q}$.

c) Let $q(x) := q_c(x) \in Q_M^+$. Using similar logic to b), (6.2.5) is false for our choice of $q(x)$. i.e. $q_c \in \overline{\mathcal{Q}}$.

Hence, both $\mathcal{Q}, \overline{\mathcal{Q}} \neq \emptyset$ for (4.1.2) with $\lambda_1 < 0$. We determine in this case for $M[P_1]$,

$$\begin{aligned} M[P_1] =& \int_0^1 r(x)\frac{1}{2\sqrt{|\lambda_1| r(x)}\cos^2(\sqrt{|\lambda_1|}g(x))}\Big[\psi\sin(2\sqrt{|\lambda_1|}g(1)) \\ &- \int_x^1 q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds\Big]dx \\ =& \frac{\psi\sin(2\sqrt{|\lambda_1|}g(1))}{2|\lambda_1|}\tan(\sqrt{|\lambda_1|}g(1)) \\ &- \frac{1}{2\sqrt{|\lambda_1|}}\int_0^1 q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds\int_0^s \frac{\sqrt{|\lambda_1| r(x)}}{\cos^2(\sqrt{|\lambda_1|}g(x))}dx \\ =& \frac{1}{|\lambda_1|}[\psi\sin^2(\sqrt{|\lambda_1|}g(1)) - \int_0^1 q(s)\sin^2(\sqrt{|\lambda_1|}g(s))ds]. \end{aligned} \tag{6.2.6}$$

6.2.2. P_2**:** If we consider (4.1.4) above we find that

$$P_2(x) = \frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s))ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x))} = \frac{\int_0^x q(s)\sinh(2i\sqrt{|\lambda_1|}g(s))ds}{2i\sqrt{|\lambda_1| r(x)}\sinh^2(i\sqrt{|\lambda_1|}g(x))}$$

$$= -\frac{\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds}{2\sqrt{|\lambda_1|r(x)}\sin^2(\sqrt{|\lambda_1|}g(x))}. \tag{6.2.7}$$

Here we require that $\sin^2() \neq 0$ along $x \in [0,1]$, so we initially proceed via L'Hopital's Rule

$$\begin{aligned}
\lim_{x\to 0}(6.2.7) &= \lim_{x\to 0} -\frac{\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds}{2\sqrt{|\lambda_1|r(x)}\sin^2(\sqrt{|\lambda_1|}g(x))} \\
&= \lim_{x\to 0}\frac{-1}{2\sqrt{|\lambda_1|r(x)}}\lim_{x\to 0}\frac{\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds}{\sin^2(\sqrt{|\lambda_1|}g(x))} \\
&=\frac{-1}{2\sqrt{|\lambda_1|r(0)}}\lim_{x\to 0}\frac{q(x)\sin(2\sqrt{|\lambda_1|}g(x))}{\sqrt{|\lambda_1|r(x)}\sin(2\sqrt{|\lambda_1|}g(x))} \\
&= -\frac{q(0)}{2\lambda_1 r(0)}.
\end{aligned}$$

Our first requirement then is that $q(0) < 0$. Our second requirement is that $|\lambda_1| < (\frac{\pi}{g(1)})^2$.

We have for $M[P_2]$ that

$$\begin{aligned}
[M[P_2] = \frac{1}{2|\lambda_1|}\bigg[&\cot(\sqrt{|\lambda_1|}g(1))\int_0^1 q(x)\sin(2\sqrt{|\lambda_1|}g(x))dx \\
&- \int_0^1 q(x)\sin(2\sqrt{|\lambda_1|}g(x))\cot(\sqrt{|\lambda_1|}g(x))dx\bigg].
\end{aligned}$$

Remark 6.1. It is immediately obvious that constants ϕ (4.0.1) and ψ (4.0.2) above will remain real numbers. To determine this for ζ defined by (4.0.3) we prove that ζ is purely imaginary.

$$\begin{aligned}
\zeta &= -\frac{\phi + \psi\cosh(2i\sqrt{|\lambda_1|}g(1)) - \int_0^1 q(s)\cosh(2i\sqrt{|\lambda_1|}g(s))ds}{\psi\sinh(2i\sqrt{|\lambda_1|}g(1)) - \int_0^1 q(s)\sinh(2i\sqrt{|\lambda_1|}g(s))ds} \\
&= -\frac{\phi + \psi\cos(2\sqrt{|\lambda_1|}g(1)) - \int_0^1 q(s)\cos(2\sqrt{|\lambda_1|}g(s))ds}{i\psi\sin(2\sqrt{|\lambda_1|}g(1)) - i\int_0^1 q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds} \\
&=i\frac{\phi + \psi\cos(2\sqrt{|\lambda_1|}g(1)) - \int_0^1 q(s)\cos(2\sqrt{|\lambda_1|}g(s))ds}{\psi\sin(2\sqrt{|\lambda_1|}g(1)) - \int_0^1 q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds} \\
&=i\zeta'.
\end{aligned}$$

Further, we define the constants ζ' and z' by

$$\zeta' := -i\zeta,$$

$$z' := \frac{1}{2}\coth^{-1}(\zeta').$$

6.2.3. P_3**:** Take C_1 and C_3 to be the constants determined in (4.1.7) and (4.1.8) above. From (4.1.7) we have that

$$C_1 = \frac{1}{2}\coth^{-1}(\zeta) = \frac{1}{2}\coth^{-1}(i\zeta') = \frac{i}{2}\cot^{-1}(\zeta').$$

We have from (4.1.9) that

$$\begin{aligned} y_1 =& \frac{1}{\sqrt{\lambda_1 k}}\sinh(\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta)) \\ =& \frac{1}{\sqrt{|\lambda_1|k}}\sin(\sqrt{|\lambda_1|}g(x) + \frac{1}{2}\cot^{-1}(\zeta')). \end{aligned}$$

We have also from (4.1.10),

$$\begin{aligned} P_3 =& \frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s) + \coth^{-1}(\zeta))ds}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta))} - \frac{\phi\sinh(\coth^{-1}(\zeta))}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta))} \\ =& \frac{\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s) + \cot^{-1}(\zeta'))ds}{2\sqrt{|\lambda_1|r(x)}\cos^2(\sqrt{|\lambda_1|}g(x) + \frac{1}{2}\cot^{-1}(\zeta'))} - \frac{\phi\sin(\cot^{-1}(\zeta'))}{2\sqrt{|\lambda_1|r(x)}\cos^2(\sqrt{|\lambda_1|}g(x) + \frac{1}{2}\cot^{-1}(\zeta'))}. \end{aligned}$$

Thus we have real-valued y_1 and P_3 for $\lambda_1 < 0$. We arrive at the positivity condition

$$\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s) + \cot^{-1}(\zeta'))ds - \phi\sin(\cot^{-1}(\zeta')) > 0, \ \forall x \in [0,1], \tag{6.2.8}$$

as well as the solvability condition

$$|\sqrt{|\lambda_1|}g(x) + \frac{1}{2}\cot^{-1}(\zeta')| < \frac{\pi}{2}, \ \forall x \in [0,1]. \tag{6.2.9}$$

It is also of course possible that for some $x_n \in [0,1]$ with $n \in \mathbb{N}$ we have that

$$\begin{cases} \int_0^{x_n} q(s)\sin(2\sqrt{|\lambda_1|}g(s) + \cot^{-1}(\zeta'))ds - \phi\sin(\cot^{-1}(\zeta')) = 0, \\ |\sqrt{|\lambda_1|}g(x) + \frac{1}{2}\cot^{-1}(\zeta')| = \frac{2n-1}{2}\pi. \end{cases} \tag{6.2.10}$$

This is similar to (6.2.1) where $P_1(x)$ may in fact be defined at x_n by continuity. We proceed via L'Hopital's Rule.

