APPLICATION OF AN OPTIMIZATION TECHNIQUE TO THE CHATTANOOGA AREA REGIONAL TRANSPORTATION AUTHORITY (CARTA) BUS SYSTEM

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ABSTRACT

The purpose of this project is to apply a mathematical model and optimization technique to the Chattanooga Area Regional Transportation Authority (CARTA).

A survey of recent urban transportation models was completed and the model demand most specific and applicable was chosen. A computer technique of sensitivity analysis was used in order to complete the model. A comparison of the analytical results with the current bus system was made.

The results obtained in this project provide a framework in which CARTA may plan for more efficient operations.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

American cities have encountered problems with urban public transportation. Privates companies provided services in the past, but within the past 10 years costs escalated to the point that efficient services could no longer be provided at reasonables fares. Public funds have steadily declined during the Reagan Administration. It has therefore become increasingly important for CARTA to more efficiently and economically manage its urban bus services.

Critical elements in a bus system must be identified if minimum levels of service and economical operation are to be achieved. From and economic point of view, one critical factor is to minimize the number of buses required to operate the system. This is not entirely compatible with providing desired levels of service. However, the balance of this opposite demands is one of the most important factors necessary to achieve a well-designed bus system.

The temptation is great to cut parts of a bus system which seem to be nonproductive in terms of passenger boarding. However, these cuts may reduce the overall service to an intolerable levels. As a result, overall system design must be recognize economically, but once designed it needs to operate as a completed system.

The purpose of this project is to apply a mathematical model and optimization technique to CARTA system and recommend changes to improve operations and minimize costs. The parameters over which CARTA has direct control are fares and number of buses, and these are critical in realizing a break-even operation. The break-even goal is basic to a non-profit entity such as CARTA.

The first part of this project was to complete a literature survey of recent urban transportation models and select one which might be modified and applied to CARTA. Results showed Kocur's optimization model to be the most specific and applicable. The model was then modified to reflect CARTA'S goals and parameters.

Computer programs were written to obtain solutions to the model and perform sensitivity analysis. Fixed parameters (constraints) could then be input and variable parameters altered tojudge effects of fare, ridership, etc..

Recommendations were then made as to operation of CARTA's bus system based on optimization of the model.

CHAPTER 2

LITERATURE SURVEY

2.1 Literature Review

Note that 25 papers and books were reviewed on the subject (see Bibliography) with the following selected as most significant :

Dispatching Policies for a Transportation Route, G.F. Newell. Transportation Science. Vol. 5, pp. 91-105. 1971.

This paper shows how to choose the dispatching time t_j, $j = 1, \ldots, n$ in order to minimize the total waiting time of all passengers. It is shown that if the capacity of the vehicles is sufficiently large to serve all waiting passengers and n (number of dispatches) is large, then the optimal flow rate of vehicles and the number of passengers served per vehicle vary with time approximately as the square root of the arrival rate of passengers. If the vehicles have limited capacity, their dispatch schedule will be distorted so that certain vehicles are dispatched as soon as they are full.

Optimum Bus Scheduling, Franz J.M. Salzborn. Transportation Science. Vol. 6, No. 2, pp. 137-148. May 1979.

This paper presents a mathematical investigation of bus scheduling. The problem is to determine the bus departure rate as a function of time when the arrival rate is known. The principal objective is to minimize the number of buses that are needed. A secondary criterion is the minimization of the passenger waiting time, for which a calculus of variation technique is used. Although this paper deals mainly with a single bus route, it also shows how the theory can be extended to the case of a pair of linked bus routes.

Public Transportation Line Position and Headways for Minimum User and System Costs in a Radial Case, Bernard F. Byrne. Transportation Research. Vol. 9, pp.97-102. 1975.

In this paper, a model of a transit system is built in polar coordinates with radial transit lines in order to find the line positions and headways which minimize user (travel time) and operating costs in response to a general population density function. It is found that the optimum line location is related to the population density and the circumferential access. The optimum headway is found to be that which causes user waiting time cost to equal theoperating cost. A method for dertermining the optimum number of lines is developed. It is proven, for optimality, lines should have equal headways (time between succesive buses passing the same point).

Some Issues Relating to the Optimal Design of Bus Routes, G.F. Newell. Transportation Science, Vol. 13, No. 1. 1979

This paper provides a discussion of some issues relating to the design of minimum cost bus routes serving a multiple-origen destination trip. In this paper the author formulates and discusses, but does not solve, a general type of bus design problem in which most of the variables are known. It is desired to design a system that minimizes the cost of transportation for a population of bus travelers.

The author remarks that the selection of an optimal route structure for a network of realistic size is a combinatorial type optimization problem of astronomical proportions. One could not hope to do more than find some heuristic algorithm which may not give good solutions.

<u>The Effects of Network Structure on Reliability of Transit Service, Mark A. Turnquist and Larry A. Bowman. Transportation Research. Vol. 14 B, pp. 79-86. 1980.</u>

This paper describes a set of simulation experiments which have been constructed to investigated the effects on service reliability of several characteristics of network structure in urban bus systems. Principal focus is on the factors which lead to vehicle bunching, and on the effect of network form and route density on transfers.

<u>A Theoretical Travel - Time Model for Flexible - Route buses, O. Adelisi. Transportation Research. Vol.14 B, pp. 319-330. 1980.</u>

In this paper a theoretical model for estimating the expectation and variance of bus running times under a flexibly routed mode of service is proposed. The model is based on a probabilistic concept that adequately accomodates the usual randomness inthe number and location of passengers served during successive vehicle trips. The proposed travel time model confirms the intuitively correct phenomenon that when the concentration of passenger trip-ends is very high, the vehicle-route degenerates into fixes-route in which the buses visit all possible loading points within the service area.

Mathematical Model of an Urban Bus Route, Per-Ake Anderson and Gian Paolo Scalia-Tomba. Transportation Research. Vol. 15 B, No. 4, pp. 249-266. 1981.

This paper provides a mathematical description of an urban bus route in peak hour traffic. "Bus route" is here used in a collective sense as a set of more or less parallel-going subroutes, called service variants. The deterministic and stochastic machanisms of bus operation are analysed, and general models for route structure, boarding and alighting events, link travel times and stop times are formulated. Various measures or goodness of fit are defined for validation and model choice. the models are primarily inteded for use in an interactive simulation program

Service Frecuency, Schedule Reliability and Passenger Wait Times at Transit Stops, Larry A. Bowman and Mark Turnsquist. Transportation Research. Vol. 15 A, No. 6, pp. 465-471. 1981

This paper develops a model used to evaluate the sensitivity of expected passenger wait time at transit stops to service frequency and schedule reliability. This model explicitly incorporates a passenger decision - making process, rather than assuming that passengers arrive at random instants in time. The implications of this model are that passenger wait time is much more sensitive to schedule reliability and much less sensitive to service frequency than previously believed.

Optimal Design of Urban Bus System with Demand Sensitive Levels, George A. Kocur, Unpublished Ph. D. Thesis, School of Urban and Public Affairs, Carnegie - Mellon University, Pittsburgh, Pa., 1981.

Urban bus service design options are analysed in this thesis, including route structure, headway, fare, vehicle size, express service, and other issues. Optimal levels of these variables are obtained analytically for three objective functions including profit maximization, maximization of a conbination of net user benefit and operator profit, and maximization of net user benefit subject to a deficit constraint. The analysis uses an equilibrium framework with transit ridership explicity sensitive to the level of service provided by the bus system. The major results consist of closed - form solutions for the optimal system design and operating policies, which are dependent upon local conditions, transit service objectives, and coefficients of the demand and cost functions. These expressions are applied directly to aid in the design and evaluation of bus services in a case study.

From all the papers and books reviewed, this unpublished

Ph. D thesis is the most specific and applicable to the current transportation system in the Chattanooga area. This work envolved from the early efforts of Newell, Hurdle, Clarents, etc. review for this project.

CHAPTER 3

MATHEMATICAL MODEL

3.1 Introduction

The mathematical model developed by Dr. George Kocur is used for the optimal design of a bus system. This model can be applied to the design of local and areawide systems.

The model is based on a demand function with transit ridership sensitive to service levels. The demand function is an approximation to a disaggregate demand model,as the problem formulation integrates (aggregates) over individual travellers explicity. A disaggregate demand model is estimated on individual behavior, as contrasted with an aggregete model, which is estimated on average behavior of all residents in an area such as a traffic analysis zone.

The objective function used in this project is stated as follows : Maximize net user benefit subject to a deficit constraint. This is most representative of the current transit objective function.

The net user benefit or consumers' surplus, the amount users would be willing to pay for a service less the total cost they actually pay, is based on a linear demand function. Besides, the deficit constraint, a bus capacity constraint is also used in the analysis.

The equations which give the optimal values for the major decision variables - route spacing (separation between routes), headway (time between successive buses passing the same point), route length, and fare- are determined by the use of calculus and the technique of Lagrange multipliers. Equations fer revenue, ridership, load factor, and cost are calculated in order to complete the model. A computer program was written as a part of this project in order to obtain the numerical values for the analysis.

3.2 Analytical Approach

The following are two types of analytical approaches for the design of a bus route:

a. Use of Cartesian Coordinates, Local Area Analysis. This is appropiate for the analysis of feeder bus routes to rapid transit lines, local areas where the areawide route structure does not need to be considered, and cities where the road network is a rectangular grid.

b. Use of Polar Coordinates, Areawide Analysis. The use of polar coordinates is appropriate for systems where the entire bus network is treated, and cities where the road network approximates a radial grid. In this approach all potential demand is modeled as being either to or from the center of the bus network.

The areawide analysis is the analytical approach used in this project due to the local area road network, and from information obtained in conversation with Mr. Mark Pritchard, former Assistant General Manager.

Table 3.1

Variable Definitions

Variable	Definition
a_1	mode choice coefficient : transit constant
a_2	mode choice coefficient : wait and walk time (1/min.)
a_3	mode choice coefficient : in vehicle travel time (1/min.)
a_4	mode choice coefficient : fare (1/dollars)
a_5	mode choice coefficient : auto time and cost (1/mi.)
b	spacing between bus stops along a route (mi.)
c	bus operating cost (cents/min.)
d	average passenger trip length (mi.)
f	fare (cents)
g	spacing between parallel bus routes (mi.)
h	headway (min.)
j	average walking speed (mi./min.)
k	ratio of expected user wait time to headway
p	trip density by all modes (trips/mi.^{2}/min.)
p_0	trip density by all modes with decreasing trip density (trips/mi.^{2}/min)
q	load factor or ratio of bus passengers to capacity
s	vehicle capacity

Table 3.1 (continued)

variable	Definition
v	average bus speed, including stops (mi./min.)
y	distance from center of area (mi.)
A	$(a_1+(a_2b/4j)+((a_3/v)+a_5)Y/2)$; invariant effect in mode choice model
C	bus operating cost over a time T(cents)
D	demand function; passengers carried in service area over time T (trips/min.)
G	net user benefit (cents)
L	route lenght (mi.)
M	maximun operating deficit in service area over a time T (cents)
P	ridership
R	revenue (cents)
T	time period of analysis (min.)
X	Cartesian coordinates : width of analysis area (mi.)
Y	Cartesian coordinates : length of analysis area (mi.) polar coordinates : radius of analysis area (mi.)
Y_2	shadow price associated with deficit constraint; every extra dollar of subsidy produces "x" dollars in extra net user benefit
Y_3	shadow price associated with capacity constraint
θ	route spacing : angle between bus routes (radians)

3.3 Decision Variables

The major decision variables treated are the route spacing, the route headway, and the fare. Secondary variables are stop spacing, route circuity, and route length. Another important variable is the vehicle capacity which in most cases is considered as a constraint because it forces adjustment in the other variables. Table 3.1 gives a list of variable definitions.

3.4 Demand Function

A linear mode share model was used in this analysis, as an approximation to logit and other demand model forms typically used in transportation planning. This demand model includes the following variables ;

1. Wait time.
2. Walk time.
3. In-vehicle travel time.
4. Fare.
5. Automobile time and cost.
6. Limited treatment of socioeconomic variables.

The coefficients in the linear model are based on a previously estimated disaggregate demand model, as the problem formulation integrates over individual travellers explicity.

Figure 3.1 shows the linear approximation used in this analysis. The linear approximation is truncated so that the mode share is bounded between zero and one. It is used instead of the logit form because of its analytical tractability; it is easily differentiated and manipulated, and it is convex

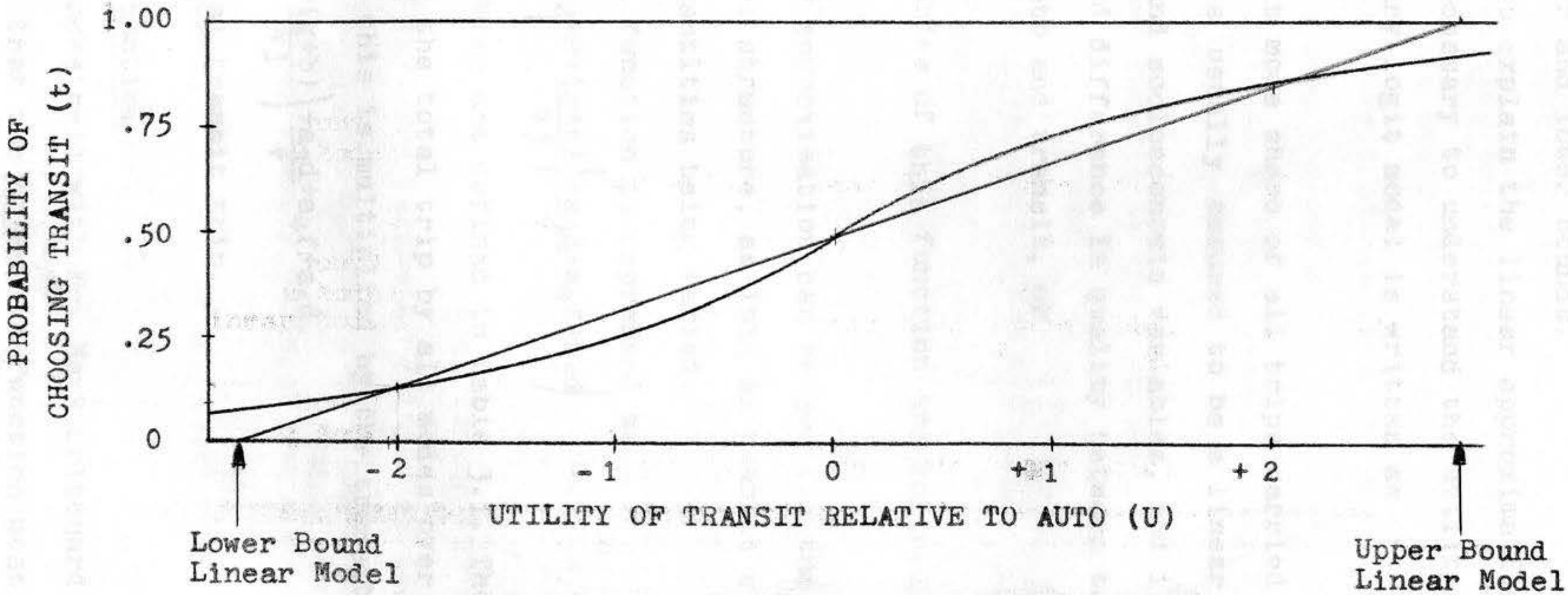

Figure 3.1 Linear Mode Share Model

Source : Kocur, George A., Optimal Design of Urban Bus Systems with Demand Sensitive Levels. Unpublished PH. D. Thesis, p. 41, 1981 within its upper and lower bounds.

In order to explain the linear approximation to the logit model, it is necessary to understand the utility variable used in it. The binary logit model is written as

$$t = 1/(1+e^{-U}) \tag{3.1}$$

where t= transit mode share of all trips carried by transit The utility U is usually assumed to be a linear function of service level and socioeconomic variables, and it is a measure of the perceived difference in quality between the competing alternatives auto and transit, or

$$U = F(h,g,f) \tag{3.2}$$

where the variables of this function are defined in Table 3.1.

The linear approximation can be used for the analysis of any demand model structure, as long as transit service levels are the only quantities being varied.

The demand function is expressed as

$$D = TpXY\left(a_1+a_2\left(kh+\frac{(g+b)}{4j}\right)+\frac{a_3d}{v}+a_4f+a_5d\right) \tag{3.3}$$

where all variables are defined in table 3.1. The quantity TpXY represents the total trip by all modes over the area XY and time T, and this is multiplied by the transit mode share

$$t = a_1+a_2\left(kh+\frac{(g+b)}{4j}\right)+\frac{a_3d}{v}+a_4f+a_5d \tag{3.4}$$

to give the total transit trip.

3.5 Objective Function

In conferences held with Mr. Mark Pritchard from CARTA, it was observed that the objective function most representative to the current transit objectives may be stated as follows :

<u>Maximize net user benefit subject to a deficit constraint.</u>

This corresponds to a model in which some broader decision fixes a budget for the transit system, and the benefits obtained by the transit system are maximized subject to that constraint. The deficit can be constrained to zero (break-even) or any other value.

The net user benefit is based on the linear demand model, and its mathematical expression is

$$G=-\frac{TpXY}{2a_4}\left(a_1+a_2\left(kh+\frac{(g+b)}{4j}\right)+\frac{a_3d}{v}+a_4f+a_5d\right)^2 \qquad (3.5)$$

This expression is the difference between the amount users would be willing to pay for a service with given values of f, h, and g and the total cost they actually pay.

The bus operating costs used for this analysis are given by

$$C=\frac{2XTcY}{ghv} \qquad (3.6)$$

They are derived as follows : there are X/g routes on the area, each operating T/h trips and taking Y/v minutes to complete, multiplied by a round trip factor (2) and the operating cost per minute (c). The parameter c includes labor, fuel, maintenance, and other costs, allocated on a bus - minute basis. Capital and operating subsidy policies can also be reflected in the parameter c.

3.6 Systemwide Analysis.

The same linear demand function and net user benefit expresions are used. However, changes have been made in order to reflect the use of polar coordinates. The demand function and the consumers' surplus have also been analysed for cases where the trip density is constant, and where the trip decreases linearly. Finally, the operating cost measure is also different in polar coordinates.

3.6.1 Demand Function and Net User Benefit with Constant Trip Density.

The expression for the demand function is given by :

$$D=2\pi Tp\int_0^L (a_1+a_2(kh+(\theta y+b)/4j)+((a_3/v)+a_5)y+a_4f)y\,dy \quad (3.7)$$

$$D=(2\pi TpL^2/2)(a_1+a_2(kh+(2\theta L+3b)/12j)+(2L/3)((a_3/v)+a_5)+a_4f)$$

The expresion for the net user benefit is given by :

$$G=-(2\pi Tp/2a_4)\int_0^L (a_1+a_2(kh+(\theta y+b)/4j+((a_3/v)+a_5)y+a_4f)^2y\,dy \quad (3.8)$$

$$G=-(2\pi TpL^2/4a_4)((a_1+a_2(kh+(b/4j))+a_4f)^2+(a_1+a_2(kh+(b/4j))+ +a_4f)((a_2\theta/4j)+(a_3/v)+a_5)(4L/3)+((a_2\theta/4j)+(a_3/v)+a_5)^2 (L^2/2))$$

3.6.2 Demand Function and Net User Benefit with Linearly Decreasing Trip Density.

In this case the expressions for the demand function and the net user benefit are slightly different, due to the decreasing trip density function

$$p=p_0(1-(y/Y)) \quad (3.9)$$

The expression for the demand function is given by :

$$D=2\pi Tp_0\int_0^L (1-(y/Y))(a_1+a_2(kh+(\theta y+b)/4j)+((a_3/v)+a_5)y+a\ f)y\,dy \quad (3.10)$$

$$D=2\pi Tp_0((a_1+a_2(kh+(b/4j))+a_4f)((L^2/2)-(L^3/3Y))+((a_2\theta/4j)+ +(a_3/v)+a_5)((L^3/3)-(L^4/3Y)))$$

The expression for the net user benefit is given by :

$$G=-(2\pi T\ p_0/2a_4)\int_0^L (1-(y/Y))(a_1+a_2(kh+(\theta y+b)/4j)+((a_3/v)+a_5)y+ +a_4f)^2 y\ dy \qquad (3.11)$$

$$G=-(2\pi Tp_0/2a_4)((a_1+a_2(kh+(b/4j)+a_4f)\ ((L^2/2)-(L^3/3Y))+ +2(a_1+a_2(kh+(b/4j)+a_4f)((a_2\theta/4j)+(a_3/v)+a_5)((L^3/3)- -(L^4/4Y))+((a_2\theta/4j)+(a_3/v)+a_5)^2((L^4/4)-(L^5/5Y)))$$

3.6.3 Operating Costs

The expression for the operating cost in polar coordinates is

$$C=\frac{4\pi TLc}{\theta hv} \qquad (3.12)$$

This equation is derived as follow : There are $2\pi/\theta$ routes, each operating T/h trips and taking L/v minutes to complete, multiplied by a round trip factor (2) and the operating cost per minute.

3.7 Results for the Objective of Maximizing Net User Benefit Subject to a Deficit Constraint.

3.7.1 Headway, Route Spacing and Fare with Uniform Trip Density.

In this case the objective function is the consumers' surplus, and the constraint is the difference between the operating cost, the demand function, and the maximun allowable deficit M. The problem is then stated as :

$$\text{max. } B=-(2\pi Tp/2a_4)\int_0^L (a_1+a_2(kh+(\theta y+b)/4j)+((a_3/v)+a_5)y+ +a_4f)^2 y\ dy \qquad (3.13)$$

subject to

$$(4\pi TLc/\theta hv)-2\pi Tpf\int_0^L (a_1+a_2(kh+(\theta y+b)/4j)+((a_3/v)+a_5)y+a_4f)y\,dy-M\leq 0$$

$$\theta,\ h,\ f\geq 0 \qquad (0\leq\theta\leq 2\pi) \qquad (3.14)$$

The problem is formulated using a Lagrange multiplier Y_2 associated with the deficit constraint. The Lagrangian for the problem is :

$$N=-(2\pi Tp/2a_4)\int_0^L (a_1+a_2(kh+(\theta y+b)/4j)+((a_3/v)+a_5)y+a_4f)^2 y\,dy$$
$$-Y_2((4\pi TLc/\theta hv)-2\pi Tpf\int_0^L (a_1+a_2(kh+(\theta y+b)/4j)+((a_3/v)+a_5)y+a_4f)y\,dy-M) \qquad (3.15)$$

From the first order condition for h, f, and θ the following optimal solutions are derived (Appendix A, Section A.2.1) :

$$\theta=((144cj^2ka_4(2Y_2-1))/(vpa_2AY_2L^3))^{1/3} \qquad (3.16)$$

$$h=((2ca_4(2Y_2-1))/(3jk^2pa_2AY_2v))^{1/3} \qquad (3.17)$$

$$f=((L-Y_2)/(2Y_2-1))(A/a_4+2((2ca_2^2k(2Y_2-1))/(3jvpa_4^2AY_2))^{1/3}) \qquad (3.18)$$

To solve for y_2, the values of θ, h, and f are substituted into the deficit equation. By using a series of approximations one obtains

$$Y_2\simeq(-n-(n^2-4mx)^{1/2})/2m \qquad (3.19)$$

where

$$m=6((2kp^2a_2^2A^2c)/(3a_4^2jv))^{1/3}+pA^2/a_4-(4M)/(\pi TL^2) \qquad (3.20)$$

$$n=-5((2kp^2a_3^2A^2c)/(3a_4^2jv))^{1/3}-pA^2/a_4+(4M/(\pi TL^2) \qquad (3.21)$$

$$x=-M/(\pi TL^2) \qquad (3.22)$$

3.7.2 Headway, Route Spacing and Fare with Linearly Decreasing Trip Density.

For this case the objective function and the constraint are the same as in the preceding section, but reflecting the linearly decresing trip density. The problem is stated as follows :

$$\max. B = -(2\pi T/2a_4)\int_0^L p_0(1-(y/Y))(a_1+a_2(kh+(\theta y+b)/4j)+ +(a_3/v+a_5)y+a_4 f)^2 y\,dy \qquad (3.23)$$

subject to :

$$(4\,Lc/\theta hv)-2\pi Tf\int_0^L p_0(1-(y/Y))(a_1+a_2(kh+(\theta y+b)/4j)+ +(a_3/v+a_5)y+a_4 f)y\,dy - M \leq 0$$

$$\theta,\ h,\ f \geqslant 0 \qquad (0 \leqslant \theta \leqslant 2\pi) \qquad (3.24)$$

The Lagrangian is :

$$N = -(2\pi T/2a_4)\int_0^L p_0(1-(y/Y))(a_1+a_2(\theta y+b)/4j)+(a_3/v+a_5)y+ +a_4 f)^2 y\,dy - Y_2((4\,Lc/\theta hv)-2\pi Tf\int_0^L p_0(1-(y/Y))(a_1+ +a_2(kh+(\theta y+b)/4j)+(a_3/v+a_5)y+a_4 f)y\,dy - M) \qquad (3.25)$$

From the first order condition for h, f, θ, and L the following optimal solutions are derived (Appendix A, Section A.2.2)

$$\theta = \left(\frac{768 j^2 kYca_4(3Y-2L)(2Y_2-1)}{vp_0 a_2 A(4Y-3L)^2 Y_2 L^3}\right)^{1/3} \qquad (3.26)$$

$$h = \left(\frac{3Yca_4(4Y-3L)(2Y_2-1)}{2jk^2 Aa_2(3Y-2L)^2 Y\,p_0}\right)^{1/3} \qquad (3.27)$$

$$f = ((1-Y_2)/(2Y_2-1))((A+2a_2 kh)/a_4) \qquad (3.28)$$

$$L = Y(1-Z) \qquad (3.29)$$

where

$$Z = \left(\frac{2Y_2-1}{2Y_2}\right)^{1/3}\left(\frac{(12ca_4a_2^2k)^{1/3}}{(9A^4jvp_0)^{1/3}+4(12ca_4a_2^2k)^{1/3}}\right) \quad (3.30)$$