$$\lim_{x\to x_n} P_3(x) = \lim_{x\to x_n} \frac{\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s) + \cot^{-1}(\zeta'))ds - \phi\sin(\cot^{-1}(\zeta'))}{2\sqrt{|\lambda_1|r(x)}\cos^2(\sqrt{|\lambda_1|}g(x) + \frac{1}{2}\cot^{-1}(\zeta'))}$$

$$
\begin{aligned}
&= \lim_{x\to x_n} \frac{1}{2\sqrt{|\lambda_1|r(x)}} \lim_{x\to x_n} \frac{\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s)+\cot^{-1}(\zeta'))ds - \phi\sin(\cot^{-1}(\zeta'))}{\cos^2(\sqrt{|\lambda_1|}g(x)+\frac{1}{2}\cot^{-1}(\zeta'))}\\
&=\frac{1}{2\sqrt{|\lambda_1|r(x_n)}} \lim_{x\to x_n} \frac{q(x)\sin(2\sqrt{|\lambda_1|}g(x)+\cot^{-1}(\zeta'))}{-\sqrt{|\lambda_1|}r(x)\sin(2\sqrt{|\lambda_1|}g(x)+\cot^{-1}(\zeta'))}\\
&=\frac{q(x_n)}{-2|\lambda_1|r(x_n)}.
\end{aligned}
$$

It would appear then that we have continuity at all $x_n \in [0,1]$ such that (6.2.10) holds true. We provide a few examples to demonstrate that both $\mathcal{Q}, \overline{\mathcal{Q}}$ are nonempty. All examples assume that we operate in the range described by (6.2.9).

a) Let $q(x) := q_a(x) \in Q_M^+$. As $\sin(2\sqrt{|\lambda_1|}g(x)+\cot^{-1}(\zeta')) > 0$ for $x \in [0,1]$ and (6.2.9), we have that (6.2.8) holds true, i.e. $q_a \in \mathcal{Q}$.

b) Let $q(x) := q_b(x) \in Q_M^-$. Using similar logic to b), we have that (6.2.9) is false for our choice of $q(x)$. i.e. $q_b \in \overline{\mathcal{Q}}$.

Hence, both $\mathcal{Q}, \overline{\mathcal{Q}} \neq \emptyset$ for (4.1.10) with $\lambda_1 < 0$. For $M[P_3]$ in this special case we find

$$
\begin{aligned}
M[P_3] &= \int_0^1 r(x)\Big[\frac{\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s)+2z')ds}{2\sqrt{|\lambda_1|r(x)}\cos^2(\sqrt{|\lambda_1|}g(x)+z')} - \frac{\phi\sin(2z')}{2\sqrt{|\lambda_1|r(x)}\cos^2(\sqrt{|\lambda_1|}g(x)+z')}\Big]dx\\
&=\frac{1}{2|\lambda_1|}[\tan(\sqrt{|\lambda_1|}g(1)+z')(\int_0^1 q(x)\sin(2\sqrt{|\lambda_1|}g(x)+2z')dx - \phi\sin(2z'))\\
&\quad - 2(\int_0^1 q(x)\sin^2(\sqrt{|\lambda_1|}g(x)+z')dx - \phi\sin^2(z'))].
\end{aligned}
$$

We find the above form for $M[P_3]$ by means of integration. It is interesting to note, however, that we achieve the same result by replacing $\sqrt{\lambda_1}$ by $i\sqrt{|\lambda_1|}$ and z by iz' in (4.3.3).

$$
\begin{aligned}
M[P_3] =&\frac{1}{2\lambda_1}[\tanh(\sqrt{\lambda_1}g(1)+z)(\int_0^1 q(x)\sinh(2\sqrt{\lambda_1}g(x)+2z)dx - \phi\sinh(2z))\\
&- 2(\int_0^1 q(x)\sinh^2(\sqrt{\lambda_1}g(x)+z)dx - \phi\sinh^2(z))]\\
=& -\frac{1}{2|\lambda_1|}[\tanh(i\sqrt{|\lambda_1|}g(1)+iz')(\int_0^1 q(x)\sinh(2i\sqrt{|\lambda_1|}g(x)+2iz')dx - \phi\sinh(2iz'))
\end{aligned}
$$

$$
\begin{aligned}
&- 2(\int_0^1 q(x)\sinh^2(i\sqrt{|\lambda_1|}g(x) + iz')dx - \phi\sinh^2(iz'))]\\
=&-\frac{1}{2|\lambda_1|}[i\tan(\sqrt{|\lambda_1|}g(1) + z')(i\int_0^1 q(x)\sin(2\sqrt{|\lambda_1|}g(x) + 2z')dx - i\phi\sin(2z'))\\
&- 2(-\int_0^1 q(x)\sin^2(\sqrt{|\lambda_1|}g(x) + z')dx + \phi\sin^2(z'))]\\
=&\frac{1}{2|\lambda_1|}[\tan(\sqrt{|\lambda_1|}g(1) + z')(\int_0^1 q(x)\sin(2\sqrt{|\lambda_1|}g(x) + 2z')dx - \phi\sin(2z'))\\
&- 2(\int_0^1 q(x)\sin^2(\sqrt{|\lambda_1|}g(x) + z')dx - \phi\sin^2(z'))].
\end{aligned}
$$

6.2.4. P_6**:** Take C_2 and C_4 to be the constants determined in (4.2.7) and (4.2.8) above above. From (4.2.7) we have that

$$
C_2 = \frac{1}{2}\coth^{-1}(\zeta) = \frac{i}{2}\cot^{-1}(\zeta).
$$

We have from (4.2.9),

$$
\begin{aligned}
y_2 =& \frac{1}{\sqrt{\lambda_1 k}}\cosh(\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta))\\
=& \frac{1}{\sqrt{|\lambda_1|k}}\cos(\sqrt{|\lambda_1|}g(x) + \frac{1}{2}\cot^{-1}(\zeta')).
\end{aligned}
$$

We have also from (4.2.10) that,

$$
\begin{aligned}
P_6 =& \frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s) + \coth^{-1}(\zeta))ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta))} - \frac{\phi\sinh(\coth^{-1}(\zeta))}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta))}\\
=& -\frac{\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s) + \cot^{-1}(\zeta'))ds}{2\sqrt{|\lambda_1|r(x)}\sin^2(\sqrt{|\lambda_1|}g(x) + \frac{1}{2}\cot^{-1}(\zeta'))}\\
&+ \frac{\phi\sin(\cot^{-1}(\zeta'))}{2\sqrt{|\lambda_1|r(x)}\sin^2(\sqrt{|\lambda_1|}g(x) + \frac{1}{2}\cot^{-1}(\zeta'))}.
\end{aligned}
$$

Thus we have real-valued y_2 and P_6 for $\lambda_1 < 0$. We arrive at the positivity condition

$$
\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s) + \cot^{-1}(\zeta'))ds - \phi\sin(\cot^{-1}(\zeta')) < 0,\ \forall x \in [0,1], \tag{6.2.11}
$$

as well as the solvability condition

$$
0 < |\sqrt{|\lambda_1|}g(x) + \frac{1}{2}\cot^{-1}(\zeta')| < \pi,\ \forall x \in [0,1]. \tag{6.2.12}
$$

We have also the possibility that at some $x_n \in [0,1]$ with $n \in \mathbb{N}$ that

$$\begin{cases} \int_0^{x_n} q(s)\sin(2\sqrt{|\lambda_1|}g(s) + \cot^{-1}(\zeta'))ds - \phi\sin(\cot^{-1}(\zeta')) = 0, \\ |\sqrt{|\lambda_1|}g(x) + \frac{1}{2}\cot^{-1}(\zeta')| = (n-1)\pi. \end{cases} \tag{6.2.13}$$

As in our analysis of (4.1.10), $P_6(x)$ may in fact be defined at x_n. We proceed via LHR.

$$\begin{aligned}
\lim_{x\to x_n} P_6(x) &= \lim_{x\to x_n} \frac{\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s) + \cot^{-1}(\zeta'))ds - \phi\sin(\cot^{-1}(\zeta'))}{2\sqrt{|\lambda_1|r(x)}\sin^2(\sqrt{|\lambda_1|}g(x) + \frac{1}{2}\cot^{-1}(\zeta'))} \\
&= \lim_{x\to x_n} \frac{1}{2\sqrt{|\lambda_1|r(x)}} \lim_{x\to x_n} \frac{\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s) + \cot^{-1}(\zeta'))ds - \phi\sin(\cot^{-1}(\zeta'))}{\sin^2(\sqrt{|\lambda_1|}g(x) + \frac{1}{2}\cot^{-1}(\zeta'))} \\
&= \frac{1}{2\sqrt{|\lambda_1|r(x_n)}} \lim_{x\to x_n} \frac{q(x)\sin(2\sqrt{|\lambda_1|}g(x) + \cot^{-1}(\zeta'))}{\sqrt{|\lambda_1|}r(x)\sin(2\sqrt{|\lambda_1|}g(x) + \cot^{-1}(\zeta'))} \\
&= \frac{q(x_n)}{2|\lambda_1|r(x_n)}.
\end{aligned}$$

It would appear then that we have continuity at all $x_n \in [0,1]$ such that (6.2.13) holds true. We provide a few examples to demonstrate that both $\mathcal{Q}, \overline{\mathcal{Q}}$ are nonempty. All examples assume that we operate in the range described by (6.2.12).

a) Let $q(x) := q_a(x) \in Q_M^+$. As $\sin(2\sqrt{|\lambda_1|}g(x) + \cot^{-1}(\zeta')) > 0$ for $x \in [0,1]$ and (6.2.9), we have that (6.2.8) holds true. i.e. $q_a \in \mathcal{Q}$.

b) Let $q(x) := q_b(x) \in Q_M^-$. Using similar logic to a), we have that 6.2.9 is false for our choice of $q(x)$. i.e. $q_b \in \overline{\mathcal{Q}}$.