To solve for Y_2, the values of θ, h, f, and L are substituted in the deficit equation. By using a series of approximations one obtains

$$Y_2 \simeq (-(n^2-4mx)^{1/2})/2m \quad (3.31)$$

where

$$m=6((3kp_0^2a_2^2A^2c)/(2a_4^2jv))^{1/3}+p_0A^2/a_4-(12M)/(\pi TL^2) \quad (3.32)$$

$$n=-5((3kp_0^2a_2^2A^2c)/(2a_4^2jv))^{1/3}-p_0A^2/a_4+(12M)/(\pi TL^2) \quad (3.33)$$

$$x=-3M/(\pi TL^2) \quad (3.34)$$

3.7.3 Headway, Route Spacing and Fare with Linearly Decreasing Trip Density when Constrained by Vehicle Capacity.

This problem is formulated as follows :

$$\max.\ B=-(2\,T/2a_4)\int_0^L p_0(1-(y/Y))(a_1+a_2(kh+(\theta y+b)/4j)+ +(a_3/v+a_5)y+a_4f)^2 y\,dy \quad (3.35)$$

subject to :

$$(4\pi Lc/\theta hv)-2\pi Tf\int_0^L p_0(1-(y/Y)(a_1+a_2(kh+\theta y+b)/4j)+(a_3/v+a_5)y+ +a_4f)y\,dy - M\leqslant 0 \quad (3.36)$$

$$\theta h\int_0^L p_0(1-(y/Y))(a_1+a_2(kh+(\theta y+b)/4j)+(a_3/v+a_5)y+a_4f)y\,dy-s\leq 0$$

$$\theta,\ h,\ f,\ L\geqslant 0 \qquad (0\leqslant\theta\leqslant 2\pi) \quad (3.37)$$

The Lagrangian is

$$N=-(2\pi T/2a_4)\int_0^L p_0(1-(y/Y))(a_1+a_2(kh+(\theta y+b)/4j)+(a_3/v+a_5)y+a_4f)^2 y\,dy-Y_2((4\pi Lc/\theta hv)-2\pi Tf\int_0^L p_0(1-(y/Y))(a_1+a_2(kh+(\theta y+b)/4j)+(a_3/v+a_5)y+a_4f)y\,dy-M)-Y_3(\theta h\int_0^L p_0(1-(y/Y))(a_1+a_2(kh+(\theta y+b)/4j)+(a_3/v+a_5)y+a_4f)y\,dy-s) \quad (3.38)$$

The objective function and the first constraint are the same as in the previous case. A second constraint equation is added to reflect the vehicle capacity. In the Lagragian, the term Y_3 represents the shadow price associated with the capacity constraint (see Table 3.1).

The Lagrangian solution for this case is much too unwieldy. As a result, a computer technique of sensitivity analysis was applied (Appendix B, Section B.3), in order to find the optimal values for the headway, route spacing, fare, revenue, cost, and ridership. This was done by following Kocur's suggestion of decreasing the route spacing and the headway by equal proportions based on the degree of overloading.

3.8 Revenue, Ridership and Bus-Load

The revenue is given by multiplying the demand function by the fare. The equation of revenue for the case when the trip density is constant is

$$R=(fW\pi TpL^2/2)(a_1+a_2(kh+(2\theta L+3b)/12j)+(2L/3)(a_3/v+a_5)+a_4f) \quad (3.39)$$

For the linearly decreasing trip density case the equation for the revenue is

$$R=W\pi fTp_0((a_1+a_2(kh+(b/4j))+a_4f)(L^2/2-L^3/3Y)+(a_2\theta/4j+a_3/v+a_5)(L^3/3-L^4/3Y)) \quad (3.40)$$

The ridership is given by the demand function, or can be calculated by using the equation :

$$P= R/f \tag{3.41}$$

The bus-load is calculated by the equation :

$$q=\frac{P\theta h}{T\ W} \tag{3.42}$$

where the value of W represents the fraction covered by the study area, Figure 3.2.

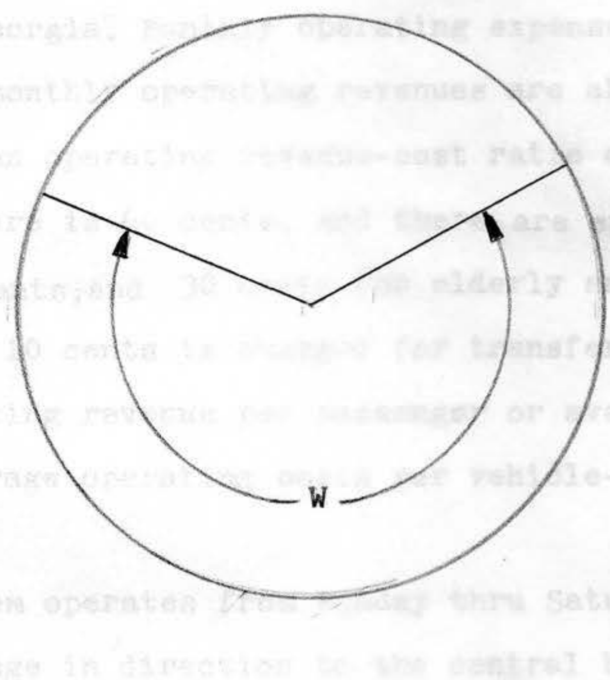

Figure 3.2 Fraction covered by the study area

CHAPTER 4

LOCAL SYSTEM RESULTS

4.1 Current System

Public transportation in the Chattanooga area is provided by a bus system operated by the Chattanooga Area Regional Transportation Authority (CARTA). The area has a population of about 250,000, from which an estimated 5 percent of this population constitute the daily ridership. The system operates a fleet of 69 buses over 19 routes, covering the city of Chattanooga and the surrounding communities of East Ridge, Lookout Mountain Red Bank, Signal Mountain and Lakesite in Tennessee and Lookout Mountain in Georgia. Monthly operating expenses are approximately $329,225 and monthly operating revenues are about $137,000, resulting in an operating revenue-cost ratio of about 42 percent. The regular fare is 60 cents, and there are special fares of 50 cents for students,and 30 cents for elderly and handicapped, an extra fare of 10 cents is charged for transfer passenger. The average operating revenue per passenger or average fare is about 48 cents. Average operating costs per vehicle-hour is approximately $36.28

The system operates from Monday thru Saturday. All buses leave the garege in direction to the central business district (CBD), Figure 4.1, which is located in downtown Chattanooga. The CBD is considered to be an area with a radius of 0.45 miles. The peak-hour periods are from 6:00 AM to 9:00 AM and from

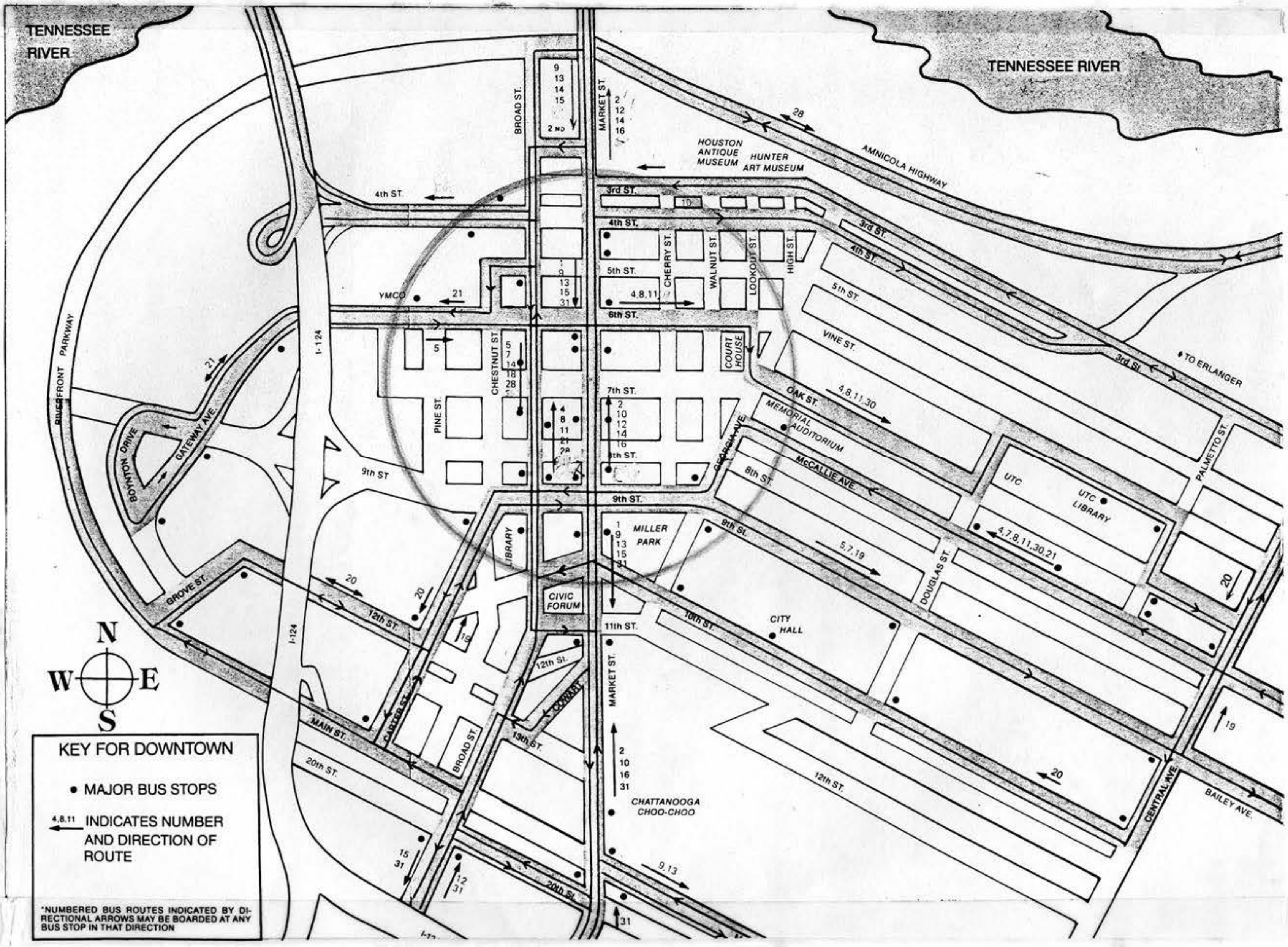

Figure 4.1 Central Business Distric (CBD)

25

3:00 PM to 6:00 PM. During the peak hours, the average headway is about 32 minutes from Monday to Friday, and approximately 42 minutes on Saturday. The routes are assumed to serve an area of $1.37\pi/19$ resulting in an average route spacing of 0.227 radians. Figure 4.2 shows the current bus services.

Peak hour operations employ approximately 33 buses per hour during weekdays, and about 11 buses per hour on Saturdays. At an operating cost of $36.28 per bus-hour, this yields a cost of approximately $3,591.72 during weekdays, and $1,197.24 on Saturdays for a peak period. The operating cost per bus hour is assumed equal for peak and offpeak periods in the absence of detailed data.

Ridership during the peak period is about 3,089. At a fare of 60 cents these riders yield about $1,853.40 in revenues. Thus in the peak period, revenues cover 52 percent of operating costs. This is higher than the overall operating ratio of 42 percent. However, this comparison between peak and offpeak operatingratios does not reflect the higher costs of peak operations.

The average number ofpassengers per bus during the peak period is about 31 and the capacity of a bus is 43 passengers. This yields a ratio of total riders to seats of 1.39.

All this information has been drawn from monthly operating statements for the period February - August, 1983, and from other sources prepared by CARTA (Appendix C). Table 4.1 summarizes current bus service.

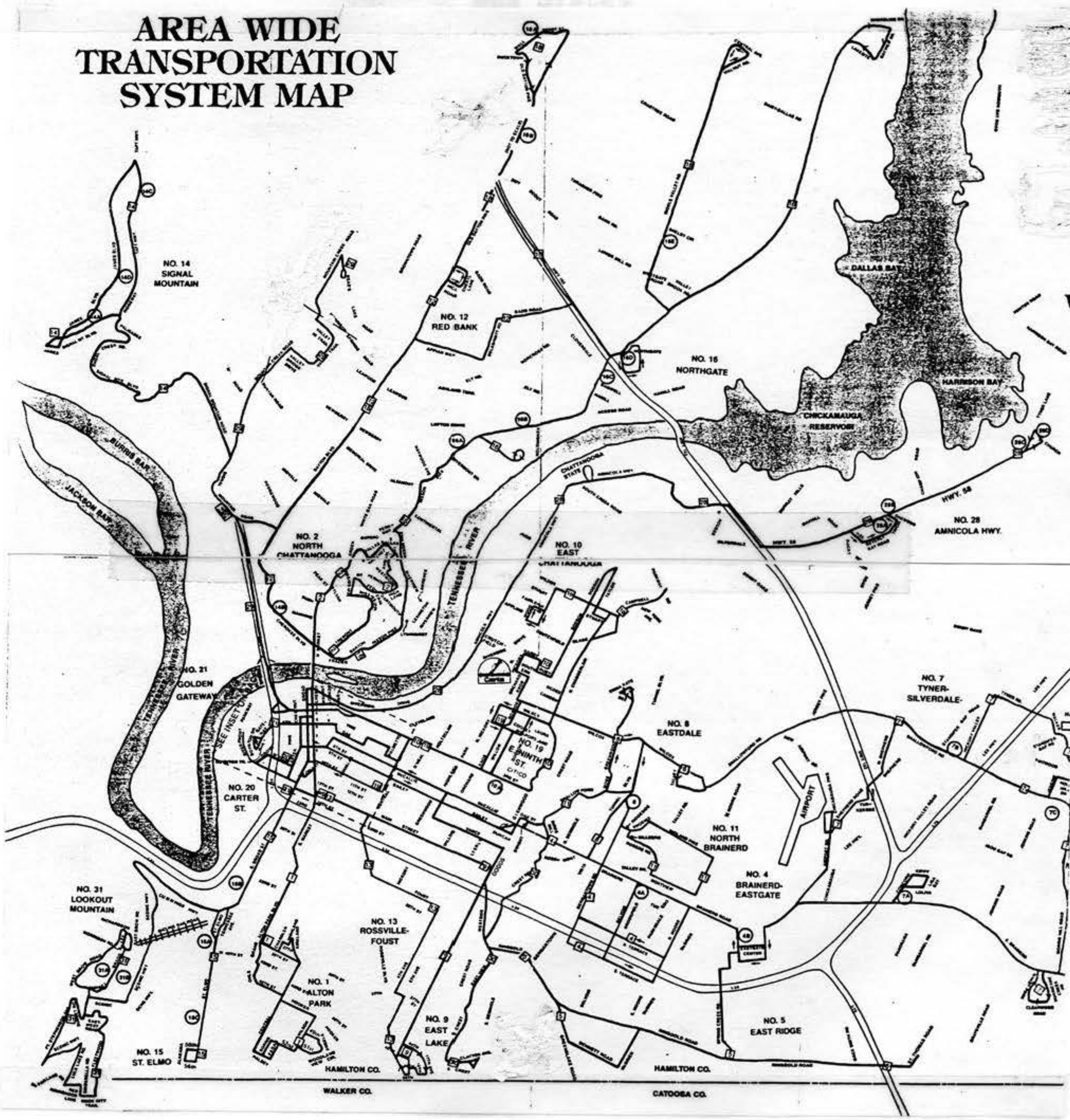

Figure 4.2 Current Bus Services.

Table 4.1

Current Bus System

	Average Values
Radius of analysis (miles)	9.3
Route spacing (radians)	0.227
Route headway (minutes)	32.0
Regular fare (cents)	60.0
Peak ridership	3,089
Peak operating revenues (dollars)	1,853.40
Peak operating costs (dollars)	3,591.72
Profit (dollars)	- 1,738.32
Passenger load per bus	31
Peak vehicles required (buses/hr)	33

Table 4.2

Summary of the Chattanooga Area Variables Values

Variable	Value	Source
a_1	0.38	CARTA data
a_2	-0.0081(1/min.)	CARTA data
a_3	-0.0033(1/min.)	CARTA data
a_4	-0.0014(1/$)	CARTA data
a_5	0.0328(1/mi.)	CARTA data
b	0.16 miles	Personal observations
c	60.36(cents/min.)	From CARTA monthly operating statements.
j	0.05(mi./min.)	Conventional assumption: 3 mph walking speed
k	0.40	Conventional assumption: wait time at slightly less than half of headway, at headway over 10 minutes.
p	$1.794\left(\frac{\text{trips}}{\text{mi.}^2\ \text{min.}}\right)$	CARTA data
p_0	$5.18\left(\frac{\text{trips}}{\text{mi.}^2\text{min.}}\right)$	CARTA data
v	0.242(mi./min.)	CARTA average bus speed including bus stops: 14.5(mi./hr)
T	180.0 minutes	CARTA departure and arrivals schedules.
Y	9.3 miles	CARTA route map and Chattanooga city map.
W	1.37 radians	CARTA route map and Chattanooga city map

4.2 Optimal Chattanooga Area Public Transportation System.

The analysis conducted in the previous chapter, and the computer programs shown in Appendix B, are now applied to the CARTA bus system. The results obtained are compared with the current operations for each case.

All variables have been previously defined in Table 3.1. Table 4.2 summarizes the Chattanooga area variables values and sources from which they were obtained.

4.2.1 Maximization of Net User Benefit Subject to a Deficit Constraint with Constant Trip Density.

Table 4.3 summarizes the results for this analysis. The values were obtained from the computer program developed for this case (Appendix B, Section B.1).

The shadow price Y_2 indicates that an extra dollar of subsidy would generate $1.36 in extra net user benefit. Route spacing of 0.228 radians is almost the same as the current 0.227 radians, resulting in an average walk distance of 0.35 miles. A headway of 17.6 minutes is shown in this case, about 45 percent lower than the current. This would decrease the waiting time from 12.8 minutes to 7.0 minutes. Fare is 7 cents lower than the current 60 cents fare.

Ridership is 27,323 in the peak period, extremely higher than the current 3,089. Peak operating revenues are $14,286.97 as opposed to the current $1,853.40, and operating costs are $13,090.25, resulting in an operating revenue - cost ratio of

Table 4.3

Maximize Net User Benefit Subject to a Deficit Constraint with Constant Trip Density

Radius of analysis (miles)	9.3
Route spacing (radians)	0.228
Route headway (minutes)	17.6
Regular fare (cents)	52.3
Peak ridership	27,323
Peak operating revenues (dollars)	14,286.97
Peak operating costs (dollars)	13,090.25
Profit (dollars)	1,196.72
Passenger load per bus	97
Shadow price (Y_2)	1.363
Peak vehicles required (buses/hr.)	282

about 1.1 percent. This indicates an operating profit of $1,196.72. Net user benefits are $29,266.95. The bus load is quite extreme reaching 97 passengers per bus. This suggests bigger or articulated buses, which would be too costly and infeasible for the density population of riders in this area.

4.2.2 Maximization of Net User Benefit Subject to a Deficit Constraint with Linearly Decreasing Trip Density.

Table 4.4 shows the results of applying an operating deficit constraint with linearly decreasing trip density to the objective of maximizing net user benefit. These results were obtained from a computer program developed for this case (Appendix B, Section B.2).

The approximate shadow price is $1.28, which means every extra dollar of subsidy produces $1.28 in extra net user benefit. The routes are spaced 0.276 radians apart, which results in an average walk time of 6.4 minutes, or an average distance of 0.32 miles. The routes operate at a 16-minute headway and charge a fare of about 46 cents, which is less than the current regular fare.

The peak period ridership is 16,122, which is about 522 percent higher than the current ridership of 3,089. peak operating revenues are $7,530.23 as opposed to the current $1,853.40. However, these would cover the peak operating costs of $7,138.36 as opposed to the current system. Net user benefits are $16,970.38. The passenger loads, as in the previous case, are quite extreme,

Table 4.4

Maximize Net User Benefit Subject to a Deficit Constraint with Linearly Decreasing Trip Density

Radius of analysis (miles)	9.3
Route spacing (radians)	0.276
Route headway (minutes)	16.0
Regular fare (cents)	46.5
Peak ridership	16,212
Peak operating revenues (dollars)	7,530.23
Peak operating costs (dollars)	7,138.36
Profit (dollars)	391.87
Passenger load per bus	93
Shadow price (Y_2)	1.284
Peak vehicles required (buses/hr)	174

Table 4.5

Maximize Net User Benefit Subject to a Deficit and Capacity Constraint with Linearly Decreasing Trip Density

Radius of analysis (miles)	9.3
Route spacing (radians)	0.179
Route headway (minutes)	10.4
Regular fare (cents)	51.2
Peak ridership	17,828
Peak operating revenues (dollars)	9,120.60
Peak operating costs (dollars)	16,895.46
Profit (dollars)	- 7,774.86
Passenger load per bus	43
Shadow price (Y_2)	1.284
Peak vehicles required (buses/hr.)	415

reaching 93 passengers per bus. This would also suggest, as in the previous case, the use of bigger or articulated buses.

4.2.3 Maximization of Net User Benefit Subject to a Deficit and Capacity Constraint with Linearly Decreasing Trip Density.

In this case passenger load per bus must be 43 or less, to meet current CARTA constraints. A computer technique of sensitivity analysis was applied in order to obtain the optimal values (Appendix B, Section B.3). Table 4.5 summarizes the vehicle size constraint.

Route spacing of 0.179 radians and headway of 10.4 minutes are the lowest of all in comparison with the previous cases and the current system. Fare is about 51 cents, which is lower than the current fare of 60 cents.

Ridership is 17,828 in the peak period. Revenues are $9,120.60 in the peak period and costs are $16,895.46, resulting in an operating revenue-cost ratio of 54 percent in the peak period, which is 2 percent higher than in the current system. Net user benefits are $20,570.12.

4.3 Sensitivity Analysis.

The results shown in the previous sections are quite extreme with respect to the actual CARTA values. As a result, a sensitivity analysis using computer programs was applied (Appendix B, Section B.4) to study the effect of the capacity constraint to the other variables. Using the previous model equations, the values of ridership, cost, revenue, and fare were obtained for the break-even condition.

From the analysis it is observed that in order to operate at a breakeven level when the capacity is 43, the fare has to be increased from its actual value of 60 cents to a value of about 96 cents,which would represent an increse of 60 percent of the current fare value. This assumes no operating subsidy.

CHAPTER 5

CONCLUSIONS

5.1 Conclusions.

The numerical results in this project apply only to a peak period, and are based on typical parameters derived from the Chattanooga area. Under actual conditions CARTA cannot presently operate at a breakeven level in peak periods (excluding capital costs) without federal subsidies. Federal funds have been declining during the Reagan administration. As a result, fare should be increased in order to compensate for this reduction.

The analysis indicates that vehicles with capacity greater that 43 passengers are sometimes warranted. Recognizing this problem, CARTA is now ordering several higher-capacity buses. However, higher-capacity buses would represent greater operating costs which at the moment are not desirable.

Average passenger loads per bus at peak load points under the optimal operating strategies of this model generally exeed the current vehicle capacity. In these cases, either headways and route spacing must be reduced and fares raised to meet capacity constraint, or the larger buses employed. Consequently, an operating profitability in peak hours appears to be possible in some cases. The results obtained in this project therefore provide a framework in which CARTA may plan for efficient operation under decreasing federal funds.

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APPENDIXES

APPENDIX A

DERIVATIONS OF ANALYTICAL RESULTS

A.1 Systemwide Analysis.

A.1.1 Demand Function and Net User Benefit with Constant Trip Density.

The demand function equation is given in (3.7), and its solution is :

$$D=2\pi Tp((a_1+a_2(kh+(b/4j))+a_4f)\int_0^L y\ dy+(a_2\theta/4j+a_3/v+a_5)\int_0^L y^2\ dy)$$

$$=2\pi Tp((a_1+a_2(kh+(b/4j))+a_4f)(L^2/2)+(a_2\theta/4j+a_3/v+a_5)(L^3/3))$$

$$D=(2\pi TpL^2/2)(a_1+a_2(kh+(2\theta L+3b)/12j+(2L/3)(a_3/v+a_5)+a_4f) \quad (A.1)$$

The net user benefit equation is given in (3.8), and its solution is :

$$G=-(2\pi Tp/2a_4)((a_1+a_2(kh+(b/4j))+a_4f)^2\int_0^L y\ dy+2(a_1+a_2(kh+(b/4j))+$$
$$+a_4f)(a_2\theta/4j+a_3/v+a_5)\int_0^L y^2\ dy+(a_2\theta/4j+a_3/v+a_5)^2\int_0^L y^3\ dy)$$

$$=-(2\pi Tp/2a_4)((a_1+a_2(kh+(b/4j))+a_4f)^2(L^2/2)+2(a_1+a_2(kh+(b/4j))+$$
$$+a_4f)(a_2\theta/4j+a_3/v+a_5)(L^3/3)+(a_2\theta/4j+a_3/v+a_5)^2(L^4/4))$$

$$G=-(2\pi TpL^2/4a_4)((a_1+a_2(hk+(b/4j))+a_4f)^2+(a_1+a_2(kh+(b/4j))+$$
$$+a_4f)(a_2\theta/4j+a_3/v+a_5)(4L/3)+(a_2\theta/4j+a_3/v+a_5)^2(L^2/2)) \quad (A.2)$$

A.1.2 Demand Function and Net User Benefit with Linearly Decreasing Trip Density.