Hence, both $\mathcal{Q}, \overline{\mathcal{Q}} \neq \emptyset$ for $P_6(x)$ with $\lambda_1 < 0$. For $M[P_6]$ in this special case we find

$$\begin{aligned}
M[P_6] &= \int_0^1 r(x)[-\frac{\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s) + 2z')ds}{2\sqrt{|\lambda_1|r(x)}\sin^2(\sqrt{|\lambda_1|}g(x) + z')} + \frac{\phi\sin(2z')}{2\sqrt{|\lambda_1|r(x)}\sin^2(\sqrt{|\lambda_1|}g(x) + z')}]dx \\
&= \frac{1}{2|\lambda_1|}[\cot(\sqrt{|\lambda_1|}g(1) + z')(\int_0^1 q(x)\sin(2\sqrt{|\lambda_1|}g(x) + 2z')dx - \phi\sin(2z')) \\
&\quad - 2(\int_0^1 q(x)\cos^2(\sqrt{|\lambda_1|}g(x) + z')dx - \phi\cos^2(z'))].
\end{aligned}$$

We find the above form for $M[P_6]$ by means of integration, however, we achieve the same result by replacing $\sqrt{\lambda_1}$ by $i\sqrt{|\lambda_1|}$ and z by iz' in (4.3.4).

$$
\begin{aligned}
M[P_6] =&\frac{1}{2\lambda_1}[\coth(\sqrt{\lambda_1}g(1)+z)(\phi\sinh(2z)-\int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s)+2z)ds)\\
&+2(\int_0^1 q(s)\cosh^2(\sqrt{\lambda_1}g(s)+z)ds-\phi\cosh^2(z))]\\
=&-\frac{1}{2|\lambda_1|}[\coth(i\sqrt{|\lambda_1|}g(1)+iz')(\phi\sinh(2iz')-\int_0^1 q(s)\sinh(2i\sqrt{|\lambda_1|}g(s)+2iz')ds)\\
&+2(\int_0^1 q(s)\cosh^2(i\sqrt{|\lambda_1|}g(s)+iz')ds-\phi\cosh^2(iz'))]\\
=&-\frac{1}{2|\lambda_1|}[-i\cot(\sqrt{|\lambda_1|}g(1)+z')(i\phi\sin(2z')-i\int_0^1 q(s)\sin(2\sqrt{|\lambda_1|}g(s)+2z')ds)\\
&+2(-\int_0^1 q(s)\cos^2(\sqrt{|\lambda_1|}g(s)+z')ds+\phi\cos^2(z'))]\\
=&\frac{1}{2|\lambda_1|}[\cot(\sqrt{|\lambda_1|}g(1)+z')(\int_0^1 q(s)\sin(2\sqrt{|\lambda_1|}g(s)+2z')ds-\phi\sin(2z'))\\
&-2(\int_0^1 q(s)\cos^2(\sqrt{|\lambda_1|}g(s)+z')ds-\phi\cos^2(z'))].
\end{aligned}
$$

CHAPTER 7

Exceptions in ϕ, ψ, ζ, and ζ'

We show that for almost all of the pathological cases of our variables, we may find both suitable and unsuitable $p(x)$.

7.1. Exceptions in ψ

We consider the case where $\beta_2 + \lambda_1 \beta_2' = 0$ in the limiting sense as $\psi \to \infty$. The sign of this ψ is immaterial as we will show.

7.1.1. P_1**:** Take P_1 as in (4.1.2), then

$$\lim_{\psi \to \pm\infty} P_1 = \pm\infty \implies P_1 \notin \mathcal{P}.$$

7.1.2. P_3**:** Observe that as $\psi \to \infty$, from (4.0.3)

$$\lim_{\psi \to \infty} \zeta = -\coth(2\sqrt{\lambda_1} g(1)).$$

Then from (4.1.7) we have,

$$C_1 = \frac{1}{2} \coth^{-1}(\zeta) = -\sqrt{\lambda_1} g(1).$$

Subsequently from (4.1.8),

$$C_3 = \frac{\phi}{\sqrt{\lambda_1 r(0)}} \tanh(\sqrt{\lambda_1} g(1)).$$

We conclude that as $\psi \to \infty$,

$$\lim_{\psi \to \infty} P_3 = \frac{-\int_0^x q(s) \sinh(2\sqrt{\lambda_1}(g(1) - g(s)))ds}{2\sqrt{\lambda_1 r(x)} \cosh^2(\sqrt{\lambda_1}(g(1) - g(x)))} + \frac{\phi \sinh(2\sqrt{\lambda_1} g(1))}{2\sqrt{\lambda_1 r(x)} \cosh^2(\sqrt{\lambda_1}(g(1) - g(x)))}.$$

At $x = 0$ the first term in the function vanishes, so we require at least that $\phi > 0$ as well as $q(x) < 0$ or to take only very small positive values. Thus both $p \in \mathcal{P}$ and $p \notin \mathcal{P}$ are possible.

7.1.3. P_6**:** Similar to our analysis of $P_3(x)$, from (4.2.7) we have

$$C_2 = \frac{1}{2}\coth^{-1}(\zeta) = -\sqrt{\lambda_1}g(1).$$

Subsequently from (4.2.8),

$$C_4 = \frac{\phi}{\sqrt{\lambda_1 r(0)}}\coth(\sqrt{\lambda_1}g(1)).$$

We conclude that as $\psi \to \infty$,

$$\lim_{\psi\to\infty} P_6 = \frac{-\int_0^x q(s)\sinh(2\sqrt{\lambda_1}(g(1)-g(x)))ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}(g(1)-g(x)))} + \frac{\phi\sinh(2\sqrt{\lambda_1}g(1))}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}(g(1)-g(x)))}.$$

Here we see a potential discontinuity at $x = 1$. We then require $q(x)$ such that

$$-\int_0^1 q(s)\sinh(2\sqrt{\lambda_1}(g(1)-g(x)))ds + \phi\sinh(2\sqrt{\lambda_1}g(1)) = 0.$$

We proceed via L'Hopital's Rule,

$$\begin{aligned}
\lim_{x\to1} P_6 =& \lim_{x\to1}\frac{-\int_0^x q(s)\sinh(2\sqrt{\lambda_1}(g(1)-g(x)))ds + \phi\sinh(2\sqrt{\lambda_1}g(1))}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}(g(1)-g(x)))},\\
=& \lim_{x\to1}\frac{1}{2\sqrt{\lambda_1 r(x)}}\lim_{x\to1}\frac{\frac{d}{dx}(-\int_0^x q(s)\sinh(2\sqrt{\lambda_1}(g(1)-g(x)))ds + \phi\sinh(2\sqrt{\lambda_1}g(1)))}{\frac{d}{dx}(\sinh^2(\sqrt{\lambda_1}(g(1)-g(x))))},\\
=& \lim_{x\to1}\frac{1}{2\sqrt{\lambda_1 r(x)}}\lim_{x\to1}\frac{-q(x)\sinh(2\sqrt{\lambda_1}(g(1)-g(x)))}{\sqrt{\lambda_1 r(x)}\sinh(2\sqrt{\lambda_1}(g(1)-g(x)))},\\
=& \frac{-q(1)}{2\lambda_1 r(1)}.
\end{aligned}$$

Thus we may guarantee continuity P_6 under these conditions, but to ensure the positivity of the design we further require $q(1) < 0$. Therefore we may have both $p(x) \in \mathcal{P}$ and $p(x) \notin \mathcal{P}$.