The expresion for the demand function in this case is given by equation (3.10). The solution is :

$$D=2\pi Tp_0(\int_0^L(a_1+a_2(kh+(\theta y+b)/4j)+(a_3/v+a_5)y+a_4f)y\ dy-$$
$$\int_0^L(a_1+a_2(kh+(\theta y+b)/4j))+(a_3/v+a_5)y+a_4f)(y^2/Y)\ dy)$$

$$=2\pi Tp_0((a_1+a_2(kh+(b/4j))+a_4f)(L^2/2) + (a_2\theta/4j+a_3/v+a_5)(L^3/3)$$
$$-(a_1+a_2(kh+(b/4j))+a_4f)(L^3/3Y) - (a_2\theta/4j+a_3/v+a_5)(L^4/4Y))$$

$$D=2\pi Tp_0((a_1+a_2(kh+(b/4j))+a_4f)(L^2/2-L^3/3Y) + (a_2\theta/4j+a_3/v+a_5)(L^3/3-L^4/3Y)) \quad (A.3)$$

The expression for the net user benefit is given by the equation (3.11), and the solution is :

$$G=-(2\pi Tp_0/2a_4)\left(\int_0^L ((a_1+a_2(kh+(b/4j))+a_4f+(a_2\theta/4j+a_3/v+a_5)y)^2 y\,dy - \int_0^L ((a_1+a_2(kh+(b/4j))+a_4f)+(a_2\theta/4j+a_3/v+a_5)y)^2(y^2/Y)\,dy\right)$$

$$G=-(2\pi Tp_0/2a_4)((a_1+a_2(kh+(b/4j))+a_4f)^2(L^2/2)+2(a_1+a_2(kh+(b/4j))+a_4f)(a_2\theta/4j+a_3/v+a_5)(L^3/3)+(a_2\theta/4j+a_3/v+a_5)^2(L^4/4)$$
$$-(a_1+a_2(kh+(b/4j))+a_4f)^2(L^3/3Y)-2(a_1+a_2(kh+(b/4j))+a_4f)(a_2\theta/4j+a_3/v+a_5)(L^4/4Y)-(a_2\theta/4j+a_3/v+a_5)^2(L^5/5Y)$$

$$G=-(2\pi Tp_0/4a_4)((a_1+a_2(kh+(b/4j))+a_4f)^2(L^2/2-L^3/3Y)+2(a_1+a_2(kh+(b/4j))+a_4f)(a_2\theta/4j+a_3/v+a_5)(L^3/3-L^4/4Y)+(a_2\theta/4j+a_3/v+a_5)^2(L^4/4-L^5/5Y)) \quad (A.4)$$

A.2 Results for the Objective of Maximizing Net User Benefit Subject to a Deficit Constraint.

A.2.1 Headway, Route Spacing and Fare with Constant Trip Density.

The problem statement is given in (3.13), and (3.14). The Lagrangian for the problem is written as :

$$N=(2\pi TpL^2/4a_4)((a_1+a_2(kh+(b/4j))+a_4f)^2+(a_1+a_2(kh+(b/4j))+a_4f)(a_2\theta/4j+a_3/v+a_5)(4L/3)+(a_2\theta/4j+a_3/v+a_5)^2(L^2/2))$$
$$-Y_2((4\pi TLc/\theta hv)-2\,Tp(L^2/2)(a_1+a_2(kh+(2\theta L+3b)/12j)+(2L/3)(a_3/v+a_5)+a_4f)-M \quad (A.5)$$

The first order conditions are :

$$\partial N/\partial h=0=-(pa_2kL^2/2a_4)(a_1+a_2(kh+(b/4j))+a_4f+(2L/3)(a_2\theta/4j+ +a_3/v+a_5))+Y_2((2cL/\theta h^2v)+(fpa_2kL^2/2)) \qquad (A.6)$$

$$\partial N/\partial\theta=0=-(pa_2L^3/12a_4)(a_1+a_2(kh+(b/4j))+a_4f+(3L/4)(a_2\theta/4j+ +a_3/v+a_5))+Y_2((2cL/\theta^2hv)-(fpa_2L^3/12j)) \qquad (A.7)$$

$$\partial N/\partial f=0=-(pL^2/2)(a_1+a_2(kh+(b/4j))+a_4f+(2L/3)(a_2\theta/4j+a_3/v+ +a_5))+(pL^2/2)(a_1+a_2(kh+(b/4j))+2a_4f+2L/3)(a_2\theta/4j+ +a_3/v+a_5)) \qquad (A.8)$$

The equations (A.6) and (A.7) yield the approximate result :

$$\theta L\simeq 6jkh \qquad (A.9)$$

This result is used to eliminate h in (A.8) which is solved for f and substituted into (A.7). Then equations (3.16)-(3.18) are found. To solve for Y_2 the optimal values of θ,h, and f, are substituted into the constraint equation, which is the first order condition $\partial N/\ Y_2$ with respect to Y_2. These substitutions yield

$$\left(\frac{p2a_2Ack}{2vja_4^2}\right)^{1/3}\left(\frac{Y_2}{2Y_2-1}\right)^{2/3}\left(\frac{p}{a_4}\right)\left(\frac{1-Y_2}{2Y_2-1}\right)\left(\frac{Y_2}{2Y_2-1}\right)$$

$$\left(A+\left(\frac{4kca_2{}^2a_4(2Y_2-1)}{vpAjY_2}\right)^{1/3}\right)^2=\left(\frac{M}{TxY}\right) \qquad (A.10)$$

Equation (A.10) is not easily solvable, as it contains terms in $Y_2{}^2$, $Y_2{}^{4/3}$, Y_2 and $Y_2{}^{2/3}$ Two approximations are used to solve (A.10). The first is as follows :

$$\left(A+\left(\frac{4kca_2\ a_4(2Y_2-1)}{vpAjY_2}\right)^{1/3}\right)^2=\left(A^2+2A\left(\frac{4kca_2^{\ 2}}{vpAj}\right)^{1/3}\left(\frac{2Y_2-1}{Y_2}\right)^{1/3}\right)$$

This approximation corresponds to :

$$(x_1+x_2)^2=x_1^{\ 2}+2x_1x_2 \qquad \text{(A.11.a)}$$

which is a good approximation, as in the case in (A.11).

The second approximation is :

$$Y_2^{\ 1/3}(2Y_2-1)^{4/3}=Y_2(2Y_2-1)(2Y_2-1/Y_2)^{1/3}=Y_2(2Y_2-1)(2-1/Y_2)^{1/3}$$

$$\simeq Y_2(2Y_2-1) \qquad \text{(A.12)}$$

because Y_2 is generally near 1 or 2, and thus the term with the 1/3 power varies little from 1.

By multiplying (A.10) through by $(2Y_2-1)^2$ and by applying (A.11) and (A.12) one obtains

$$\left(\frac{p^2a_2^{\ 2}A^2ck}{2vja_4^{\ 2}}\right)^{1/3}Y_2(2Y_2-1)+\left(\frac{pA^2}{a_4}\right)-\left(\frac{32kca_2^{\ 2}p^2A^2}{vja_4^{\ 2}}\right)^{1/3}Y_2(1-Y_2)$$

$$-\left(\frac{M}{TxY}\right)(4Y_2^{\ 2}-4Y_2+1) \qquad \text{(A.13)}$$

By collecting terms, a quadratic equation Y_2 is obtained, with the solutions shown in (3.19)-(3.22).

A.2.2 Headway, Route Spacing and Fare with Linearly Decreasing Trip Density.

The problem is stated in (3.23) and (3.24). The Lagrangian is given in (3.25). This problem is solved exactly as the previous one. The approximation $L\simeq Y$ is used to simplify the derivation.

APPENDIX B

COMPUTER PROGRAMS

B.1 Results of Maximizing Net User Benefit Subject to a Deficit Constraint with Constant Trip Density.

```
      DIMENSION RL(100),Y2(100),TTA(100),OH(100),OF(100)
      DIMENSION RE(100),PA(100),CO(100),UB(100),PAL(100)
      DATA A1,A2,A3,A4,A5/0.38,-.0081,-.0033,-.0014,0.0328/
      DATA B,C,WJ,CK,P/0.16,60.36,0.05,0.4,1.795/
      DATA V,DM,Y,T,PI/0.2417,0.0,6.0,180.0,3.14159/
      WRITE(6,200)
      WRITE(6,210)
      WRITE(6,230)
      DO 2 I=1,41
      A=A1+(A2*B/(4*WJ))+((A3/V+A5)*Y/2.)
      RL(I)=Y
      YM=6.*(((2.*CK*(P**2)*(A2**2)*(A**2)*C)/(3.*(A4**2)*WJ*V
     C))**(1./3.))+(P*(A**2)/A4)-(4.*DM/(PI*T*RL(I)**2))
      YN=-5.*(((2.*CK*(P**2)*(A2**2)*(A**2)*C)/(3.*(A4**2)*WJ*
     CV))**(1./3.))-(P*(A**2)/A4)+(4.*DM/(T*RL(I)**2))
      X=-DM/(PI*RL(I)**2*T)
      Y2(I)=(-YN-(((YN**2)-4.*YM*X)**.5))/(2.*YM)
      TTA(I)=((144*C*(WJ**2)*CK*A4*(2.*Y2(I)-1.))/(V*P*A2*A*Y2
     C(I)*(RL(I)**3)))**(1./3.)
      OH(I)=((2.*C*A4*(2.*Y2(I)-1.))/(3.*WJ*(CK**2)*V*P*A2*A*Y
     C2(I)))**(1./3.)
      OF(I)=((1.-Y2(I))/(2.*Y2(I)-1.))*((A/A4)+2.*((2.*C*(A2**
     C2)*CK*(2.*Y2(I)-1.))/(3*WJ*V*P*(A4**2)*A*Y2(I)))**(1./3.
     C))
      RE(I)=(OF(I)*PI*T*P*(RL(I)**2))*(A1+A2*(CK*OH(I)+(2.*TTA
     C(I)*RL(I))/(12.*WJ))+(2.*RL(I)/3.)*(A3/V+A5)+A4*OF(I))/1
     C00.
      PA(I)=(RE(I)/OF(I))*100.
      CO(I)=(4.*PI*RL(I)*C*T)/(TTA(I)*OH(I)*V*100.)
      UB(I)=(-PI*T*P*(RL(I)**2)/(2.*A4*100.))*(((A1+A2*(CK*OH(
     CI)+B/(4.*WJ))+A4*OF(I))**2)+((4./3.)*RL(I)*(A1+A2*(CK*OH
     C(I)+B/(4.*WJ))+A4*OF(I))*(A2*TTA(I)/(4.*WJ)+A3/V+A5))+(
     C(1./2.)*(RL(I)**2)*((A2*TTA(I)/(4.*WJ)+A3/V+A5)**2)))
      PAL(I)=(PA(I)*TTA(I)*OH(I))/(T*PI*2.)
      Y=Y+0.1
    2 CONTINUE
      WRITE(6,240) DM
      WRITE(6,250)
      WRITE(6,260)(I,RL(I),Y2(I),TTA(I),OH(I),OF(I),I=1,41)
      WRITE(6,270)
      WRITE(6,280) (I,RE(I),PA(I),CO(I),UB(I),PAL(I),I=1,41)
  200 FORMAT(//,15X,38('*'),/)
  210 FORMAT(14X,' RESULTS OF MAXIMIZING NET USER BENEFIT',/,14X,
     C' SUBJECT TO A DEFICIT CONSTRAINT WITH',14X,/,
     C14X,' CONSTANT TRIP DENSITY',/)
  230 FORMAT(15X,38('*'),/)
  240 FORMAT(/,20X,'OPERATING DEFICIT=',1X,F6.2,' (CENTS)',/)
  250 FORMAT(8X,' ROUTE LENGTH',2X,' SHADOW PRICE',2X,' ANGLE'
     C,2X,' HEADWAY',2X,' FARE',/,10X,' (MILES)',19X,' (RAD.)',2X,' (M
  IN
     C.)',2X,' (CENTS)',/)
  260 FORMAT(1X,I2,10X,F4.1,10X,F6.3,6X,F6.3,3X,F5.2,3X,F6.2)
  270 FORMAT(/,6X,' REVENUE',3X,' PASSENGERS',4X,' COST',5X,' NET USER
   B
     CENEF.',2X,' BUS-LOAD',/,5X,' (DOLLARS)',15X,' (DOLLARS)',5X,' (D
  OL
     CLARS)',/)
  280 FORMAT(1X,I2,1X,F9.2,5X,F8.0,4X,F9.2,5X,F10.2,8X,F5.0)
      STOP
      END
```

Results (continued)

****

RESULTS OF MAXIMIZING NET USER BENEFIT
SUBJECT TO A DEFICIT CONSTRAINT WITH
CONSTANT TRIP DENSITY

****

OPERATING DEFICIT= .00 (CENTS)

	ROUTE LENGTH (MILES)	SHADOW PRICE	ANGLE (RAD.)	HEADWAY (MIN.)	FARE (CENTS)
1	6.0	1.508	.368	18.39	56.11
2	6.1	1.502	.361	18.36	55.98
3	6.2	1.496	.355	18.34	55.85
4	6.3	1.490	.349	18.31	55.72
5	6.4	1.485	.343	18.28	55.60
6	6.5	1.479	.337	18.26	55.47
7	6.6	1.474	.332	18.24	55.34
8	6.7	1.468	.326	18.21	55.22
9	6.8	1.463	.321	18.19	55.10
10	6.9	1.458	.316	18.16	54.97
11	7.0	1.453	.311	18.14	54.85
12	7.1	1.448	.306	18.11	54.73
13	7.2	1.444	.302	18.09	54.61
14	7.3	1.439	.297	18.07	54.49
15	7.4	1.435	.293	18.04	54.37
16	7.5	1.430	.288	18.02	54.26
17	7.6	1.426	.284	18.00	54.14
18	7.7	1.421	.280	17.98	54.03
19	7.8	1.417	.276	17.95	53.91
20	7.9	1.413	.272	17.93	53.80
21	8.0	1.409	.269	17.91	53.68
22	8.1	1.405	.265	17.89	53.57
23	8.2	1.401	.261	17.87	53.46
24	8.3	1.397	.258	17.84	53.35
25	8.4	1.394	.255	17.82	53.24
26	8.5	1.390	.251	17.80	53.13
27	8.6	1.386	.248	17.78	53.02
28	8.7	1.383	.245	17.76	52.92
29	8.8	1.379	.242	17.74	52.81
30	8.9	1.376	.239	17.72	52.70
31	9.0	1.372	.236	17.70	52.60
32	9.1	1.369	.233	17.68	52.50
33	9.2	1.366	.230	17.66	52.39
34	9.3	1.363	.228	17.64	52.29
35	9.4	1.359	.225	17.62	52.19
36	9.5	1.356	.222	17.60	52.09
37	9.6	1.353	.220	17.58	51.99
38	9.7	1.350	.217	17.56	51.89
39	9.8	1.347	.215	17.54	51.79
40	9.9	1.344	.212	17.52	51.69
41	10.0	1.342	.210	17.50	51.59