We also consider the case $\beta_1 + \lambda_1\beta_1' = 0$, i.e. $\phi = 0$.

7.1.4. P_1**:** Take P_1 as in (4.1.2), then

$$\psi = 0 \implies P_1(x) = -\frac{1}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x))}\int_x^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds.$$

Here we see that at $x = 1$, $P_1(x) = 0$. We conclude that there is no design under these conditions.

7.1.5. P_3 **and** P_6**:** Observe that at $\psi = 0$, from (4.0.3) and (4.0.4)

$$\zeta = \frac{\phi - \int_0^1 q(s)\cosh(2\sqrt{\lambda_1 g(s)})ds}{\int_0^1 q(s)\sinh(2\sqrt{\lambda_1 g(s)})ds},\ z = \frac{1}{2}\coth^{-1}(\zeta).$$

We conclude that while the exact value of these constants will be changed, the overall character of predesigns P_3 and P_6 will remain unchanged. Therefore we may still have both $p(x) \in \mathcal{P}$ and $p(x) \notin \mathcal{P}$.

7.2. Exceptional ζ

In this section, we refer to ζ by

$$\zeta := -\frac{\phi + \psi\cosh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\cosh(2\sqrt{\lambda_1}g(s))ds}{\psi\sinh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds} := \frac{\zeta_1}{\zeta_2}.$$

In Ch 5, we discuss the case brought about by $\zeta_2 = 0$ in the sense that $\zeta \to \infty$. It is, however, possible that as $\zeta_2 = 0$, we have also that $\zeta_1 = 0$. This section is devoted to a discussion of this special case, i.e. $\zeta_1 = \zeta_2 = 0$. For a discussion on the main restrictions on $q(x)$ brought on by $\zeta \to \infty$ and the domain of arccoth(ζ), see Ch 5 in full. We examine here a small section of the derivation of ζ found in Ch 4. Upon equating (4.1.5) and (4.1.6), we find,

$$\begin{aligned} -\phi\sinh(2C_1) = &\cosh(2C_1)[\psi\sinh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds] \\ &+ \sinh(2C_1)[\psi\cosh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\cosh(2\sqrt{\lambda_1}g(s))ds] \end{aligned}$$

$$\implies 0 = \sinh(2C_1)[\phi + \psi\cosh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\cosh(2\sqrt{\lambda_1}g(s))ds]$$

$$\implies 0 = \sinh(2C_1) * 0.$$

Hence we may draw no conclusions about C_1 or subsequently C_3. However, we may at least reduce the form of $P_3(x)$ to one with only the constant C_1 using (4.1.5).

$$P_3(x) = \frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s) + 2C_1)ds}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x) + C_1)} - \frac{\phi\sinh(2C_1)}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x) + C_1)}.$$

Here we may make some commentary on the conditions necessary on C_1 to ensure the positivity of the design, but without extensive knowledge of a particular $q(x)$, we may go no further. The dilemma for $P_6(x)$ is similar.

7.3. Exceptional ζ'

In this section, we refer to ζ' by

$$\zeta' := \frac{\phi + \psi\cos(2\sqrt{|\lambda_1|}g(1)) - \int_0^1 q(s)\cos(2\sqrt{|\lambda_1|}g(s))ds}{\psi\sin(2\sqrt{|\lambda_1|}g(1)) - \int_0^1 q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds} = \frac{\zeta_1'}{\zeta_2'}.$$

We examine first the case where $\zeta_2' = 0$ in each **Case 3** and **Case 6**.

7.3.1. P_3**:** The function $\cot^{-1}()$ at infinity is defined as 0, which transforms our representation for $p_1(x)$ for $\lambda_1 < 0$ into

$$p_1(x) = \frac{\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds}{2\sqrt{|\lambda_1|r(x)}\cos^2(\sqrt{|\lambda_1|}g(x))},$$

which is continuous at $x = 0$ but $p_1(0) = 0$. We therefore have no suitable design for this case with these conditions and conclude $p_1 \notin \mathcal{P}$.

7.3.2. P_6**:** Similarly, the function $p_2(x)$ for $\lambda_1 < 0$ is transformed to

$$p_2(x) = \frac{\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds}{2\sqrt{|\lambda_1|r(x)}\sin^2(\sqrt{|\lambda_1|}g(x))},$$

which has a potential discontinuity at $x = 0$. To determine if this is the case, we proceed via L'Hopital's Rule.

$$\begin{aligned}
\lim_{x\to 0} p_2(x) &= \lim_{x\to 0} \frac{\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds}{2\sqrt{|\lambda_1|r(x)}\sin^2(\sqrt{|\lambda_1|}g(x))}, \\
&= \lim_{x\to 0} \frac{1}{2\sqrt{|\lambda_1|r(x)}} \lim_{x\to 0} \frac{\int_0^x q(s)\sin(2\sqrt{|\lambda_1|}g(s))ds}{\sin^2(\sqrt{|\lambda_1|}g(x))}, \\
&= \frac{1}{2\sqrt{|\lambda_1|r(0)}} \lim_{x\to 0} \frac{q(x)\sin(2\sqrt{|\lambda_1|}g(x))}{\sqrt{|\lambda_1|r(x)}\sin(2\sqrt{|\lambda_1|}g(x))}, \\
&= \frac{q(0)}{2|\lambda_1|r(0)}.
\end{aligned}$$

We can therefore guarantee the continuity of p_2 under these conditions, but to ensure positivity we require at least that $q > 0$. Therefore we may have both $p_2 \in \mathcal{P}$ and $p_2 \notin \mathcal{P}$.

Furthermore, we must consider the potential case that both $\zeta_2' = 0$ and $\zeta_1' = 0$. We arrive at the same dilemma as in Sec 6.2 above. We may simply not conclude about C_1 or C_2 respectively if this is the case.

CHAPTER 8

Comparison with the results of [**3**].

Though our study works for any continuous q this section examines the special case $q \equiv 0$ in all cases above and compares those special cases to the findings of [**3**]. Here, the goal is to find equivalence between them. The authors of [**3**] assumed $q \equiv 0$ and we expect to rediscover the findings of that paper here if we make the same assumption.

All Cases: First, we will compare the constants used in [**3**] and the current text which are shared by all cases. Characters and symbols without the hat symbol denote constants derived from the current project, while characters with the hat symbol denote constants derived in [**3**].

$$\phi = \cot\alpha = \hat{\alpha}; \quad \psi = \frac{\beta_1 + \lambda_1\beta_1'}{\beta_2 + \lambda_2\beta_2'} = \hat{B}.$$

$$\begin{aligned}
\zeta &= -\frac{\phi + \psi\cosh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\cosh(2\sqrt{\lambda_1}g(s))ds}{\psi\sinh(2\sqrt{\lambda_1}g(1)) - \int_0^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds}, \\
&= -\frac{\phi + \psi\cosh(2\sqrt{\lambda_1}g(1)) - \int_0^1 0 * \cosh(2\sqrt{\lambda_1}g(s))ds}{\psi\sinh(2\sqrt{\lambda_1}g(1)) - \int_0^1 0 * \sinh(2\sqrt{\lambda_1}g(s))ds}, \\
&= -\frac{\phi + \psi\cosh(2\sqrt{\lambda_1}g(1))}{\psi\sinh(2\sqrt{\lambda_1}g(1))} = -\frac{\psi}{\psi}\frac{\phi + \psi\cosh(2\sqrt{\lambda_1}g(1))}{\psi\sinh(2\sqrt{\lambda_1}g(1))}, \\
&= -\frac{\phi/\psi + \cosh(2\sqrt{\lambda_1}g(1))}{\sinh(2\sqrt{\lambda_1}g(1))} = (\hat{\zeta})^{-1}.
\end{aligned}$$

These ζ do not match perfectly, but we will arrive at the same result. Note that

$$\coth^{-1}(\zeta) = \coth^{-1}((\hat{\zeta})^{-1}) = \tanh^{-1}(\hat{\zeta}).$$