Results (continued)

	REVENUE (DOLLARS)	PASSENGERS	COST (DOLLARS)	NET USER BENEF. (DOLLARS)	BUS-LOAD
1	5308.36	9460.	5012.78	8318.95	57.
2	5508.40	9839.	5195.82	8710.47	58.
3	5712.69	10228.	5382.54	9114.62	59.
4	5921.25	10626.	5572.97	9531.63	60.
5	6134.12	11033.	5767.10	9961.72	61.
6	6351.32	11450.	5964.96	10405.14	62.
7	6572.88	11876.	6166.58	10862.14	63.
8	6798.80	12312.	6371.97	11332.97	65.
9	7029.15	12758.	6581.14	11817.85	66.
10	7263.90	13213.	6794.12	12317.06	67.
11	7503.12	13679.	7010.91	12830.82	68.
12	7746.80	14154.	7231.54	13359.39	69.
13	7994.99	14640.	7456.03	13903.04	71.
14	8247.70	15136.	7684.39	14462.02	72.
15	8504.96	15642.	7916.63	15036.55	73.
16	8766.78	16158.	8152.79	15626.95	74.
17	9033.19	16685.	8392.87	16233.45	75.
18	9304.23	17222.	8636.88	16856.27	77.
19	9579.90	17770.	8884.84	17495.73	78.
20	9860.22	18329.	9136.79	18152.12	79.
21	10145.25	18898.	9392.72	18825.61	80.
22	10434.96	19478.	9652.66	19516.55	82.
23	10729.41	20070.	9916.61	20225.18	83.
24	11028.62	20672.	10184.62	20951.80	84.
25	11332.60	21286.	10456.66	21696.60	85.
26	11641.38	21910.	10732.79	22459.96	87.
27	11954.98	22546.	11013.00	23242.09	88.
28	12273.42	23194.	11297.31	24043.32	89.
29	12596.72	23853.	11585.74	24863.89	90.
30	12924.92	24523.	11878.30	25704.07	92.
31	13258.02	25206.	12175.03	26564.21	93.
32	13596.03	25899.	12475.91	27444.54	94.
33	13939.02	26605.	12780.97	28345.36	96.
34	14286.97	27323.	13090.25	29266.95	97.
35	14639.93	28053.	13403.73	30209.60	98.
36	14997.88	28794.	13721.44	31173.66	100.
37	15360.88	29548.	14043.40	32159.37	101.
38	15728.95	30315.	14369.63	33167.00	102.
39	16102.08	31093.	14700.11	34196.91	104.
40	16480.34	31884.	15034.89	35249.34	105.
41	16863.71	32688.	15374.00	36324.66	106.

B.2 Results of Maximizing Net User Benefit Subject to a Deficit Constraint with Linearly Decreasing Trip Density

```
      DIMENSION Y(60),Y2(60),RL(60),TTA(60),OH(60),OF(60)
      DIMENSION RE(60),PA(60),CO(60),UB(60),PAL(60),F(60),F2(60)
      DATA A1,A2,A3,A4,A5/0.38,-0.0081,-0.0033,-0.0014,0.0328/
      DATA B,C,WJ,CK,P0,P/0.16,60.36,0.05,0.40,5.18,1.795/
      DATA V,DM,Y(1),T,W,PI/0.2417,0.0,6.0,180.0,1.37,3.14159/
      WRITE(6,200)
      WRITE(6,210)
      WRITE(6,220)
    1 DO 2 I=1,41
      A=A1+(A2*B/(4.*WJ))+((A3/V+A5)*Y(I)/2.)
      D=Y(I)
      YN=-5.*((3.*CK*(P0**2)*(A2**2)*(A**2)*C/(2.*(A4**2)*WJ*V
     C))**(1./3.))-P0*(A**2)/A4+12.*DM/(PI*T*(D**2))
      YM=6.*((3.*CK*(P0**2)*(A2**2)*(A**2)*C/(2.*(A4**2)*WJ*V
     C))**(1./3.))+P0*(A**2)/A4-12.*DM/(PI*T*(D**2))
      X=-3.*DM/(PI*T*D**2)
      Y2(I)=(-YN-(((YN**2)-(4.*YM*X))**.5))/(2.*YM)
      CM=(((2.*Y2(I)-1.)/(2.*Y2(I)))**(2./3.))*(((-12.*C*A4*(A
     C2**2)*CK)**(1./3.))/(((9.*(A**4)*WJ*V*P0)**(1./3.))-4.*(
     C(-12.*C*A4*(A2**2)*CK)**(1./3.))))
      RL(I)=Y(I)*(1.-CM)
      TTA(I)=((768.*(WJ**2)*CK*Y(I)*C*A4*(3.*Y(I)-2.*D)*(2.*Y2
     C(I)-1.))/(V*P0*A2*A*((4.*Y(I)-3.*D)**2)*Y2(I)*(D**3)))**
     C(1./3.)
      OH(I)=((3.*Y(I)*C*A4*(4.*Y(I)-3.*D)*(2.*Y2(I)-1.))/(2.*V
     C*P0*WJ*(CK**2)*A*A2*((3.*Y(I)-2.*D)**2)*Y2(I)))**(1./3.)
      OF(I)=((1.-Y2(I))/(2.*Y2(I)-1.))*(A/A4+2.*A2*CK*OH(I)/A4
     C)
      RE(I)=(W*PI*OF(I)*P0*T/100.)*(((A1+A2*(CK*OH(I)+B/(4.*WJ
     C))+A4*OF(I))*((RL(I)**2)/2.-(RL(I)**3)/(3.*Y(I))))+((A2*
     CTTA(I)/(4.*WJ)+A3/V+A5)*((RL(I)**3)/3.-(RL(I)**4)/(4.
     C*Y(I)))))
      PA(I)=(RE(I)/OF(I))*100.
      CO(I)=(2.*W*PI*RL(I)*C*T)/(TTA(I)*OH(I)*V*100.)
      UB(I)=(-W*PI*T*P0/(2.*A4*100.))*(((A1+A2*(CK*OH(I)+B/(4.
     C*WJ))+A4*OF(I))**2)*((RL(I)**2)/2.-(RL(I)**3)/(3.*Y(I)))+2.*(
     CA1+A2*(CK*OH(I)+B/(4.*WJ))+A4*OF(I))*(A2*TTA(I)/(4.*WJ)+
     CA3/V+A5)*((RL(I)**3)/3.-((RL(I)**4)/(4.*Y(I))))+((A2*TTA
     C(I)/(4.*WJ)+A3/V+A5)**2)*((RL(I)**4)/4.-(RL(I)**5)/(5.*Y
     C(I))))
      PAL(I)=PA(I)*TTA(I)*OH(I)/(T*PI*W)
      Y(I+1)=Y(I)+0.1
    2 CONTINUE
      WRITE(6,230) DM
      WRITE(6,240)
      WRITE(6,250)  (I,Y(I),RL(I),Y2(I),TTA(I),OH(I),OF(I),I=1,41)
      WRITE(6,260)
      WRITE(6,270)(I,RE(I),PA(I),CO(I),UB(I),PAL(I),I=1,41)
  200 FORMAT(//,18X,38('*'),/)
  210 FORMAT(17X,' RESULTS OF MAXIMIZING NET USER BENEFIT',/,17X,' SUB
  JE
     CCT TO A DEFICIT CONSTRAINT WITH',/,17X,' LINEARLY DECREASING TRI
  P
     CDENSITY',/)
  220 FORMAT(18X,38('*'),/)
  230 FORMAT(/,20X,' OPERATING DEFIC.=',1X,F6.2,' (CENTS)',/)
  240 FORMAT(4X,' RADIUS OF ANALY.',2X,' ROUTE LENGTH',2X,
     C' SHADOW PRICE',1X,' ANGLE',1X,' HEADWAY',3X,' FARE'/
     C7X,' (MILES)',10X,' (MILES)',19X,' (RAD)',
     C2X,' (MIN.)',2X,' (CENTS)',/)
  250 FORMAT(1X,I2,8X,F4.1,13X,F4.1,9X,F6.3,5X,F6.3,3X,F5.2,4X,F6.2)
  260 FORMAT(/,6X,' REVENUE',3X,' PASSENGERS',4X,' COST',5X,' NET USER
   B
     CENEF.',2X,' BUS-LOAD',/,5X,' (DOLLARS)',15X,' (DOLLARS)',5X,' (D
  OL
     CLARS)',/)
  270 FORMAT( 1X,I2,2X,F9.2,4X,F8.0,6X,F9.2,3X,F10.2,7X,F5.0)
      STOP
      END
```

****

RESULTS OF MAXIMIZING NET USER BENEFIT
SUBJECT TO A DEFICIT CONSTRAINT WITH
LINEARLY DECREASING TRIP DENSITY

****

OPERATING DEFIC.= .00 (CENTS)

	RADIUS OF ANALY. (MILES)	ROUTE LENGTH (MILES)	SHADOW PRICE	ANGLE (RAD)	HEADWAY (MIN.)	FARE (CENTS)
1	6.0	5.1	1.376	.444	16.65	49.54
2	6.1	5.2	1.373	.436	16.63	49.43
3	6.2	5.3	1.369	.429	16.61	49.33
4	6.3	5.4	1.366	.421	16.59	49.22
5	6.4	5.5	1.362	.414	16.57	49.12
6	6.5	5.6	1.359	.407	16.55	49.02
7	6.6	5.7	1.356	.401	16.53	48.92
8	6.7	5.8	1.352	.394	16.51	48.82
9	6.8	5.8	1.349	.388	16.49	48.72
10	6.9	5.9	1.346	.382	16.47	48.62
11	7.0	6.0	1.343	.376	16.45	48.52
12	7.1	6.1	1.340	.370	16.43	48.42
13	7.2	6.2	1.337	.365	16.41	48.33
14	7.3	6.3	1.334	.359	16.39	48.23
15	7.4	6.4	1.331	.354	16.37	48.14
16	7.5	6.5	1.328	.349	16.36	48.04
17	7.6	6.6	1.326	.344	16.34	47.95
18	7.7	6.7	1.323	.339	16.32	47.85
19	7.8	6.8	1.320	.334	16.30	47.76
20	7.9	6.9	1.318	.330	16.28	47.67
21	8.0	7.0	1.315	.325	16.26	47.58
22	8.1	7.0	1.312	.321	16.25	47.49
23	8.2	7.1	1.310	.317	16.23	47.40
24	8.3	7.2	1.307	.312	16.21	47.31
25	8.4	7.3	1.305	.308	16.19	47.22
26	8.5	7.4	1.303	.304	16.18	47.13
27	8.6	7.5	1.300	.301	16.16	47.04
28	8.7	7.6	1.298	.297	16.14	46.96
29	8.8	7.7	1.296	.293	16.12	46.87
30	8.9	7.8	1.293	.290	16.11	46.78
31	9.0	7.9	1.291	.286	16.09	46.70
32	9.1	8.0	1.289	.283	16.07	46.62
33	9.2	8.1	1.287	.279	16.06	46.53
34	9.3	8.2	1.284	.276	16.04	46.45
35	9.4	8.3	1.282	.273	16.02	46.36
36	9.5	8.4	1.280	.270	16.01	46.28
37	9.6	8.4	1.278	.266	15.99	46.20
38	9.7	8.5	1.276	.263	15.97	46.12
39	9.8	8.6	1.274	.261	15.96	46.04
40	9.9	8.7	1.272	.258	15.94	45.96
41	10.0	8.8	1.270	.255	15.92	45.88

Results (continued)

	REVENUE (DOLLARS)	PASSENGERS	COST (DOLLARS)	NET USER BENEF. (DOLLARS)	BUS-LOAD
1	2845.21	5744.	2675.94	5201.12	55.
2	2950.56	5969.	2775.77	5430.63	56.
3	3058.09	6200.	2877.69	5666.86	57.
4	3167.80	6436.	2981.71	5909.97	58.
5	3279.72	6677.	3087.86	6160.03	59.
6	3393.84	6924.	3196.13	6417.17	60.
7	3510.18	7176.	3306.54	6681.50	61.
8	3628.75	7433.	3419.11	6953.15	62.
9	3749.56	7697.	3533.84	7232.21	64.
10	3872.63	7965.	3650.74	7518.81	65.
11	3997.96	8240.	3769.84	7813.06	66.
12	4125.57	8520.	3891.13	8115.09	67.
13	4255.46	8806.	4014.63	8425.00	68.
14	4387.66	9097.	4140.36	8742.91	69.
15	4522.15	9395.	4268.32	9068.95	70.
16	4658.98	9698.	4398.52	9403.23	71.
17	4798.12	10007.	4530.97	9745.88	73.
18	4939.61	10322.	4665.70	10097.01	74.
19	5083.46	10643.	4802.70	10456.74	75.
20	5229.65	10971.	4941.99	10825.22	76.
21	5378.23	11304.	5083.57	11202.53	77.
22	5529.18	11643.	5227.47	11588.83	78.
23	5682.53	11989.	5373.70	11984.22	80.
24	5838.28	12341.	5522.25	12388.84	81.
25	5996.46	12699.	5673.14	12802.79	82.
26	6157.04	13064.	5826.39	13226.25	83.
27	6320.06	13434.	5982.00	13659.31	84.
28	6485.53	13812.	6139.99	14102.09	85.
29	6653.46	14195.	6300.36	14554.71	87.
30	6823.84	14586.	6463.12	15017.35	88.
31	6996.70	14982.	6628.30	15490.12	89.
32	7172.05	15386.	6795.89	15973.11	90.
33	7349.89	15796.	6965.91	16466.48	91.
34	7530.23	16212.	7138.36	16970.38	93.
35	7713.10	16636.	7313.27	17484.94	94.
36	7898.47	17066.	7490.63	18010.26	95.
37	8086.38	17503.	7670.45	18546.49	96.
38	8276.84	17946.	7852.77	19093.78	97.
39	8469.85	18397.	8037.56	19652.25	99.
40	8665.43	18855.	8224.85	20222.04	100.
41	8863.58	19319.	8414.66	20803.29	101.