The most general forms of p_1, p_2 in our study correspond to the previous work as follows,

$$\begin{aligned}
p_1 &= \frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s)+2C_1)ds}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x)+C_1)} + C_3\frac{\sqrt{r(0)}\cosh^2(C_1)}{\sqrt{r(x)}\cosh^2(\sqrt{\lambda_1}g(x)+C_1)},\\
&=\frac{\int_0^x 0*\sinh(2\sqrt{\lambda_1}g(s)+2C_1)ds}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x)+C_1)} + C_3\frac{\sqrt{r(0)}\cosh^2(C_1)}{\sqrt{r(x)}\cosh^2(\sqrt{\lambda_1}g(x)+C_1)},\\
&=C_3\frac{\sqrt{r(0)}\cosh^2(C_1)}{\sqrt{r(x)}\cosh^2(\sqrt{\lambda_1}g(x)+C_1)} = \hat{p}_1.\\
p_2 &= \frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s)+2C_2)ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x)+C_2)} + C_4\frac{\sqrt{r(0)}\sinh^2(C_2)}{\sqrt{r(x)}\sinh^2(\sqrt{\lambda_1}g(x)+C_2)},\\
&=\frac{\int_0^x 0*\sinh(2\sqrt{\lambda_1}g(s)+2C_2)ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x)+C_2)} + C_4\frac{\sqrt{r(0)}\sinh^2(C_2)}{\sqrt{r(x)}\sinh^2(\sqrt{\lambda_1}g(x)+C_2)},\\
&=C_4\frac{\sqrt{r(0)}\sinh^2(C_2)}{\sqrt{r(x)}\sinh^2(\sqrt{\lambda_1}g(x)+C_2)} = \hat{p}_2.
\end{aligned}$$

Predesign 1 and 4:

$$\begin{aligned}
P_1 =&\frac{1}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x))}[\psi\sinh(2\sqrt{\lambda_1}g(1)) - \int_x^1 q(s)\sinh(2\sqrt{\lambda_1}g(s))ds],\\
=&\frac{1}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x))}[\psi\sinh(2\sqrt{\lambda_1}g(1)) - \int_x^1 0*\sinh(2\sqrt{\lambda_1}g(s))ds],\\
=&\frac{\psi\sinh(2\sqrt{\lambda_1}g(1))}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x))} = \hat{P}_1.
\end{aligned}$$

We observe that the authors of [**3**] found no result for P_4.

Predesign 2 and 5:

$$P_2 =\frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s))ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x))} = \frac{\int_0^x 0*\sinh(2\sqrt{\lambda_1}g(s))ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x))} = 0.$$

We observe that the authors of [**3**] found no result for P_2 or P_5. We infer now that this is because if $q \equiv 0$, then our solution vanishes entirely.

Predesign 3:

$$P_3 =\frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s)+\coth^{-1}(\zeta))ds}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x)+\frac{1}{2}\coth^{-1}(\zeta))} - \frac{\phi\sinh(\coth^{-1}(\zeta))}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x)+\frac{1}{2}\coth^{-1}(\zeta))},$$

$$
\begin{aligned}
=&\frac{\int_0^x 0 * \sinh(2\sqrt{\lambda_1}g(s) + \coth^{-1}(\zeta))ds}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta))} - \frac{\phi\sinh(\coth^{-1}(\zeta))}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta))},\\
=&-\frac{\phi\sinh(\coth^{-1}(\zeta))}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta))},\\
=&-\frac{\phi\sinh(\tanh^{-1}(\hat{\zeta}))}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}g(x) + \frac{1}{2}\tanh^{-1}(\hat{\zeta}))} = \hat{P}_3.
\end{aligned}
$$

Predesign 6:

$$
\begin{aligned}
P_6 =&\frac{\int_0^x q(s)\sinh(2\sqrt{\lambda_1}g(s) + \coth^{-1}(\zeta))ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta))} - \frac{\phi\sinh(\coth^{-1}(\zeta))}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta))},\\
=&\frac{\int_0^x 0 * \sinh(2\sqrt{\lambda_1}g(s) + \coth^{-1}(\zeta))ds}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta))} - \frac{\phi\sinh(\coth^{-1}(\zeta))}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta))},\\
=&-\frac{\phi\sinh(\coth^{-1}(\zeta))}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x) + \frac{1}{2}\coth^{-1}(\zeta))},\\
=&-\frac{\phi\sinh(\tanh^{-1}(\hat{\zeta}))}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}g(x) + \frac{1}{2}\tanh^{-1}(\hat{\zeta}))} = \hat{P}_6.
\end{aligned}
$$

Since $p(x)$ and $r(x)$ in the calculations above match, we have that the integral of their product, $M[p]$, matches as well. We conclude that the results of this section of our work match the results of the corresponding section of [**3**] with the condition that $q \equiv 0$. However, our study goes further with the discussion of design versus predesign and with the additional results at P_2, P_4, and P_5.

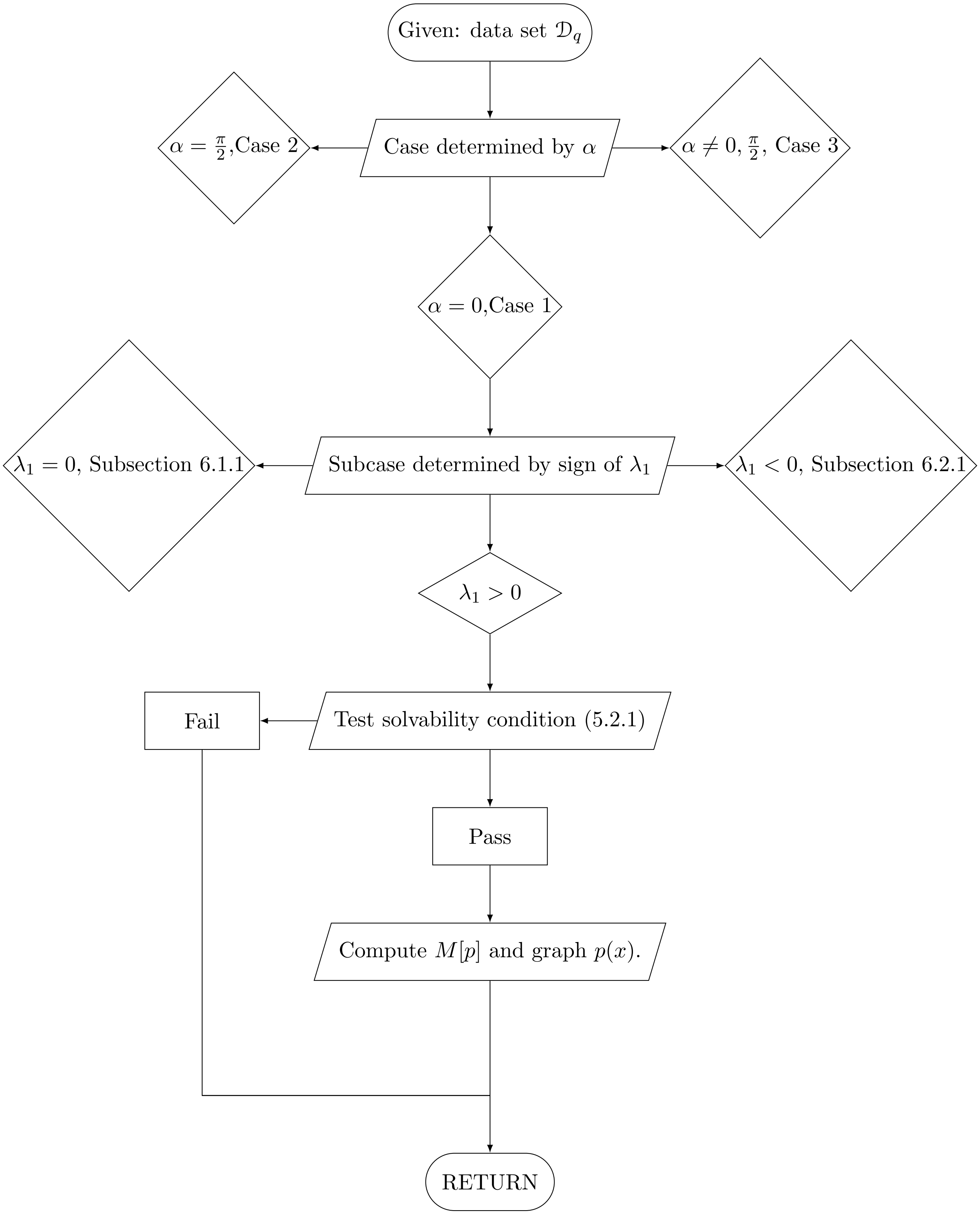
CHAPTER 9

Flowchart and Numerical Examples

We find it illustrative to include a flowchart describing the use of our solvability conditions for a given data set as well as a few examples of $p(x)$ and $M[p]$ for a given data set. We find that it would not be useful to include here the full text of the Python code used to generate the graphs of $p(x)$ and values for $M[p]$ due to the length of the algorithm. However, for the full text of the code used, see [**25**].

9.1. Flowchart

Our previous analysis shows that we have many cases and subcases to consider and that they are branching in regard to the region of α and the positivity of λ_1. We see that we may have $p \in \mathcal{P}$ or $p \in \overline{\mathcal{P}}$ depending on the particulars of the data set $\mathcal{D}$. We use a flowchart to describe a code that allows us to for the given functions $\{q, r\}$ and numbers $\{\alpha, \beta_1, \beta_1', \beta_2, \beta_2', \lambda_1\}$: a) check conditions of solvability; b) plot the function $p(x)$; and c) calculate the value of functional $M[p]$.


Given: data set $\mathcal{D}_q$
$\alpha = \frac{\pi}{2}$,Case 2
Case determined by α
$\alpha \neq 0, \frac{\pi}{2}$, Case 3
$\alpha = 0$,Case 1
$\lambda_1 = 0$, Subsection 6.1.1
Subcase determined by sign of λ_1
$\lambda_1 < 0$, Subsection 6.2.1
$\lambda_1 > 0$
Fail
Test solvability condition (5.2.1)
Pass
Compute $M[p]$ and graph $p(x)$.
RETURN

9.2. Numerical Examples

To further illustrate the nature of our critical functions $p(x)$, we present several numerical examples of potential designs which do (do not) meet the solvability conditions we have described.

(1) $\mathcal{D}_q := \{q(x) = 0, r(x) = 1, \alpha = 0, \beta_1 = 0, \beta_1' = 1, \beta_2 = 1, \beta_2' = 0, \lambda_1 = \lambda\}$. This was the system that Turner studied and our formula (4.1.2) returns the same result. Namely, for our critical $p(x)$ we find

$$p(x) = \frac{\sqrt{\lambda_1}\sinh(2\sqrt{\lambda_1})}{2\cosh^2(\sqrt{\lambda_1}x)},$$

and for our mass we find

$$M[p(x)] = \sqrt{\lambda_1}\sinh^2(\sqrt{\lambda_1}).$$

If we go further and assign a numerical value to λ_1, say $\lambda_1 = 2$, we can present a numerical value for $M[p]$ and the graph of $p(x)$ (Fig 9.1).

(2) $\mathcal{D}_q := \{q(x) = -x^2, r(x) = 1, \alpha = 0, \beta_1 = 0, \beta_1' = 1, \beta_2 = 1, \beta_2' = 0, \lambda_1 = 2\}$. Here we introduce a nonzero $q(x)$ to Turner's work and return a design and mass (Fig 9.2).

(3) $\mathcal{D}_q := \{q(x) = x + 10, r(x) = 1, \alpha = 0, \beta_1 = 0, \beta_1' = 1, \beta_2 = 1, \beta_2' = 0, \lambda_1 = 2\}$. In this example, our data fails to meet the requirements of solvability condition (5.2.1) and the program returns a negative mass.

(4) Let us retain the same data set as in (2), but vary ψ from this data set slightly to illustrate how we may radically alter $p(x)$ and $M[p]$ by breaking our solvability conditions. We include two graphs (Fig 9.3 and Fig 9.4), which illustrate this idea. In both figures we multiply the ψ given in the data set by a small scalar c. In Fig

9.3, we take small variations of positive ψ with $c \in [0.5, 1.5]$, while not breaking the solvability conditions on our $p(x)$. We consequently observe very little deformation of the curve. However, in Fig 9.4 we vary between $c \in [-1, 1]$. These negative $c\psi$ do violate our solvability conditions and, consequently, show a dramatic deformation of the graph.

(5) $\mathcal{D}_q := \{q(x) = -x, r(x) = 1 + 2x, \alpha = \frac{\pi}{2}, \beta_1 = 1, \beta_1' = 5, \beta_2 = 1, \beta_2' = 2, \lambda_1 = 2\}$. In this example, our data fails our solvability condition that $q(x) > 0$ along the domain and the program returns a negative mass.

(6) $\mathcal{D}_q := \{q(x) = r(x) = x^2 + x + 1, \alpha = \frac{\pi}{2}, \beta_1 = 1, \beta_1' = 5, \beta_2 = 1, \beta_2' = 2, \lambda_1 = 2\}$. Notably, this example would not return a nonzero result under the earlier assumption (see [**3**]) that $q \equiv 0$. Here, however, we meet the solvability condition and return a positive mass (Fig 9.5).

(7) $\mathcal{D}_q := \{q(x) = -x, r(x) = 1 + 2x, \alpha = \frac{\pi}{4}, \beta_1 = 1, \beta_1' = 10, \beta_2 = 5, \beta_2' = 10, \lambda_1 = 2\}$. At $\alpha \notin \{0, \frac{\pi}{2}\}$, our optimal $p(x)$ is $P_3(x)$. This data set meets our solvability conditions and our algorithm returns a positive mass (Fig 9.6).

(8) $\mathcal{D}_q := \{q(x) = -x, r(x) = 1 + 2x, \alpha = 3\frac{\pi}{4}, \beta_1 = 1, \beta_1' = 10, \beta_2 = 5, \beta_2' = 10, \lambda_1 = 2\}$. Here, all of our data is the same as in the previous example (7) except we change α from $\frac{\pi}{4}$ to $3\frac{\pi}{4}$. As a result, we are not able to meet our solvability conditions and our algorithm does not return a positive mass.