B.3 Results of Maximizing Net User Benefit Subject to a Deficit and Capacity Constraint with Linearly Decreasing Trip Density.

```
      DIMENSION Y(10),Y2(10),RL(10),TTA(20),OH(20),OF(20)
      DIMENSION RE(20),PA(20),CO(20),UB(20),PAL(20),F(20),F2(20)
      DATA A1,A2,A3,A4,A5/0.38,-0.0081,-0.0033,-0.0014,0.0328/
      DATA B,C,WJ,CK,P0,P/0.16,60.36,0.05,0.40,5.18,1.795/
      DATA V,DM,Y(1),T,W,PI/0.2417,0.0,9.3,180.0,1.37,3.14159/
      WRITE(6,200)
      WRITE(6,210)
      WRITE(6,220)
      F(1)=0.70
      J=1
      A=A1+(A2*B/(4.*WJ))+((A3/V+A5)*Y(J)/2.)
      D=Y(J)
      YN=-5.*((3.*CK*(P0**2)*(A2**2)*(A**2)*C/(2.*(A4**2)*WJ*V
     C))**(1./3.))-P0*(A**2)/A4+12.*DM/(PI*T*(D**2))
      YM=6.*((3.*CK*(P0**2)*(A2**2)*(A**2)*C/(2.*(A4**2)*WJ*V
     C))**(1./3.))+P0*(A**2)/A4-12.*DM/(PI*T*(D**2))
      X=-3.*DM/(PI*T*D**2)
      Y2(J)=(-YN-(((YN**2)-(4.*YM*X))**.5))/(2.*YM)
      CM=(((2.*Y2(J)-1.)/(2.*Y2(J)))**(2./3.))*(((-12.*C*A4*(A
     C2**2)*CK)**(1./3.))/(((9.*(A**4)*WJ*V*P0)**(1./3.))-4.*(
     C(-12.*C*A4*(A2**2)*CK)**(1./3.))))
      RL(J)=Y(J)*(1.-CM)
      DO 2 I=1,11
      TTA(I)=((768.*(WJ**2)*CK*Y(J)*C*A4*(3.*Y(J)-2.*D)*(2.*Y2
     C(J)-1.))/(V*P0*A2*A*((4.*Y(J)-3.*D)**2)*Y2(J)*(D**3)))**
     C(1./3.)*F(I)
      OH(I)=((3.*Y(J)*C*A4*(4.*Y(J)-3.*D)*(2.*Y2(J)-1.))/(2.*V
     C*P0*WJ*(CK**2)*A*A2*((3.*Y(J)-2.*D)**2)*Y2(J)))**(1./3.)
     C*F(I)
      OF(I)=((1.-Y2(J))/(2.*Y2(J)-1.))*(A/A4+2.*A2*CK*OH(I)/A4
     C)
      RE(I)=(W*PI*OF(I)*P0*T/100.)*(((A1+A2*(CK*OH(I)+B/(4.*WJ
     C))+A4*OF(I))*((RL(J)**2)/2.-(RL(J)**3)/(3.*Y(J))))+((A2*
     CTTA(I)/(4.*WJ)+A3/V+A5)*((RL(J)**3)/3.-(RL(J)**4)/(4.
     C*Y(J)))))
      PA(I)=(RE(I)/OF(I))*100.
      CO(I)=(2.*W*PI*RL(J)*C*T)/(TTA(I)*OH(I)*V*100.)
      UB(I)=(-W*PI*T*P0/(2.*A4*100.))*(((A1+A2*(CK*OH(I)+B/(4.
     C*WJ))+A4*OF(I))**2)*((RL(J)**2)/2.-(RL(J)**3)/(3.*Y(J)))+2.*(
     CA1+A2*(CK*OH(I)+B/(4.*WJ))+A4*OF(I))*(A2*TTA(I)/(4.*WJ)+
     CA3/V+A5)*((RL(J)**3)/3.-((RL(J)**4)/(4.*Y(J))))+((A2*TTA
     C(I)/(4.*WJ)+A3/V+A5)**2)*((RL(J)**4)/4.-(RL(J)**5)/(5.*Y
     C(J))))
      PAL(I)=PA(I)*TTA(I)*OH(I)/(T*PI*W)
      F2(I)=1-F(I)
      F(I+1)=F(I)-0.01
    2 CONTINUE
      WRITE(6,230) DM,Y(1),Y2(1),RL(1)
      WRITE(6,240)
      WRITE(6,250)  (I,F2(I),TTA(I),OH(I),OF(I),I=1,11)
      WRITE(6,260)
      WRITE(6,270)(I,RE(I),PA(I),CO(I),UB(I),PAL(I),I=1,11)
  200 FORMAT(//,18X,38('*'),/)
  210 FORMAT(17X,' RESULTS OF MAXIMIZING NET USER BENEFIT',/,17X,' SUB
  JE
     CCT TO A DEFICIT AND CAPACITY CON-',/,17X,' STREINTS WITH LINEARL
  Y
     CDECREASING TRIP',/,17X,' DENSITY',/)
  220 FORMAT(18X,38('*'),/)
  230 FORMAT(/,22X,' OPERATING DEFIC.=',F4.1,1X,' (CENTS)',/,22X,
     C' RADIUS OF ANALY.=',F4.1,1X,' (MILES)',/,22X,
     C' SHADOW PRICE     =',F6.3,1X,' (DOLLARS)',/,22X,
     C' ROUTE LENGTH     =',F4.1,1X,' (MILES)',/)
  240 FORMAT(15X,' DECREASING FACTOR',2X,' ANGLE',1X,' HEADWAY',
     C3X,' FARE',/,21X,' (%)',10X,' (RAD)',2X,' (MIN)',2X,' (CENTS)',/
  )
```

```
250 FORMAT(1X,I2,18X,F5.4,10X,F6.3,3X,F5.2,3X,F6.2)
260 FORMAT(/,6X,' REVENUE',3X,' PASSENGERS',4X,' COST',5X,' NET USER
 B
   CENEF.',2X,' BUS-LOAD',/,5X,' (DOLLARS)',15X,' (DOLLARS)',5X,' (D
OL
   CLARS)',/)
270 FORMAT( 1X,I2,2X,F9.2,4X,F8.0,6X,F9.2,3X,F10.2,7X,F5.0)
    STOP
    END
```

****

RESULTS OF MAXIMIZING NET USER BENEFIT SUBJECT TO A DEFICIT AND CAPACITY CONSTREINTS WITH LINEARLY DECREASING TRIP DENSITY

****

OPERATING DEFIC.= .0 (CENTS)
RADIUS OF ANALY.= 9.3 (MILES)
SHADOW PRICE = 1.284 (DOLLARS)
ROUTE LENGTH = 8.2 (MILES)

	DECREASING FACTOR (%)	ANGLE (RAD)	HEADWAY (MIN)	FARE (CENTS)
1	.3000	.193	11.23	50.49
2	.3100	.190	11.07	50.62
3	.3200	.188	10.91	50.76
4	.3300	.185	10.75	50.89
5	.3400	.182	10.59	51.02
6	.3500	.179	10.43	51.16
7	.3600	.177	10.26	51.29
8	.3700	.174	10.10	51.43
9	.3800	.171	9.94	51.56
10	.3900	.168	9.78	51.70
11	.4000	.166	9.62	51.83

	REVENUE (DOLLARS)	PASSENGERS	COST (DOLLARS)	NET USER BENEF. (DOLLARS)	BUS-LOAD
1	8884.08	17597.	14568.03	20033.86	49.
2	8931.13	17643.	14993.36	20140.52	48.
3	8978.31	17689.	15437.57	20247.48	47.
4	9025.62	17736.	15901.84	20354.73	45.
5	9073.05	17782.	16387.36	20462.28	44.
6	9120.60	17828.	16895.46	20570.12	43.
7	9168.27	17874.	17427.57	20678.24	42.
8	9216.08	17920.	17985.22	20786.67	41.
9	9264.00	17966.	18570.06	20895.39	39.
10	9312.05	18013.	19183.91	21004.41	38.
11	9360.22	18059.	19828.70	21113.71	37.

B.4 Sensitivity Analysis

```
      DATA T,W,PI,TTA,C/180.,1.37,3.14159,.239,60.36/
      DATA V,RL/.2417,8.2/
      H=34
    1 H=H-2
      DO 2 J=1,13
      Q=44-J
      PA=Q*T*PI*W/(TTA*H)
      CO=2*W*PI*RL*C*T/(TTA*H*V*100.)
      WRITE(6,100) H,Q,PA,CO
      WRITE(6,101)
      DO 2 I=1,90,5
      F=49.+I
      RE=PA*F/100.
      POL=RE-CO
      WRITE(6,102)F,RE,POL
    2 CONTINUE
      IF (H.EQ.32) GO TO 3
      GO TO 1
  100 FORMAT(/,16X,' HEADWAY= ',F5.2,1X,' (MIN.)',/,16X,' BUS CAP.= ',
     CF5.2,/,16X,' RIDERSHIP= ',F5.0,/,16X,' COST= ',F9.2,1X,' (DOLLAR
S)
     C',/)
  101 FORMAT(/,10X,' FARE ',7X,' REVENUE ',7X,' PROFIT ',/,9X,
     C' (CENTS)',5X,' (DOLLARS) ',5X,' (DOLLARS) ',/)
  102 FORMAT(10X,F6.2,6X,F9.2,6X,F9.2)
    3 STOP
      END
```

Results (continued)

HEADWAY= 32.00 (MIN.)
BUS CAP.= 43.00
RIDERSHIP= 4356.
COST= 4148.69 (DOLLARS)

FARE (CENTS)	REVENUE (DOLLARS)	PROFIT (DOLLARS)
50.00	2177.88	-1970.82
55.00	2395.66	-1753.03
60.00	2613.45	-1535.24
65.00	2831.24	-1317.45
70.00	3049.03	-1099.67
75.00	3266.81	-881.88
80.00	3484.60	-664.09
85.00	3702.39	-446.30
90.00	3920.18	-228.52
95.00	4137.96	-10.73
100.00	4355.75	207.06
105.00	4573.54	424.85
110.00	4791.33	642.63
115.00	5009.12	860.42
120.00	5226.90	1078.21
125.00	5444.69	1296.00
130.00	5662.48	1513.79
135.00	5880.26	1731.57

HEADWAY= 32.00 (MIN.)
BUS CAP.= 40.00
RIDERSHIP= 4052.
COST= 4148.69 (DOLLARS)

FARE (CENTS)	REVENUE (DOLLARS)	PROFIT (DOLLARS)
50.00	2025.93	-2122.76
55.00	2228.52	-1920.17
60.00	2431.12	-1717.57
65.00	2633.71	-1514.98
70.00	2836.30	-1312.39
75.00	3038.90	-1109.80
80.00	3241.49	-907.20
85.00	3444.08	-704.61
90.00	3646.68	-502.02
95.00	3849.27	-299.42
100.00	4051.86	-96.83
105.00	4254.46	105.76
110.00	4457.05	308.36
115.00	4659.64	510.95
120.00	4862.24	713.54
125.00	5064.83	916.14
130.00	5267.42	1118.73
135.00	5470.01	1321.32

Results (continued)

HEADWAY= 32.00 (MIN.)
BUS CAP.= 37.00
RIDERSHIP= 3748.
COST= 4148.69 (DOLLARS)

FARE (CENTS)	REVENUE (DOLLARS)	PROFIT (DOLLARS)
50.00	1873.99	-2274.71
55.00	2061.38	-2087.31
60.00	2248.78	-1899.91
65.00	2436.18	-1712.51
70.00	2623.58	-1525.11
75.00	2810.98	-1337.71
80.00	2998.38	-1150.31
85.00	3185.78	-962.92
90.00	3373.17	-775.52
95.00	3560.57	-588.12
100.00	3747.97	-400.72
105.00	3935.37	-213.32
110.00	4122.77	-25.92
115.00	4310.17	161.48
120.00	4497.57	348.87
125.00	4684.96	536.27
130.00	4872.36	723.67
135.00	5059.76	911.07