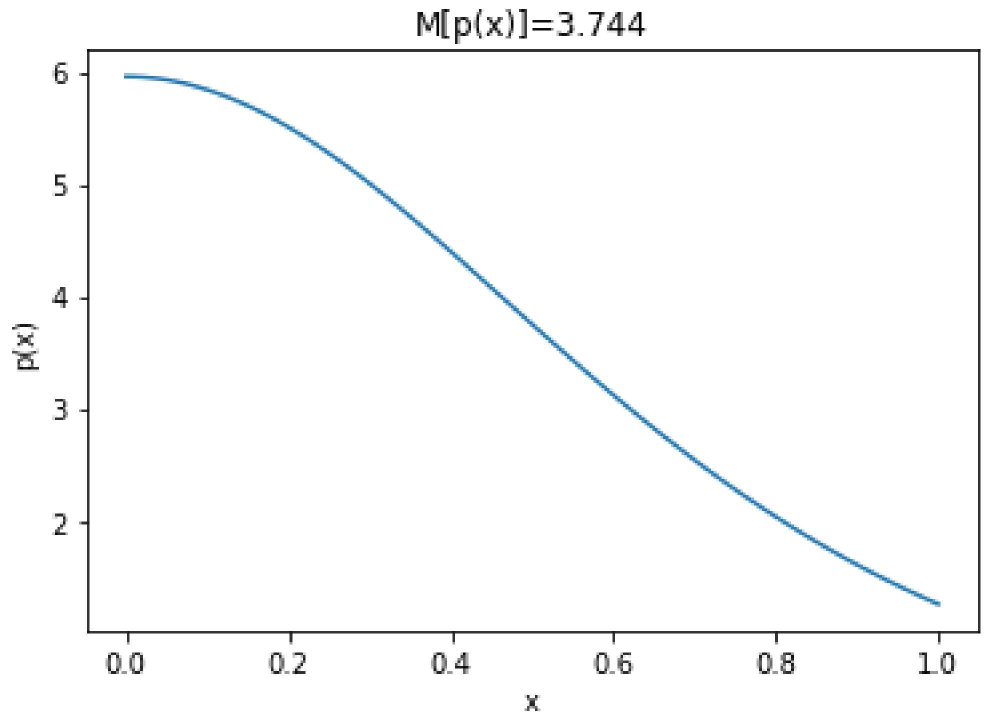


FIGURE 9.1 Example 9.2.1

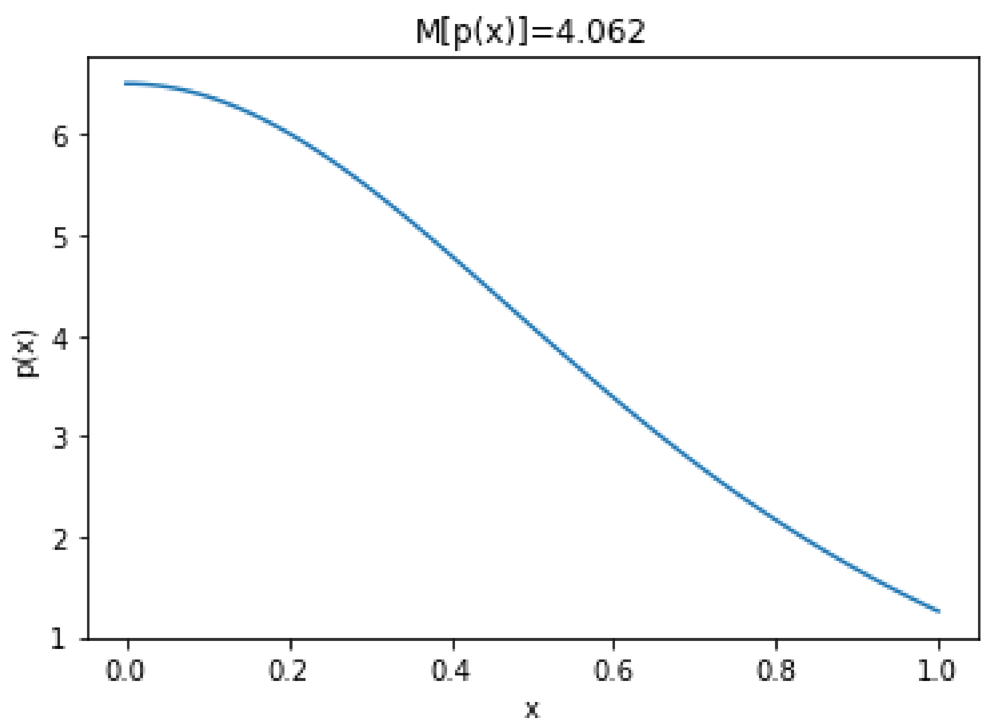


FIGURE 9.2 Example 9.2.2

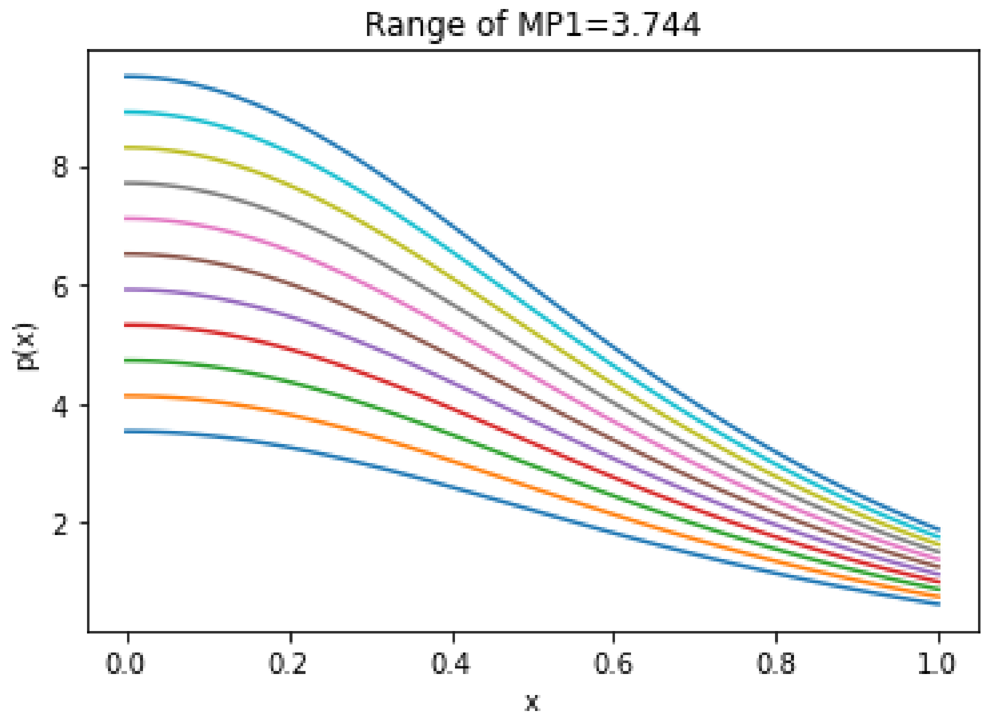


FIGURE 9.3 Example 9.2.4a

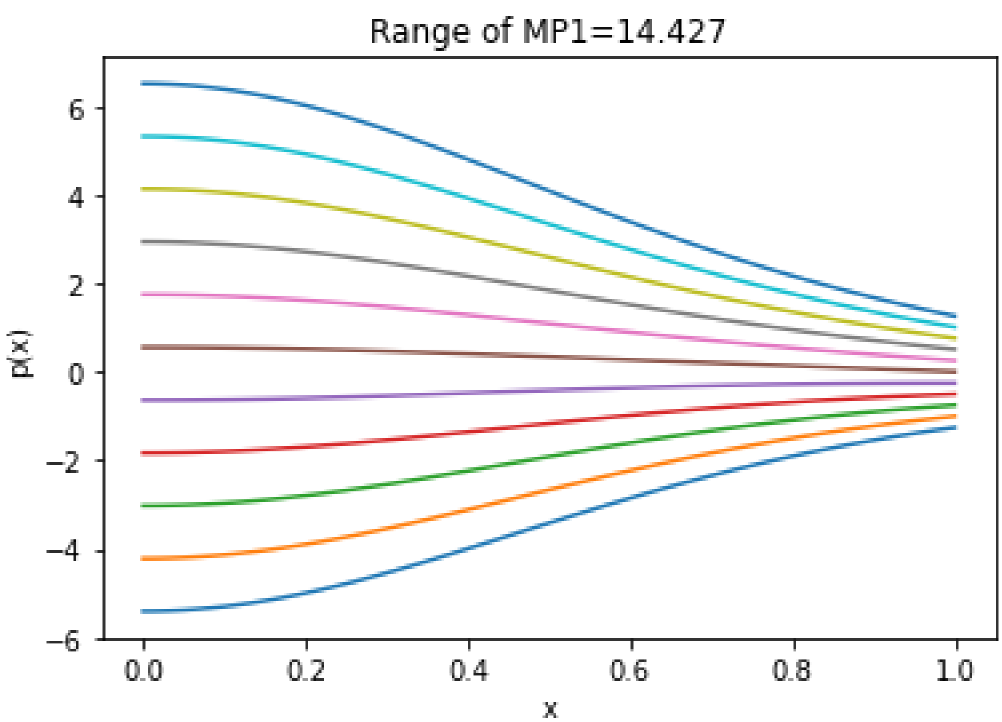


FIGURE 9.4 Example 9.2.4b

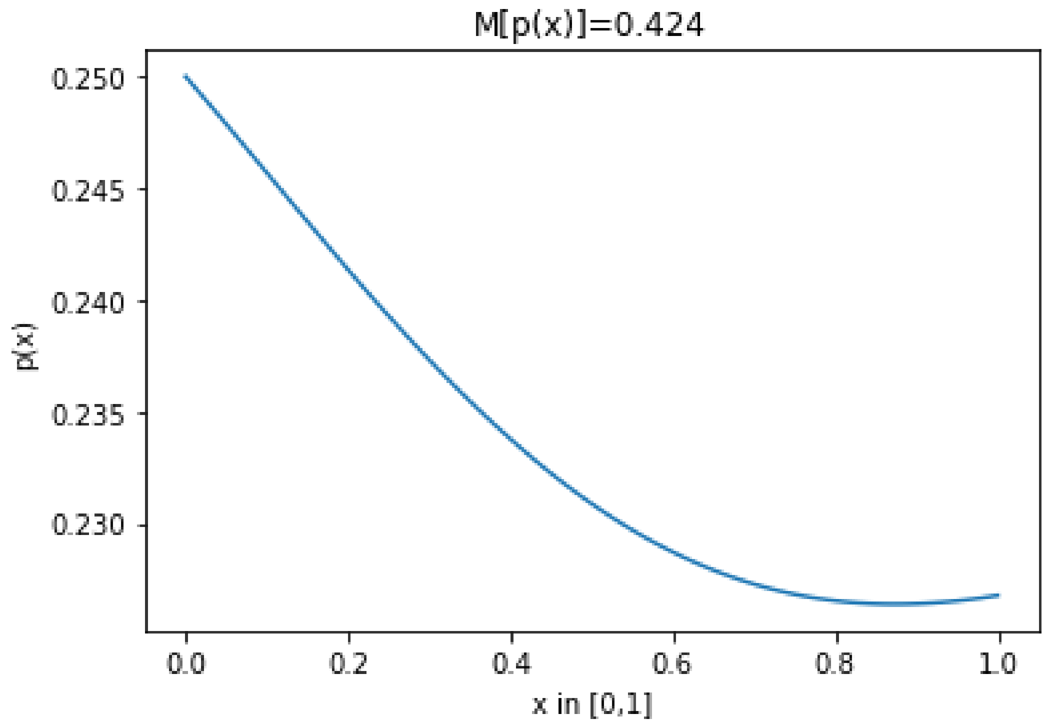


Figure 9.5 Example 9.2.6

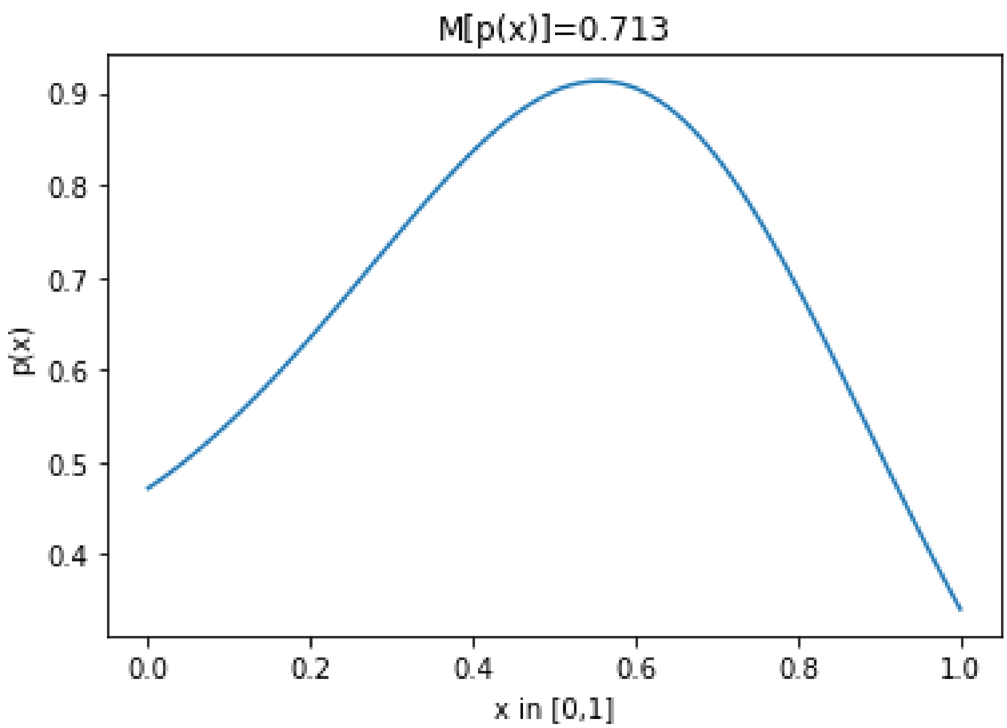


Figure 9.6 Example 9.2.7

CHAPTER 10

Conclusion and Continuing Work

We consider an S-L problem with the spectral parameter in the boundary conditions and solve the optimization problem for a functional that is similar to a mass functional in corresponding engineering problems. We generalize the results of the paper [**3**], which was in turn a generalization of [**6**] and [**5**], which dealt with the optimization of real engineering models. Earlier works made the simplifying assumption that $q \equiv 0$ while we make no such assumption. We similarly to previous work assume that the eigenparameter, i.e. the minimal eigenvalue, is given and positive. We then proceed based on a calculus of variations philosophy. The functional we introduce appears from the mass functional but also takes the S-L problem into consideration. Evaluation of the first variation of the functional and equating it to zero in accordance with the Fundamental Lemma of Calculus of Variations results in a system of equations we may solve explicitly for $y(x)$. We come to a nonlinear ODE that we solve explicitly for $p(x)$, though the solutions contain arbitrary constants and quadrature. As a result, we find points $\{y_j, p_j\}$ such that the first variation of our functional vanishes. The constants are found with the help of our boundary conditions. We finally come up with the set of predesigns by which we understand our solutions which have the potentiality of producing a guaranteed positive mass. A predesign which can satisfy such a guarantee we refer to as a design, and indeed our next goal is to determine whether any of these aforementioned solutions are actually designs. To this end we construct a series of what we refer to as solvability conditions on each predesign to ensure that they meet our criteria.

We finally find the solvability conditions on the designs and the corresponding values of the mass functional explicitly. The conditions that guarantee that a predesign is actually a design appear to be very sensitive to the set of parameters of the problem (data set). In particular, we show that it is possible under certain circumstances to perturbate the data set slightly and either spoil a design or transform an unsuitable predesign into one which satisfies our solvability conditions. We further introduce the lemma that all of our predesigns are capable of producing designs for a proper data set. All aforementioned considerations have been made under the assumption that the given eigenvalue is positive. We further extend the aforementioned formulas to the case when the eigenvalue is nonpositive. We study how designs vary for the limiting cases of the parameters. The results of the study are compared with the results of [**3**] and we are able to determine that our results reduce to the results of both [**3**] and [**5**] under the proper assumptions on the data set. As a result of the aforementioned instability combined with the branching of the cases of our problem, we find that it is helpful to create an algorithm that allows, for the given data set, to identify a predesign or design, and evaluate the corresponding mass functional. We suggest a flowchart that describes the algorithm and provide a link to the code of the algorithm as well as several examples of using the algorithm at work. In each example, we assume the data set to be given and show the result. The results vary. It may be that solution DNE. If it exists, we find the corresponding mass functional as well as graph the corresponding $p(x)$ on the interval $[0, 1]$. We provide several examples that demonstrate all possible outcomes of optimization; i.e. nonexistence, existence of predesign that is not in the desired class, the example given in Turner [**5**], an example where the approach given in [**3**] (i.e. $q \equiv 0$) does not return results, an example of the instability of the predesigns and the infinite closeness between data sets that produce designs and those that do not. We may claim that the presence of the term $q(x)$ in the differential equation changes the results quantitatively, i.e we have the new designs not present in earlier work on the subject. In Taylor [**6**] and Turner [**5**], which is the simplest form of the 2017 model [**3**] where $q \equiv 0$, the expression for the optimal $p(x)$ is explicit. In our more general model, with relaxed assumptions on $q(x)$, the expression for the optimal $p(x)$ is also explicit but contains quadratures.

We now present a list of problems that represent the natural continuation of our findings:

a) We fix λ_1 and minimize $M[p]$ - but we may reverse the problem, i.e. for the given $M[p]$, maximize λ_1. The philosophy of calculus of variations shows that generally speaking, the critical functions will be the same, but it is not obvious, i.e. it might be easier to do calculations independently than to justify the result through the calculus of variations.

b) In [**3**] the authors also considered the optimization problem on graphs. We may do the same but go further - introducing $q \neq 0$ and our language of predesign vs design.

c) Repeat our work, but on a disk instead of on a string. This would represent a shift from ODE to PDE.

d) We may generalize our problem to the case when the spectral parameter appears in both boundary conditions.

The following problems were suggested to the author by the members of the UTC Department of Mathematics during the presentation of the preliminary results at the Math Colloquium in May of 2022.

a) Investigate time dependency and higher dimensions (recommended by Dr. Cox).

b) Think about whether it is possible to use the results in the model of quantum physics (recommended by Dr. Nichols). Actually, we have briefly surveyed such a paper where the S-L problem with the spectral parameter in the boundary conditions appears. However, a further search of the literature is required here.

c) Consider the S-L for higher-order DE and think about the necessity of y'''s existence since in the literature it is not guaranteed y'' exists (recommended by Dr. Kong).

d) Add a numerical simulation (recommended by Dr. Wang). Actually, this is done in Ch 9.

e) Perfect the code to publish via GitHub and support our findings with numerical experiments. (recommended by Dr. Gao). This is also done in Ch 9.

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