HEADWAY= 32.00 (MIN.)
BUS CAP.= 31.00
RIDERSHIP= 3140.
COST= 4148.69 (DOLLARS)

FARE (CENTS)	REVENUE (DOLLARS)	PROFIT (DOLLARS)
50.00	1570.10	-2578.60
55.00	1727.11	-2421.59
60.00	1884.12	-2264.58
65.00	2041.13	-2107.57
70.00	2198.14	-1950.56
75.00	2355.15	-1793.55
80.00	2512.15	-1636.54
85.00	2669.16	-1479.53
90.00	2826.17	-1322.52
95.00	2983.18	-1165.51
100.00	3140.19	-1008.50
105.00	3297.20	-851.49
110.00	3454.21	-694.48
115.00	3611.22	-537.47
120.00	3768.23	-380.46
125.00	3925.24	-223.45
130.00	4082.25	-66.44
135.00	4239.26	90.57

```
      DATA W,V,T,PI,TTA,RL,H,C/1.37,.2417,180,3.14159,.239,8.2,32,60.3
6./
      PA=2900
      CO=2*W*PI*RL*C*T/(TTA*H*V*100.)
      DO 1 I=1,15
      PA=PA+100
      WRITE(6,100)PA,CO,H
      WRITE(6,101)
      DO 1 J=1,100,5
      F=49+J
      RE=F*PA/100.
      POD=RE-CO
      BC=PA*TTA*H/(T*PI*W)
      WRITE(6,102)F,RE,POD,BC
    1 CONTINUE
  100 FORMAT(/,20X,' RIDERSHIP= ',F6.0,/,20X,' COST= ',F9.2,/,
     C20X,' HEADWAY= ',F5.2,/)
  101 FORMAT(/,10X,' FARE ',7X,' REVENUE ',7X,' PROFIT ',3X,' BUS CAP.
  '
     C,/,9X,' (CENTS) ',4X,' (DOLLARS) ',5X,' (DOLLARS) ',/)
  102 FORMAT(10X,F6.2,6X,F9.2,7X,F9.2,4X,F5.0)
      STOP
      END
```

RIDERSHIP= 3100.
COST= 4148.69
HEADWAY= 32.00

FARE (CENTS)	REVENUE (DOLLARS)	PROFIT (DOLLARS)	BUS CAP.
50.00	1550.00	-2598.69	31.
55.00	1705.00	-2443.69	31.
60.00	1860.00	-2288.69	31.
65.00	2015.00	-2133.69	31.
70.00	2170.00	-1978.69	31.
75.00	2325.00	-1823.69	31.
80.00	2480.00	-1668.69	31.
85.00	2635.00	-1513.69	31.
90.00	2790.00	-1358.69	31.
95.00	2945.00	-1203.69	31.
100.00	3100.00	-1048.69	31.
105.00	3255.00	-893.69	31.
110.00	3410.00	-738.69	31.
115.00	3565.00	-583.69	31.
120.00	3720.00	-428.69	31.
125.00	3875.00	-273.69	31.
130.00	4030.00	-118.69	31.
135.00	4185.00	36.31	31.
140.00	4340.00	191.31	31.
145.00	4495.00	346.31	31.

RIDERSHIP= 3700.
COST= 4148.69
HEADWAY= 32.00

FARE (CENTS)	REVENUE (DOLLARS)	PROFIT (DOLLARS)	BUS CAP.
50.00	1850.00	-2298.69	37.
55.00	2035.00	-2113.69	37.
60.00	2220.00	-1928.69	37.
65.00	2405.00	-1743.69	37.
70.00	2590.00	-1558.69	37.
75.00	2775.00	-1373.69	37.
80.00	2960.00	-1188.69	37.
85.00	3145.00	-1003.69	37.
90.00	3330.00	-818.69	37.
95.00	3515.00	-633.69	37.
100.00	3700.00	-448.69	37.
105.00	3885.00	-263.69	37.
110.00	4070.00	-78.69	37.
115.00	4255.00	106.31	37.
120.00	4440.00	291.31	37.
125.00	4625.00	476.31	37.
130.00	4810.00	661.31	37.
135.00	4995.00	846.31	37.
[illegible]	[illegible]	[illegible]	[illegible]

Results (continued)

RIDERSHIP= 4400.
COST= 4148.69
HEADWAY= 32.00

FARE (CENTS)	REVENUE (DOLLARS)	PROFIT (DOLLARS)	BUS CAP.
50.00	2200.00	-1948.69	43.
55.00	2420.00	-1728.69	43.
60.00	2640.00	-1508.69	43.
65.00	2860.00	-1288.69	43.
70.00	3080.00	-1068.69	43.
75.00	3300.00	-848.69	43.
80.00	3520.00	-628.69	43.
85.00	3740.00	-408.69	43.
90.00	3960.00	-188.69	43.
95.00	4180.00	31.31	43.
100.00	4400.00	251.31	43.
105.00	4620.00	471.31	43.
110.00	4840.00	691.31	43.
115.00	5060.00	911.31	43.
120.00	5280.00	1131.31	43.
125.00	5500.00	1351.31	43.
130.00	5720.00	1571.31	43.
135.00	5940.00	1791.31	43.
140.00	6160.00	2011.31	43.

APPENDIX C

SUPPORTING DATA FOR CASE STUDY

Table C.1

Weekday Buses per Hour
(6:00 A.M.-12:00 Midnight)

Route	6	7	8	9	10	11	12	1	2	3	4	5	6	7-12
Alton Park East Chattanooga North Chattanooga	5	11	11	8	8	6	5	5	5	10	11	13	10	2
Ross. Foust	1	2	2	2	1	1	1	1	1	1	2	2	1	0
Brained East Ridge	1	2	2	2	2	2	2	2	1	1	2	2	2	1
Golden Gateway	1	2	2	2	2	2	2	2	2	2	2	2	2	0
Tyner Silverdale Morris Hill Hurricane Creek	1	3	2	1	0	0	0	0	0	1	2	3	2	0
Eastdale	0	2	2	1	1	1	1	1	1	2	2	2	2	0
North Brained	0	1	1	1	0	0	0	0	1	1	1	1	1	0
East Lake Northgate	2	7	8	3	3	3	3	3	3	3	5	6	4	0
M.L. King Carter	0	1	1	1	1	1	1	1	1	1	1	1	1	0
Red Bank ST. Elmo	1	4	3	3	2	1	1	1	2	1	3	4	3	0
Signal Mtn.	0	1	1	1	0	0	0	0	1	1	1	1	1	0
Amnicola Hwy. Murray Hills	0	3	3	1	0	0	0	1	1	1	2	3	2	0
Lookout Mtn.	0	1	1	1	0	0	0	1	1	1	1	1	1	0
Total	12	41	40	28	21	18	17	20	20	27	36	42	33	3

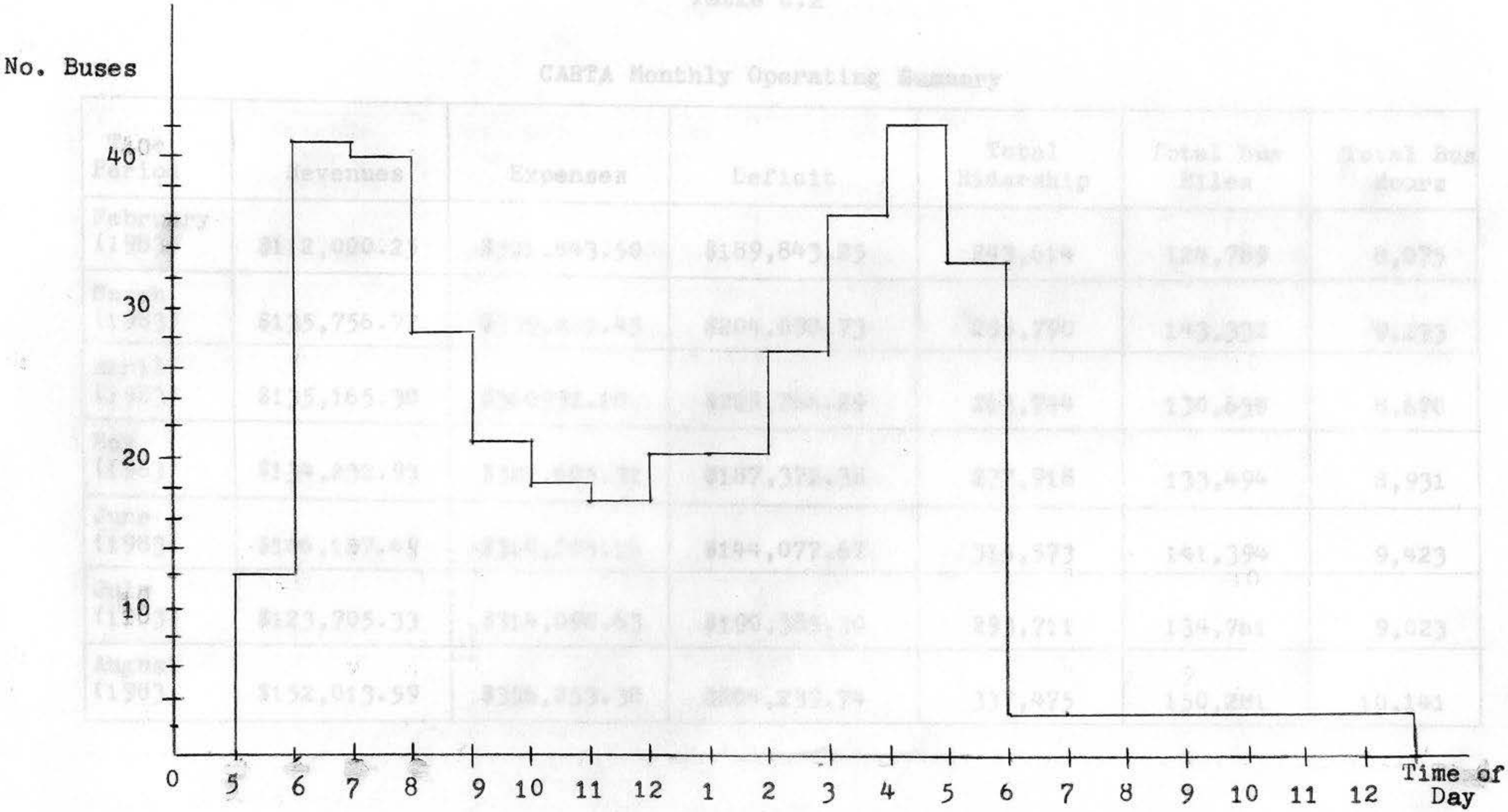

Figure C.1 Weekday Buses per Hour
6:00 AM-12:00 Midnight

Table C.2

CARTA Monthly Operating Summary

Time Period	Revenues	Expenses	Deficit	Total Ridership	Total Bus Miles	Total Bus Hours
February (1983)	$112,000.25	$301,843.50	$189,843.25	243,614	124,789	8,075
March (1983)	$135,756.72	$339,855.45	$204,098.73	285,790	143,332	9,273
April (1983)	$135,165.30	$360932.10	$225,766.29	263,744	130,638	8,670
May (1983)	$134,232.93	$321,605.31	$187,372.38	277,918	133,494	8,931
June (1983)	$166,127.49	$310,205.16	$144,077.67	315,573	141,394	9,423
July (1983)	$123,705.33	$314,090.63	$190,385.30	293,711	134,761	9,023
August (1983)	$152,013.59	$356,253.30	$204,239.74	337,475	150,281	10,141

Table C.2 (Continued)

Time Period	Revenue per Passenger	Revenue per Mile	Revenue per Hour	Expense per Passenger	Expense per Mile	Expense per Hour
February (1983)	$0.46	$0.90	$13.87	$1.24	$2.42	$37.38
March (1983)	$0.48	$0.95	$14.64	$1.19	$2.31	$36.65
April (1983)	$0.51	$1.04	$15.59	$1.37	$2.76	$41.63
May (1983)	$0.483	$1.01	$15.03	$1.16	$2.41	$36.01
June (1983)	$0.53	$1.18	$17.63	$0.98	$2.19	$32.92
July (1983)	$0.42	$0.92	$13.71	$1.07	$2.33	$34.81
August (1983)	$0.45	$1.01	$14.99	$1.06	$2.37	$35.13

Table C.3

Route Average Peak Period Headway

Route	Average Peak Period Headway (minutes)
No.1 Alton Park	11.30
No.2 North Chattanooga	39.17
No.4 Brainerd-Eastgate	29.50
No.5 East Ridge	45.71
No.7 Morris Hill Road	95.00
No.8 Eastdale	29.29
No.9 East Lake	22.27
No.10 East Chattanooga	12.12
No.11 North Brainerd	29.29
No.12 Red Bank	34.24
No.13 Rossville Foust	40.00
No.14 Signal Mountain	29.50
No.15 St.Elmo	33.33
No.16 Northgate	26.11
No.19 M.L. King Blv.	50.00
No.20 Carter Street	50.00
No.21 Golden Gateway	46.00
No.28 Amnicola Hwy.	28.00
No.31 Lookout Mountain	38.53